# Classification and Ordering of Portfolios and of New Insured Unities of Risks

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# Introduction

**The classical definitions of classification and ordering of risks.**

- The **classification** of risks is used to group individual risks to which it must be applied the same premium.

The aim is the protection of the insurance system's financial soundness.

- The **ordering** of risks is a comparison of risks belonging to two different classes.

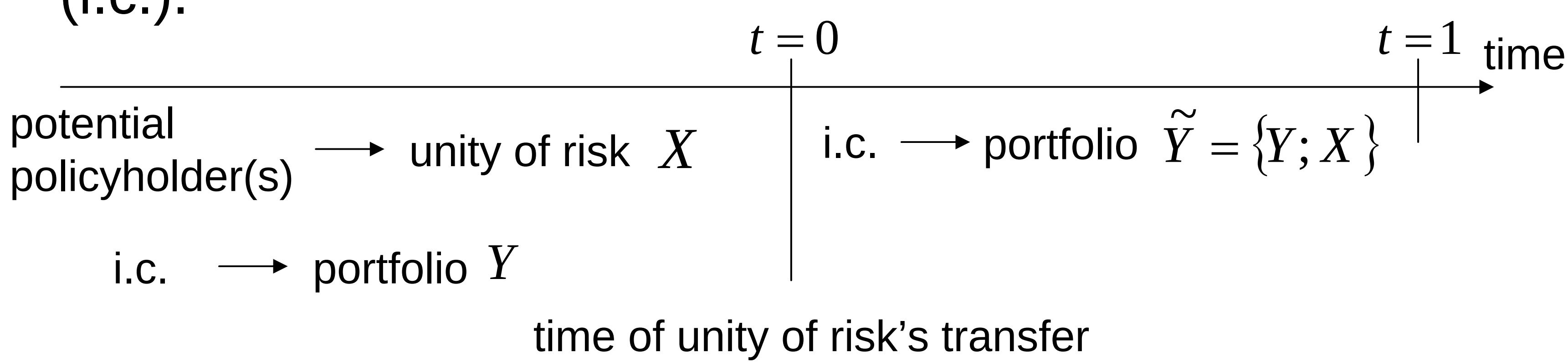
The aim is to establish to which risk it must be applied the greater premium.

- Both the classification and the ordering are based on the risks' measures.

# Introduction

The basic ideas of our model.

- The classification and the ordering are made after the risks are insured (the purpose is the outline of a reinsurance strategy).
- The classification and the ordering are based on the changes produced in the state of the business in the passage from a generic portfolio $Y$ to the new portfolio $\widetilde{Y}$ managed by a property-casualty insurance company (i.c.).



# Introduction

Tools used in our model.

- The (*actuarial*) ***business*** of the i.c.:

  Business={Portfolio, Operative structure}.

- The ***operative structure****:*

  it is the set of the constraints and rules imposed on the portfolio's management by the insurance market and the regulatory authority and of the criteria adopted by the i.c.

- The ***state of the business****:*

  it is described by using the loss exceedance probability (LEP) curve and some vectors defined as state variables.

- The ***state variables***' measures:

  they are used to establish the ordering criteria.

# Outline of the paper

- Portfolio of risks $Y$ run by the i.c. at time $t=0^-$.
- The description of the business $B(Y)$.
- The state of the business $B(Y)$.
- The graphic representation of the state of the business $B(Y)$.
- The classification of $Y$.
- The ordering of $Y$.
- The new unity of risk $X$ introduced at time $t=0$.
- The new portfolio $\tilde{Y}$.
- The state of the business $B(\tilde{Y})$.
- The graphic representation of $B(Y)$ and $B(\tilde{Y})$.
- The classification of $X$.
- The ordering of $X$.

# Portfolio of risks $Y$ run by the i.c. at time $t=0^{-}$

Notations and assumptions.

- $Y=\{Y_1,\ldots,Y_l\},\quad l>>1,\quad Y_i$ independent components.
- The risk's analysis is over a time horizon of one year.
- The composition of $Y$ does not change over $[0,1]$.
- The total claim amount in one year is represented by $S_Y$.
- $F_{S_Y}$ is known and defined on $[0,M(S_Y)],\quad M(S_Y)\in(0,\infty)$.
- The i.c. operates out of a continuous time economic environment. In particular the model does not include taxes, commissions, investment incomes, dividend payout to shareholders, inflation.

# The description of the business $B(Y)$

*The* ***business*** relative to portfolio $Y$ :

$$B(Y) = \left(Y, \Theta(Y)\right).$$

- $\Theta(Y)$ is the ***operative structure*** relative to portfolio $Y$.
- Criteria for defining $\Theta(Y)$.

1. Insurance market's constraint.
2. Criteria adopted by the i.c.
3. Conditions imposed on the i.c. by the regulatory authority (r.a.).

# The description of the business $B(Y)$

Independently of the principles used to assess the single pure premiums $P(Y_i)$, the premium income over one year can be expressed by

$$P(Y) = E[S_Y] + c(Y), \quad c(Y) = \eta(Y)E[S_Y].$$

1. Insurance market's constraint.

$$\eta(Y) \equiv \eta \in (0, M_\eta],$$

where $M_\eta > 0$ is the maximum of $\eta$ in the competitive insurance maket (i.m.) during $[0,1]$.

# The description of the business $B(Y)$

2. Criteria adopted by the i.c.

- The i.c. fixes $\varepsilon_0 \in (0,1)$ as the maximum acceptable ruin probability per year.
- The i.c. selects the premium calculation principles.
- Having expressed the free reserve proportional to the pure premium,

$$u(Y) = \alpha P(Y), \; \alpha \equiv \alpha(Y) > 0,$$

the i.c. fixes the maximum for $\alpha$, $M_\alpha > 0$, lower than a value $M_M$ linked to the i.m.

# The description of the business $B(Y)$

3. Conditions imposed on the i.c by the r.a. ( I ).

- The i.c. must operate within the maximum acceptable ruin probability per year $\varepsilon^* \in (0, \varepsilon_0]$ to which corresponds the maximum acceptable loss $MAL^*(Y)$.
- The i.c. must have a minimum free reserve

$$u^* = \alpha^*(1+\eta^*)E[S_Y].$$

The ***minimum acceptable capital structure*** is

$$CS^*(Y) = u^* + (1+\eta^*)E[S_Y] = h^* E[S_Y],$$

where $h^* = (1+\alpha^*)(1+\eta^*)$ is the ***minimum acceptable actuarial capitalization factor***.

# The description of the business $B(Y)$

3. Conditions imposed on the i.c by the r.a. ( II ).

- The only authorized state of the business is the ***acceptable state:***

$$\frac{CS(Y)}{CS^*(Y)} \geq 1, (\alpha,\eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta].$$

As particular case, it includes the ***stable state:***

$$\frac{CS(Y)}{CS^*(Y)} \geq 1+\varphi_0, (\alpha,\eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta],$$

where $\varphi_0 > 0$ is independent on the portfolio and $CS(Y) = u(Y) + P(Y)$ is the capital structure of the i.c.

# The description of the business $B(Y)$

- The acceptable and the stable states of the business are equivalent to a non negative and to a strictly positive **capacity**, respectively, having defined the capacity of the i.c. relative to the portfolio $Y$ by

$$C_Y(\alpha,\eta) = CS(Y) - CS^*(Y),$$

where

$$(\alpha,\eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta].$$

# The description of the business $B(Y)$

- The ***operative structure*** $\Theta(Y)$ expresses all the previous constraints, criteria and rules and it can be represented by the following set

$$\Theta(Y) = \left\{\varepsilon^*, M_\alpha, M_\eta, \varphi_0; \alpha(Y), \eta(Y), h(Y), \varepsilon \in (0, \varepsilon^*]\right\}.$$

# The graphic representation of the state of the business $B(Y)$

- To each d.f. $F_{S_Y}$ corresponds a loss exceedance probability (LEP) curve.
- We trace the LEP relative to $Y$ on the $(H, 0, \varepsilon)$ -plane, where

$$H = S_Y / E[S_Y].$$

- We represent the state of the business $B(Y) = (Y, \Theta(Y))$ on the LEP curve through the points

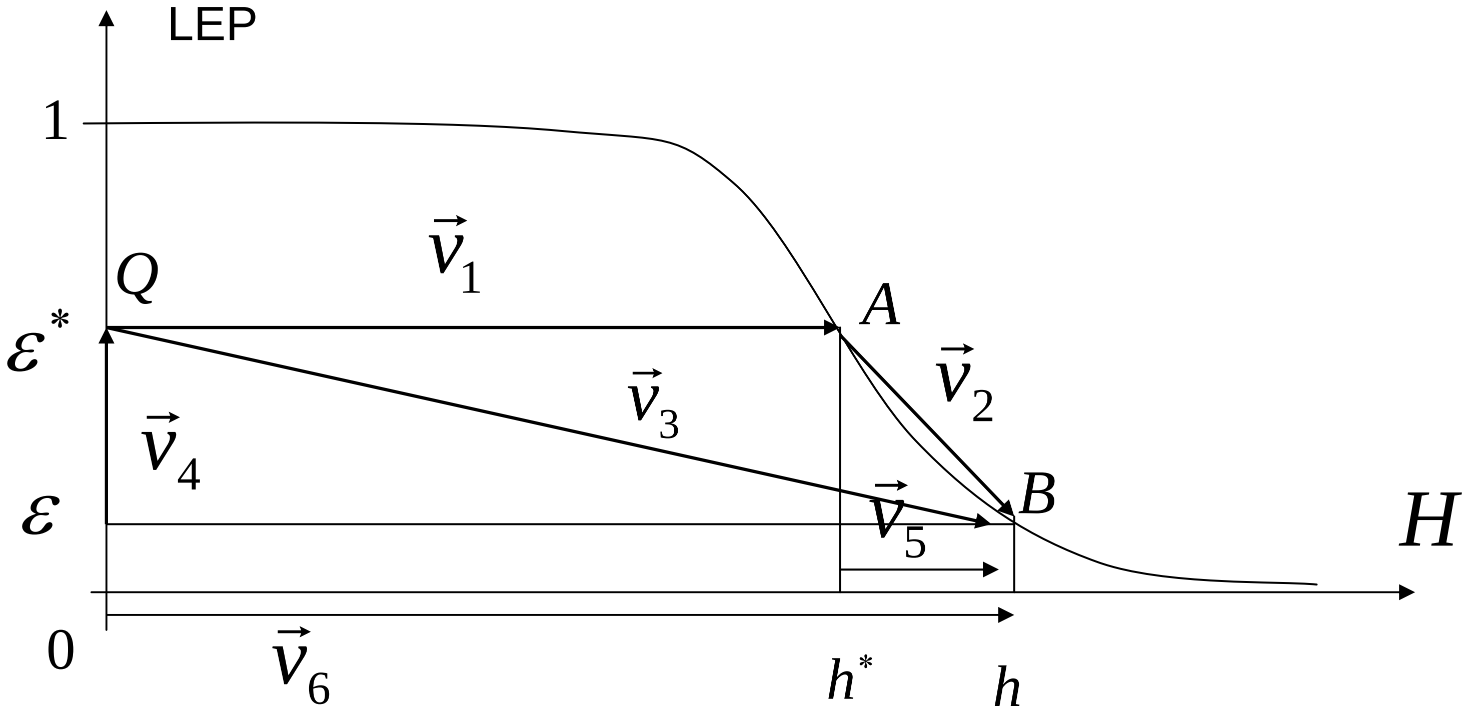
$$A = (h^*, \varepsilon^*), \qquad B = (h, \varepsilon), \qquad Q = (0, \varepsilon^*).$$

# The graphic representation of the state of the business $B(Y)$

$A, B, Q$ define on the $(H, 0, \varepsilon)$ -plane the ***state variables***, that is, the following six vectors:

$$\begin{cases} \vec{v}_1 = (h^*, 0), \\ \vec{v}_2 = (h - h^*, \varepsilon - \varepsilon^*), \\ \vec{v}_3 = (h, \varepsilon - \varepsilon^*), \\ \vec{v}_4 = (0, \varepsilon^* - \varepsilon), \\ \vec{v}_5 = (h - h^*, 0), \\ \vec{v}_6 = (h, 0). \end{cases}$$

# The graphic representation of the state of the business $B(Y)$



# The graphic representation of the state of the business $B(Y)$

- One of the three possible systems of basic state equations is

$$\begin{cases} \vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0, \\ v_2 + \vec{v}_4 - \vec{v}_5 = 0, \\ \vec{v}_3 + \vec{v}_4 - \vec{v}_6 = 0. \end{cases}$$

# The classification of $Y$

- The classification is based on the state of the business and is graphically expressed only through the vector $\vec{v}_4$.

  Definition.

  The management of portfolio $Y$

1. is ***authorized*** if either $\operatorname{sgn}\vec{v}_4 > 0$ (the stable state) or $|\vec{v}_4| = 0$ (the acceptable state),
2. It is ***not authorized*** if $\operatorname{sgn}\vec{v}_4 < 0$.

# The ordering of $Y$

- The ordering criteria $oc_i = oc_i(\Theta)$ for portfolio $Y$ are defined by the measures $\rho(\vec{v}_i)$ of the vectors $\vec{v}_i, i = 1, \ldots, 6.$
- The measures of the vectors are:

$$\rho(\vec{v}_i) = |\vec{v}_i|, i = 1,6,$$

$$\rho(\vec{v}_i) = \operatorname{sgn} \vec{v}_4 \cdot |\vec{v}_i|, i = 2,3,4,$$

if either $\operatorname{sgn} \vec{v}_4 > 0$ or $\operatorname{sgn} \vec{v}_4 < 0.$

$$\rho(\vec{v}_5) = \operatorname{sgn} \vec{v}_5 \cdot |\vec{v}_5|,$$

where $\operatorname{sgn} \vec{v}_5 > [<] 0$ when $h > [<] h^*.$

If $|\vec{v}_4| = 0,$ then $\vec{v}_1 = \vec{v}_3 = \vec{v}_6, \quad \vec{v}_2 = \vec{v}_5 = 0.$

# The ordering of $Y$

- Let $Y_1, Y_2$ be two portfolios whose states of the business are defined by the vectors $\vec{v}_i(Y_1), \vec{v}_i(Y_2), i = 1,\ldots,6.$

  Definition. $Y_1$ precedes $Y_2$ in the $oc_i(\Theta(\cdot))$ - order, i.e. $Y_1 \leq_{oc_i(\Theta(\cdot))} Y_2$,

  iff

$$\rho(\vec{v}_i(Y_1)) \leq \rho(\vec{v}_i(Y_2)), i = 1,\ldots,6.$$

  $Y_1$ precedes $Y_2$ in the $\Theta(\cdot)$ -order, i.e. $Y_1 \leq_{\Theta(\cdot)} Y_2$,

  iff

$$\rho(\vec{v}_i(Y_1)) \leq \rho(\vec{v}_i(Y_2)), \quad \forall\, i = 1,\ldots,6.$$

# The new unity of risk $X$ introduced at time $t=0$

## Notations and assumptions.

- $X=\{X_1,\ldots,X_m\}, \; m\geq 1, \; X_h$ dependent components.
- $S_X$ is the total claim amount of $X$ per year.
- $F_{S_X}$ is known and defined on

$$[0, M(S_X)], \; M(S_X)>0.$$

# The new portfolio $\tilde{Y}$

- The i.c. runs the portfolio $\tilde{Y} = \{Y; X\}$ in $[0,1]$.
- The state of the business is

$$B(\tilde{Y}) = (\tilde{Y}, \tilde{\Theta}), \quad \tilde{\Theta} = \Theta(\tilde{Y}).$$

- The operative structure relative to portfolio $\tilde{Y}$ is represented by

$$\tilde{\Theta} = \left\{\varepsilon^*, M_\alpha, \tilde{M}_\eta, \varphi_0; \alpha(\tilde{Y}), \eta(\tilde{Y}), h(\tilde{Y}), \tilde{\varepsilon} \in (0, \varepsilon^*]\right\}.$$

# The state of the business $B(\tilde{Y})$

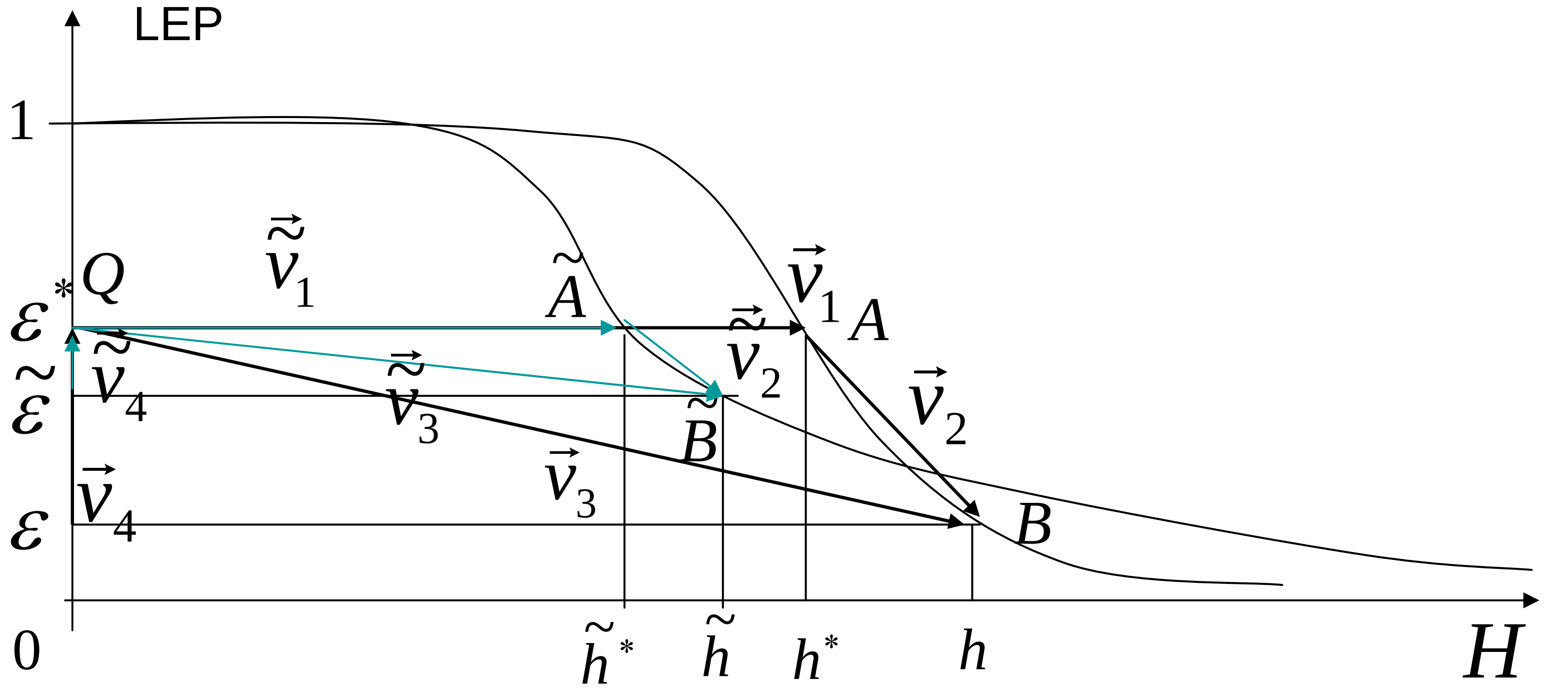
- The state of the business $B(\tilde{Y})$ is represented on the $(H, 0, \varepsilon)$- plane through the points

$$\tilde{A} = \left(\tilde{h}^*, \varepsilon^*\right), \tilde{B} = \left(\tilde{h}, \tilde{\varepsilon}\right), Q = \left(0, \varepsilon^*\right),$$

by which we define the new state variables

$$\begin{cases} \tilde{\vec{v}}_1 = \vec{v}_1 + \overrightarrow{A\tilde{A}}, & \tilde{\vec{v}}_2 = \vec{v}_2 + \overrightarrow{BF} + \overrightarrow{BG} - \overrightarrow{A\tilde{A}}, \\ \tilde{\vec{v}}_3 = \vec{v}_3 + \overrightarrow{BF} + \overrightarrow{BG}, & \tilde{\vec{v}}_4 = \vec{v}_4 - \overrightarrow{BG}, \\ \tilde{\vec{v}}_5 = \vec{v}_5 + \overrightarrow{BF} - \overrightarrow{A\tilde{A}}, & \tilde{\vec{v}}_6 = \vec{v}_6 + \overrightarrow{BF}. \end{cases}$$

# The graphic representation of $B(Y)$ and $B(\tilde{Y})$



# The classification of $X$

Assumption. The state of the business $B(Y)$ is authorized and $X$ is completely retained by the i.c.

- The classes and sub-classes of $X$ are

| | |
|---|---|
| normal risk | capacity generator risk |
| | capacity risk |
| dangerous risk | great risk |
| | catastrophic risk |
| | mega-catastrophic risk |

# The classification of $X$

Definition.

- $X$ is a normal risk iff $B(\tilde{Y})$ is authorized, i.e., $\tilde{h} \geq \tilde{h}^*$.

- $X$ is a dangerous risk iff $\tilde{h} < \tilde{h}^*$.

Definition. Let $X$ be a normal risk.

- $X$ is a capacity generator risk iff

$$\tilde{h} > \tilde{h}^* + \tau(h - h^*),$$

- $X$ is capacity risk iff

$$\tilde{h}^* \leq \tilde{h} < \tilde{h}^* + \tau(h - h^*).$$

where

$$\tau = E[S_Y] / E[S_{\tilde{Y}}].$$

# The classification of $X$

Definition. Let $X$ be a dangerous risk.

- $X$ is a great risk iff

$$\tilde{h} < \tilde{h}^{*} \leq h(M_{\alpha}, \tilde{M}_{\eta}).$$

- $X$ is a catastrophic risk iff

$$h(M_{\alpha}, \tilde{M}_{\eta}) < \tilde{h}^{*} \leq h(M_{M}, \tilde{M}_{\eta}).$$

- $X$ is a mega-catastrophic risk iff

$$h(M_{M}, \tilde{M}_{\eta}) < \tilde{h}^{*}.$$

# The ordering of $X$

- Let $B(Y)$ be a state of the business.
- Let $X_1, X_2$ be two unities of risk and let $\tilde{Y}_1 = \{Y; X_1\}$, $\tilde{Y}_2 = \{Y; X_2\}$ be the corresponding new portfolios.

Definition. $X_1$ precedes $X_2$ in the $oc_i(B(Y))$ - order, i.e. $X_1 \leq_{oc_i(B(Y))} X_2$,

iff

$$\tilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \tilde{Y}_2, \quad i = 1, \ldots, 6.$$

$X_1$ precedes $X_2$ in the $B(Y)$ -order, i.e. $X_1 \leq_{B(Y)} X_2$,

iff

$$\rho(\vec{v}_i(\tilde{Y}_1)) \leq \rho(\vec{v}_i(\tilde{Y}_2)), \quad \forall\, i = 1, \ldots, 6.$$

# The ordering of $X$

Proposition. If $\tilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \tilde{Y}_2, \quad i = 1,4,5,6,$
then

$$\rho(\vec{v}_i(\tilde{Y}_1)) \leq \rho(\vec{v}_i(\tilde{Y}_2)), \quad \forall\, i = 1,\ldots,6,$$

that is,

$$X_1 \leq_{B(Y)} X_2.$$