Classification and Ordering of Portfolios and of New Insured Unities of Risks

Augusto Freddi*, Giulia Sargenti*

April 30, 2005

Abstract

In this paper we suggest a model to classify and order portfolios and just insured unities of risks that are managed by a property–casualty insurance company. The examination of these risks is provided through what we define the "actuarial business" of the company, that is, the couple made by the distribution function of the total claim amount of the portfolio together with the set of constraints and rules that the insurance company must follow to manage such portfolio. Under this new approach, the classification and the ordering we have performed are made with respect to such actuarial business through some suitable measures. The results may be used for reinsurance purposes.

Key words: Classification, ordering, actuarial business, state of the business, operative structure.

1 Introduction

The classification and the ordering of risks are so far two topics widely discussed in the actuarial literature. Classification is used to group individual risks to which it must be applied the same premium. On the contrary, when two risks belong to two different classes, the ordering is used to establish to which one it must be applied a greater premium.

As remarked in [1] and [17], any risk classification should possess certain standards in order to achieve its primary purposes, such as the protection of the insurance system's financial soundness.

As regards the ordering of risks, in the monograph [12], Kaas et al. gave a survey of the principal mathematical tools for comparing risks.

Both the classification and the ordering of risks need some suitable measures. Indeed, risk measures aim at quantifying risks, providing a criterion to determine whether they

*Università degli Studi di Roma "La Sapienza", Dipartimento di Scienze Attuariali e Finanziarie, via Nomentana 41, 00161 Roma. Tel.: +39-06-49919552; fax: +39-06-44250289. E–mail: augusto.freddi@uniroma1.it, giulia.sargenti@uniroma1.it.

are "acceptable or not" [7] and transforming "risky situations into risk–free equivalent ones" [3]. The problem of defining an adequate risk measure seems still to be far from finding an ultimate solution. Important contributions are contained in the discussion by Garrido [8], in the paper by Ramsay [15] and in the seminal work by Artzner et al. [2]. Recently, in Goovaerts et al. [3], it has been examined the problem of the selection of the measures that are more relevant for the analysis of risks and what properties they have to hold to reflect the basic economic underlying reality: under this approach, new classes of "consistent risk measures" have been introduced [9].

In the present paper we propose the operations of classification and ordering after the risk has been insured by a property–casualty insurance company (briefly, i.c.). The scope is to suggest a new approach to the "quantification" of actuarial portfolios run by the i.c. and of unities of risks just insured by the same i.c. The model we introduce basically consists on taking the couple (portfolio, *operative structure*) into account, that is, on considering the distribution function of the total claim amount of the portfolio together with the set of constraints and rules that the i.c. must follow for the portfolio's management (Definition 2.6). The above–mentioned couple, that we call the *actuarial* (or *technical*) *business* (briefly, business) of the i.c., is the main object of our analysis, since it represents the "tool" by which we quantify the "position" of the company, and classify and order the portfolios and the unities of risks. The definition of the business is performed through the following steps: a) we consider only two subsets of the set containing the real random vectors, that represent portfolios and unities of risks, and we assume that all the distribution functions of the total claim amounts corresponding to such vectors are known; b) we define the operative structure by introducing a set of constraints and rules such that results to be both simple and sufficiently consistent with the practice in insurance; c) we assess the business of the i.c. by analyzing its *state*. All these aims have been realized in a static setting, more precisely considering a fixed time horizon of one year.

As concerns the point a) basically, any i.c. has to deal with: i) the portfolio, say Y, constituted by a very large number of independent risks; ii) the unity of risk, say X, that consists in either a single risk or a set constituted by dependent risks.

The second point describes in detail the operative structure $\Theta(\cdot)$ (Assumption 2.3 and 2.4). As regards the third point, we impose that the "authorized states" of the business are the *acceptable state*, characterized by a non negative capacity (eq. 7)) and the more restrictive *stable state*, characterized by a strictly positive capacity (eq. 8)). Furthermore, step c) is performed introducing six vectors whose components depend on the operative structure relative to the portfolio and that are linked together by the three basic equations that express the "state of the business" (eq. (15)). We emphasize that such vectors, which represent the "state variables" of our model, are established in order to define different complete ordering criteria (Definition 2.9) through their measure (eq. (16)) and all these criteria let to introduce a partial order which is relative to the business (Definition 2.10).

Having fixed $(Y, \Theta(Y))$, the ordering of X is performed similarly to the ordering of Y with the measures of the new state variables which heavily depend on the variation that is realized in the passage from the "old" state of the business to the new one (§3.4).

As regards the classification, assuming that a fair price for insuring the portfolios and the unities of risks has already been done by the i.c. following other premium calculation principles, we classify Y according to the authorization, or not, of the business related to it (Definition 2.8) and the classification of X is made considering the variation it produces to the business as we pass from Y to the new portfolio $\{ Y; X \}$. In particular, as concerns X, our classification is provided before any reinsurance strategy is realized and whether the new state of the business is authorized or not provides a first distinction between the *normal risks* and the *dangerous risks* (Definition 3.4). The decomposition of these classes into the subclasses of *capacity generator risks*, *capacity risks* (Definition 3.5) and *great*, *catastrophic*, *mega–catastrophic risks* (Definition 3.6) is based on the variation of some of the dependent variables that define the operative structure.

Furthermore, both the classification and the ordering, as performed here, may be used for reinsurance purposes.

2 The model for classifying and ordering portfolios

Let $\mathcal{V}$ be the set of real random vectors characterized by $l \geq 1$ components, say $Y = \{ Y_1, \ldots, Y_l \}$. In order to introduce a model to classify and order the elements $Y \in \mathcal{V}$, we suppose, for simplicity, that the i.c. is operating over a fixed time horizon of one year which starts from $t = 0^-$. Now, let denote by $\mathcal{V}_1$, the subset of $\mathcal{V}$ constituted by real random vectors of $l \gg 1$ independent components[1], which, from now on, we refer to as "portfolios", and by $\mathcal{V}_2$ the subset of $\mathcal{V}$ constituted by "unities of risks" that are real random vectors of $l \geq 1$ dependent components.

To classify and order the elements of $\mathcal{V}_1$, we assume that any other risk will be introduced during the time interval $[0, 1]$.

We begin by describing the framework of this model, when we consider $\mathcal{V}_1$, through the following essential points.

2.1. The portfolio Y run by the i.c. at time $t = 0^-$.

2.2. The description of the business.

2.3. The state of the business and its graphic representation.

2.4. The classification and the ordering of Y.

2.1. The portfolio Y run by the i.c. at time $t = 0^-$. Let Y be a generic element of $\mathcal{V}_1$. We start with the following assumptions.

Assumption 2.1. *The composition of* $Y \in \mathcal{V}_1$ *(i.e. both the number and the distribution functions (d.f.'s) of its components* $Y_1, \ldots, Y_l$*) does not change over the whole year. The total claim amount in one year is denoted by the r.v.* S_Y *whose d.f.* F_{S_Y} *is known and defined on* $[0, M(S_Y)]$, $M(S_Y) \in (0, \infty)$.

[1]This includes the case in which some of the components may be unities of risks.

Assumption 2.2. *The i.c. operates out of a continuous–time economic environment, that is the model does not include: (a) expenses, taxes, commissions; (b) investment incomes; (c) dividend payout to shareholders; (d) inflation.*

The premium income over one year, that we suppose to be collected at time $t = 0^-$, is given by $P(Y) = E[S_Y] + c(Y)$, where $c(Y)$ is the safety loading. In particular, independently of the principles used to assess the single pure premiums $P(Y_i)$, $i = 1, \ldots, l$, it is always possible to set $c(Y) = \eta E[S_Y]$, where $\eta = \eta(Y) = \sum_{i=1}^{l} \theta_i \eta(Y_i)$, $\theta_i = E[S_{Y_i}]/E[S_Y]$.

2.2. The description of the business. The business of the i.c. is based on the portfolio Y, that it runs from time $t = 0^-$, and on the operative structure $\Theta(\cdot)$, namely, on the set of constraints imposed on the portfolio's management, that is: a) the insurance market's constraints, b) the criteria adopted by the i.c., and c) the conditions imposed on the i.c. by an external regulatory authority. In particular, the business relative to portfolio Y, $B(Y)$, is defined by the couple $(Y, \Theta(Y))$. Now, we express the above points in detail.

a) $M_\eta(> 0)$ is the maximum for η due to the competitive insurance market.

b) As regards the criteria adopted by the i.c., they basically derive from its risk aversion, its commercial politics (with respect to the policyholders) and the maximum capital it chooses to invest as free reserve in function of $E[S_Y]$.

Assumption 2.3. (i) *The i.c. fixes $\varepsilon_0 \in (0, 1)$ as the maximum acceptable probability of ruin per year.*

(ii) *The i.c. selects the premium calculation principles.*

(iii) *Having expressed the free reserve proportional to the pure premium $P(Y)$*[2]*, that is, $u(Y) = \alpha P(Y)$, $\alpha = \alpha(Y) > 0$, the i.c. establishes that the optimal value for α must be depending on the risk Y through a given formula, say $\alpha(Y) = f(P(Y))$ where f is fixed, and chooses the maximum for α, say $M_\alpha(> 0)$.*

Obviously, the value $M_\alpha(> 0)$ is lower than a maximum, say M_M, linked to the insurance market as a whole and that will be defined later (see eq. (23)).

c) We list all the conditions imposed by the regulator into the following assumption.

Assumption 2.4. (i) *The insurer must operates within the "maximum acceptable ruin probability per year" $\varepsilon^* \in (0, \varepsilon_0]$.*

Relating to ε^*, the *maximum acceptable loss* for portfolio Y is the value $MAL^*(Y)$ satisfying

$$MAL^*(Y) < M(S_Y), \qquad \int_{MAL^*(Y)}^{M(S_Y)} dF_{S_Y}(x) = \varepsilon^*. \tag{1}$$

For the second point of the assumption, we follow the model introduced in [4].

[2]Because of the 'standard rules' of Solvency I, the free reserve must be determined in relation to either the pure premium or the mathematical average of the liability due to the claims of the last three years.

(ii) *In order to minimize the shortfall (or insolvency) risk, measured by the expected value $E[(S_Y - P(Y) - u(Y))_+]$, the regulatory authority imposes on the i.c. a minimum free reserve, the solvency capital requirement, to be used to cover future policyholder claims*[3] *and takes the cost due to this capital into account by the term $\varepsilon^* u(Y)$. The solvency capital requirement, u^*, must be a solution of the following minimization problem*

$$\min_{u(Y)} \{E[(S_Y - P(Y) - u(Y))_+] + \varepsilon^* u(Y)\}. \tag{2}$$

We introduce $h(\alpha, \eta) = (\alpha+1)(1+\eta)$, which we call the *actual actuarial capitalization factor*, and we denote by $(\alpha^*, \eta^*) \equiv (\alpha(\varepsilon^*; Y), \eta(\varepsilon^*; Y))$ a pair (α, η) such that $u^* = \alpha^*(1+\eta^*)E[S_Y]$ and $h(\alpha^*, \eta^*) = (1+\alpha^*)(1+\eta^*) < M(S_Y)/E[S_Y]$. We call $h(\alpha^*, \eta^*)$ the *minimal acceptable actuarial capitalization factor* (see also Remark 2.5). Furthermore, taking into account the maxima M_α and M_η, we introduce the minimum of α, denoted by m_α, and the minimum of η, denoted by m_η, that is,

$$m_\alpha = \frac{h(\alpha, \eta)}{1 + M_\eta} - 1, \qquad m_\eta = \frac{h(\alpha, \eta)}{1 + M_\alpha} - 1. \tag{3}$$

Then we introduce the *capital structure* of the company for portfolio Y,

$$CS(\alpha, \eta; Y) = u(Y) + P(Y) = h(\alpha, \eta)E[S_Y], \qquad \alpha \in (m_\alpha, M_\alpha],\ \eta \in (m_\eta, M_\eta], \tag{4}$$

the corresponding *actual probability of ruin* per year $\varepsilon = \varepsilon(Y)$,

$$\int_{CS(\alpha,\eta;Y)}^{M(S_Y)} dF_{S_Y}(x) = \varepsilon, \tag{5}$$

and the *minimal acceptable capital structure*

$$CS(\alpha^*, \eta^*; Y) = MAL^*(Y) = h(\alpha^*, \eta^*)E[S_Y]. \tag{6}$$

(iii) *The regulator establishes that the only authorized states of the business are the "acceptable state" and the "stable state". This means that the i.c. must have at its disposal a capital structure $CS(\alpha, \eta; Y)$ satisfying the following constraint:*

$$\frac{CS(\alpha, \eta; Y)}{CS(\alpha^*, \eta^*; Y)} = \frac{h(\alpha, \eta)}{h(\alpha^*, \eta^*)} \geq 1, \qquad (\alpha, \eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta], \tag{7}$$

[3] The solvency capital requirement is the minimum amount of capital needed for the risk inherent to the insurer's portfolio. We emphasize the fact that the present study is concerned only with the technical risk carried by the insurer: our aim is neither to analyze the overall solvency problem nor to take into account the Solvency II project's rules for assessing the overall financial position of the i.c. and any other capital requirement, necessary for all the different risks involved in the operation of the whole insurance business (see, e.g. [14]), is not take into account.

for the acceptable state,

$$\frac{h(\alpha,\eta)}{h(\alpha^*,\eta^*)} \geq (1+\psi_0), \qquad \psi_0(>0) \text{ independent of } Y, \tag{8}$$

for the stable state.

Remark 2.5. *1.* Eq. (4) and (6) suggest a comparison with the basic formula of the interest rate theory, with $E[S_Y]$ to be intended as the original amount (the principal), $h(\alpha,\eta)$ and $h(\alpha^*,\eta^*)$ as the capitalization factors. We point out that, differently from the economic setting, where the capitalization factor is charged to a single "individual" (for instance, the bank), both $h(\alpha,\eta)$ and $h(\alpha^*,\eta^*)$ results from the product of two factors: one is charged to the policyholders and the other one is charged to the shareholders.

2. The constraint (7) is equivalent to condition $C_Y \geq 0$, having introduced the capacity

$$C_Y \equiv C_Y(\alpha,\eta) = \{h(\alpha,\eta) - h(\alpha^*,\eta^*)\}E[S_Y]. \tag{9}$$

3. Having defined the VaR at the probability level $1-\varepsilon^*$ by

$$VaR_{1-\varepsilon^*}[S_Y - P(Y)] = \inf\{\, u > 0 \;:\; Pr\{\, S_Y \leq u + P(Y)\,\} \geq 1-\varepsilon^*\,\}, \tag{10}$$

then in [4] it has been showed that $VaR_{1-\varepsilon^*}[S_Y - P(Y)]$ is a solution $u(Y) = u^*$ of eq. (2). Then the minimal acceptable capital structure (6) is explicitly expressed by

$$CS(\alpha^*,\eta^*;Y) = VaR_{1-\varepsilon^*}[S_Y - (1+\eta^*)E[S_Y]] + (1+\eta^*)E[S_Y]. \tag{11}$$

We remark that α^* and η^* are linked together by the following equation

$$\alpha^* = \frac{VaR_{1-\varepsilon^*}[S_Y - (1+\eta^*)E[S_Y]]}{(1+\eta^*)E[S_Y]} \tag{12}$$

which explicitly shows that the more η^* increases, the more α^* decreases (and vice versa).

∎

Now we can formally define the *operative structure* $\Theta(\cdot)$.

Definition 2.6. *The operative structure $\Theta(\cdot)$ expresses all the constraints and rules listed in Assumptions 2.3 and 2.4 which synthesize all the basic features of portfolio's management. Thus, it can be formally represented by the following set of parameters and dependent variables:*

$$\Theta(\cdot) = \{\, \varepsilon^*, M_\alpha, M_\eta, \psi_0; \alpha(\cdot), \eta(\cdot), h(\alpha(\varepsilon^*;\cdot), \eta(\varepsilon^*;\cdot)), \varepsilon(\cdot) \in (0,\varepsilon^*]\,\}. \tag{13}$$

In particular, the values assumed in Y by the dependent variables appearing in (13) are the ones defined by (4), (5), (6), and (8).

Remark 2.7. Definition 2.6 emphasizes that the riskiness of the economic situation of the i.c. depends heavily on the portfolio and on all the parameters and dependent variables concerning the management of Y: it may differ considering the same portfolio of risks but two different operative structures, and, in the same fashion, considering the same operative structure but two different portfolios. Furthermore, this suggests an interesting comparison with the classical utility theory. Any decision maker, having an initial wealth w, needs a "procedure" for ranking random future cash flows and uses an utility function U for comparing them, by means of the expected utility values $E[U(w;\cdot)]$. Before a risk is insured, the i.c., in the role of decision maker, uses the utility function for the premium calculation and orders the risks through the expected utility. After the insuring has been realized, it uses the business $(\cdot, \Theta(\cdot))$ to "weigh" the portfolios and/or the unities of risks and, therefore, to order them by means of some measures we will introduce later (see §2.3.).

∎

2.3. The state of the business and its graphic representation. For the graphic representation of the business $B(\cdot) = (\cdot, \Theta(\cdot))$ we need some preliminaries. To any portfolio $(\cdot)$, that is, to each d.f. $F_{S_{(\cdot)}}$, corresponds a loss exceedance probability (briefly, LEP) curve which can be depicted on the $(\text{total claim amount per year}, 0, \text{ruin probability per year})$–plane (see [5] for details). Any operative structure $\Theta(\cdot)$ defines on the LEP a couple of points $P^* = (CS^*(\cdot), \varepsilon^*)$ and $P = (CS(\cdot), \varepsilon)$. Now, for defining suitable measures for the ordering we have in mind, we need to introduce an a–dimensional real random variable (r.v.) $H = S_{(\cdot)}/E[S_{(\cdot)}]$, the *relative total claim amount*, whose outcomes are denoted by h. On the $(H, 0, \varepsilon)$–plane, correspondingly to P^* and P, we can depict two points, say A and B, respectively.

The state of the business $B(Y)$ can be graphically represented on the $(H, 0, \varepsilon)$–plane, where the LEP curve relative to portfolio Y has been traced, through the points $A = (h^*, \varepsilon^*)$, $B = (h, \varepsilon)$, where $h = h(\alpha, \eta)$ and $h^* = h(\alpha^*, \eta^*)$, and by means of a third point $Q = (0, \varepsilon^*)$, which does not belong to the LEP. We remark that Q can be considered as a "reference" point since it does not depend on Y. Then, the state of the business is defined by three points and by the following six two–dimensional vectors (see Figure 1),

$$\begin{cases} \overrightarrow{v}_1 = \overrightarrow{QA} = (h^*, 0), \\ \overrightarrow{v}_2 = \overrightarrow{AB} = (h - h^*, \varepsilon - \varepsilon^*), \\ \overrightarrow{v}_3 = \overrightarrow{QB} = (h, \varepsilon - \varepsilon^*), \\ \overrightarrow{v}_4 = \overrightarrow{EQ} = (0, \varepsilon^* - \varepsilon), \\ \overrightarrow{v}_5 = \overrightarrow{CB} = (h - h^*, 0), \\ \overrightarrow{v}_6 = \overrightarrow{EB} = (h, 0), \end{cases} \tag{14}$$

The components of the aforesaid vectors are the basic features of the operative structure $\Theta(Y)$: in this sense we refer to them as *state variables* and the equations that link

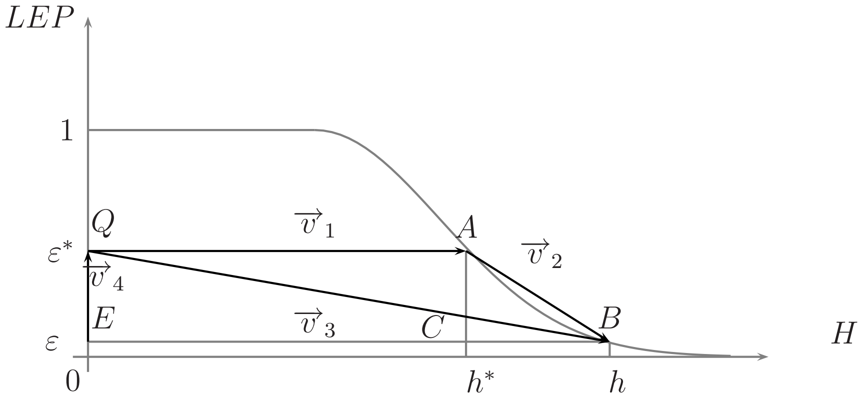

Figure 1: The state of the business $B(Y)$.

them together as *state equations*. It can be easily state (see Proposition 4.1 in the Appendix) that a system of "basic" state equations of the business $B(Y)$ can be provided by considering the following three independent equations:

$$\overrightarrow{v}_1 + \overrightarrow{v}_2 - \overrightarrow{v}_3 = 0, \qquad \overrightarrow{v}_2 + \overrightarrow{v}_4 - \overrightarrow{v}_5 = 0, \qquad \overrightarrow{v}_3 + \overrightarrow{v}_4 - \overrightarrow{v}_6 = 0. \tag{15}$$

2.4. The classification and the ordering of Y. The classification of Y is relative to the state of the business and it is "graphically" expressed only through the vector $\overrightarrow{v}_4$. In the following we write $\text{sgn}\overrightarrow{v}_4 > [<]\, 0$ when $\varepsilon^* > [<]\, \varepsilon$.

Definition 2.8. *We say that the management of portfolio Y is authorized if either $sgn\overrightarrow{v}_4 > 0$ or $|\overrightarrow{v}_4| = 0$, not authorized if $sgn\overrightarrow{v}_4 < 0$. Thus, the i.c. is authorized to manage portfolio Y if either condition (7) or condition (8) is fulfilled. If the reverse of condition (7) holds, we say that the i.c. is not authorized to manage portfolio Y.*

Figure 1 represents a case in which the state of the business is authorized.

Now, the ordering criteria for portfolios can be realized using the measures of the vectors $\overrightarrow{v}_i$, $i = 1, \ldots, 6$, and we will denote such criteria by oc_i, respectively. More precisely, to each $\overrightarrow{v}_i$, $i = 1, \ldots, 6$, we associate a measure, say $\rho(\overrightarrow{v}_i)$, defined by

$$\begin{cases} \rho(\overrightarrow{v}_i) = |\overrightarrow{v}_i|, \ \ i = 1, 6, \\ \rho(\overrightarrow{v}_i) = \text{sgn}(\overrightarrow{v}_4) \cdot |\overrightarrow{v}_i|, \ \ i = 2, 3, 4, \qquad \text{if either } \text{sgn}(\overrightarrow{v}_4) > 0 \text{ or } < 0, \\ \rho(\overrightarrow{v}_5) = \text{sgn}(\overrightarrow{v}_5) \cdot |\overrightarrow{v}_5|, \\ \overrightarrow{v}_1 = \overrightarrow{v}_3 = \overrightarrow{v}_6, \qquad \overrightarrow{v}_2 = \overrightarrow{v}_5 = \overrightarrow{0}, \qquad \text{if } |\overrightarrow{v}_4| = 0, \end{cases} \tag{16}$$

where $\text{sgn}\overrightarrow{v}_5 > [<]\, 0$ when $h > [<]\, h^*$.

We then denote by Y_1 and Y_2 two different portfolios and we simplify the notations by writing $h_{Y_1} = CS(\alpha_1, \eta_1; Y_1)/E[S_{Y_1}]$, $h^*_{Y_1} = CS^*(\alpha^*_1, \eta^*_1; Y_1)/E[S_{Y_1}]$, $h_{Y_2} = CS(\alpha_2, \eta_2; Y_2)/E[S_{Y_2}]$, $h^*_{Y_2} = CS^*(\alpha^*_2, \eta^*_2; Y_2)/E[S_{Y_2}]$. We emphasize that, on the $(H, 0, \varepsilon)$–plane, the two states of the business are determined by five points A_1, B_1, A_2, B_2, and

Q, which define the twelve vectors $\overrightarrow{v}_i(Y_1)$, $\overrightarrow{v}_i(Y_2)$, $i = 1, \ldots, 6$, representing the state variables of $B(Y_1)$ and $B(Y_2)$, respectively. The ordering of the portfolios depends on all the ordering criteria $oc_i = oc_i(\Theta(\cdot))$.

Definition 2.9. *We say that Y_1 precedes Y_2 in the "$oc_i(\Theta(\cdot))$–order", $i = 1, \ldots, 6$, and we write $Y_1 \leq_{oc_i(\Theta(\cdot))} Y_2$, if and only if $\rho(\overrightarrow{v}_i(Y_1)) \leq \rho(\overrightarrow{v}_i(Y_2))$ for $i = 1, \ldots, 6$.*

In the present paper we do not provide an economic interpretation to the positions of the risks based on the above–mentioned orderings. Such orderings, that result to be complete, let us to introduce the following partial order.

Definition 2.10. *We say that Y_1 precedes Y_2 in the "$\Theta(\cdot)$–order", and we write $Y_1 \leq_{\Theta(\cdot)} Y_2$, if and only if $Y_1 \leq_{oc_i(\Theta(\cdot))} Y_2$ for all $i = 1, \ldots, 6$.*

Now, using (16) and Definition 2.9, it is easy to check that if Y_1 precedes Y_2 in both the $oc_i(\Theta(\cdot))$–order, $i = 4, 5$, then Y_1 precedes Y_2 in the $oc_2(\Theta(\cdot))$–order, whereas for $i = 4, 6$, the $oc_i(\Theta(\cdot))$–order yield the $oc_3(\Theta(\cdot))$–order. Thus, the following result holds.

Proposition 2.11. *If $Y_1 \leq_{oc_i(\Theta(\cdot))} Y_2$ for $i = 1, 4, 5, 6$, then $Y_1 \leq_{\Theta(\cdot)} Y_2$.*

3 Classification and ordering of a new insured unities of risks

Now we describe the framework of the model when a new unity of risks $X \in \mathcal{V}_2$ is added to portfolio Y through the following essential points.

3.1. The new risk X introduced at time $t = 0$.

3.2. The new portfolio $\widetilde{Y}$ run by the i.c. in the interval $[0, 1]$.

3.3. The graphic representation of $B(\widetilde{Y})$.

3.4. The classification and the ordering of X.

3.1. The new risk X introduced at time $t = 0$. We start with the following Assumption.

Assumption 3.1. *The state of the business $B(Y)$ is authorized (in the sense of Assumption 2.4–(iii)).*

At time $t = 0$ the i.c. takes over a new unity of risks $X \in \mathcal{V}_2$, whose valuation we suppose starting an instant later the passage from the policyholders to the i.c., namely, when $t = 0^+$. We remark that such risk may be either a single risk or a unity of risks but, in both cases, we consider it as it were a single one. We denote by S_X the total claim amount of X per year.

Assumption 3.2. *S_X has a bounded support $[0, M(S_X)]$ and its d.f. F_{S_X} is known.*

3.2. The new portfolio $\widetilde{Y}$ run by the i.c. in the interval $[0,1]$**.** From time $t=0$ up until $t=1$, having introduced only X, the insurer runs a new portfolio $\widetilde{Y}=\{Y;X\}$. As Assumptions 2.1 and 3.2 hold, the total claim amount in one year, given by the r.v. $S_{\widetilde{Y}}$, has a bounded support $[0,M(S_{\widetilde{Y}})]$, $M(S_{\widetilde{Y}})=M(S_Y)+M(S_X)$, and the d.f. $F_{S_{\widetilde{Y}}}$ is known. The new premium income over one year is given by

$$P(\widetilde{Y})=(1+\widetilde{\eta})E[S_{\widetilde{Y}}], \qquad \widetilde{\eta}=\eta(\widetilde{Y})=\eta(X)\frac{E[S_X]}{E[S_{\widetilde{Y}}]}+\eta(Y)\frac{E[S_Y]}{E[S_{\widetilde{Y}}]}. \tag{17}$$

Since the value M_α is fixed as in Assumption 2.3–(i) and M_η is exchanged with $\widetilde{M}_\eta$, $\widetilde{M}_\eta=M_\eta(\widetilde{Y})=M_\eta\frac{E[S_X]}{E[S_{\widetilde{Y}}]}+\eta(Y)\frac{E[S_Y]}{E[S_{\widetilde{Y}}]}$, analogously to eq. (3), we have the new minima $\widetilde{m}_\alpha$ and $\widetilde{m}_\eta$ defined using $h(\widetilde{\alpha},\widetilde{\eta})$ instead of $h(\alpha,\eta)$ and $\widetilde{M}_\eta$ instead of M_η. We introduce $(\widetilde{\alpha}^*,\widetilde{\eta}^*)\equiv(\widetilde{\alpha}(\varepsilon^*;\widetilde{Y}),\eta(\varepsilon^*;\widetilde{Y}))$ as a pair (α,η) such that $h(\widetilde{\alpha}^*,\widetilde{\eta}^*)<M(S_{\widetilde{Y}})/E[S_{\widetilde{Y}}]$ and we define $CS(\widetilde{\alpha}^*,\widetilde{\eta}^*;\widetilde{Y})=MAL^*(\widetilde{Y})$. The capital structure of the i.c. for portfolio $\widetilde{Y}$ is now given by

$$CS(\widetilde{\alpha},\widetilde{\eta};\widetilde{Y})=h(\widetilde{\alpha},\widetilde{\eta})E[S_{\widetilde{Y}}], \qquad (\widetilde{\alpha},\widetilde{\eta})\in[\widetilde{m}_\alpha,M_\alpha]\times[\widetilde{m}_\eta,\widetilde{M}_\eta] \tag{18}$$

and the capacity is defined by

$$C_{\widetilde{Y}}=C_{\widetilde{Y}}(\widetilde{\alpha},\widetilde{\eta})=\{h(\widetilde{\alpha},\widetilde{\eta})-h(\widetilde{\alpha}^*,\widetilde{\eta}^*)\}E[S_{\widetilde{Y}}]. \tag{19}$$

3.3. The graphic representation of $B(\widetilde{Y})$**.** Let consider the business $B(\widetilde{Y})=(\widetilde{Y},\widetilde{\Theta})$, where

$$\widetilde{\Theta}=\Theta(\widetilde{Y})=\{\,\varepsilon^*,M_\alpha,\widetilde{M}_\eta,\psi_0;\alpha(\widetilde{Y}),\eta(\widetilde{Y}),h(\alpha(\varepsilon^*;\widetilde{Y}),\eta(\varepsilon^*;\widetilde{Y})),\widetilde{\varepsilon}=\varepsilon(\widetilde{Y})\in(0,\varepsilon^*]\,\}.$$

is the operative structure relative to portfolio $\widetilde{Y}$, which defines on the new LEP curve the points $\widetilde{P}^*=CS(\widetilde{\alpha}^*,\widetilde{\eta}^*,\widetilde{Y})$ and $\widetilde{P}=CS(\widetilde{\alpha},\widetilde{\eta},\widetilde{Y})$. Similarly to what made in §2.3, we simply write $\widetilde{h}=h(\widetilde{\alpha},\widetilde{\eta})$, $\widetilde{h}^*=h(\widetilde{\alpha}^*,\widetilde{\eta}^*)$ and on the $(H,0,\varepsilon)$–plane we trace the points corresponding to $\widetilde{P}^*$ and $\widetilde{P}$, say $\widetilde{A}=(\widetilde{h}^*,\varepsilon^*)$ and $\widetilde{B}=(\widetilde{h},\varepsilon)$, and we still consider the reference point $Q=(0,\varepsilon^*)$. The description of $B(\widetilde{Y})$ can now be performed by the points $\widetilde{A}$, $\widetilde{B}$ and Q, and the six vectors obtained from the ones defined by (14), namely,

$$\begin{cases}
\overrightarrow{\widetilde{v}}_1=\overrightarrow{Q\widetilde{A}}=\overrightarrow{v}_1+\overrightarrow{A\widetilde{A}},\\
\overrightarrow{\widetilde{v}}_2=\overrightarrow{\widetilde{A}\widetilde{B}}=\overrightarrow{v}_2+\overrightarrow{BF}+\overrightarrow{BG}-\overrightarrow{A\widetilde{A}},\\
\overrightarrow{\widetilde{v}}_3=\overrightarrow{Q\widetilde{B}}=\overrightarrow{v}_3+\overrightarrow{BF}+\overrightarrow{BG},\\
\overrightarrow{\widetilde{v}}_4=\overrightarrow{\widetilde{E}Q}=\overrightarrow{v}_4-\overrightarrow{BG}\\
\overrightarrow{\widetilde{v}}_5=\overrightarrow{\widetilde{C}\widetilde{B}}=\overrightarrow{v}_5+\overrightarrow{BF}-\overrightarrow{A\widetilde{A}},\\
\overrightarrow{\widetilde{v}}_6=\overrightarrow{\widetilde{E}\widetilde{B}}=\overrightarrow{v}_6+\overrightarrow{BF},
\end{cases} \tag{20}$$

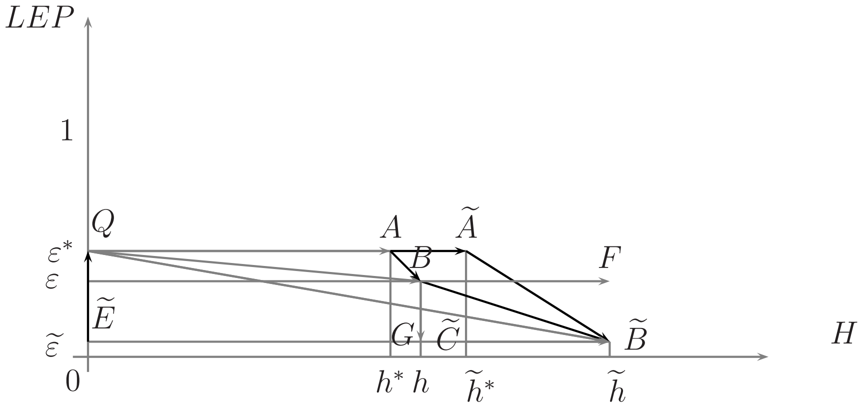

Figure 2: The states of the business $B(Y)$ and $B(\widetilde{Y})$.

where $\widetilde{C} = (\widetilde{h}^*, \widetilde{\varepsilon})$, $\widetilde{E} = (0, \widetilde{\varepsilon})$, $F = (\widetilde{h}, \varepsilon)$ and $G = (h, \widetilde{\varepsilon})$. Figure 2 illustrates the basic vectors defined in (20) when we consider on the $(H, 0, \varepsilon)$–plane both the LEP curves corresponding to portfolios Y and $\widetilde{Y}$.

Arguing as made for the vectors $\{\overrightarrow{v}_i\}_{i=1,\ldots,6}$, it is easy to see that a system of basic state equations can now be expressed through the following ones:

$$\overrightarrow{\widetilde{v}}_1 + \overrightarrow{\widetilde{v}}_2 - \overrightarrow{\widetilde{v}}_3 = 0, \qquad \overrightarrow{\widetilde{v}}_2 + \overrightarrow{\widetilde{v}}_4 - \overrightarrow{\widetilde{v}}_5 = 0, \qquad \overrightarrow{\widetilde{v}}_3 + \overrightarrow{\widetilde{v}}_4 - \overrightarrow{\widetilde{v}}_6 = 0. \tag{21}$$

The variation of the state of the business is defined by the point A, $\widetilde{A}$, B and $\widetilde{B}$ and it may be expressed through the following equation:

$$\overrightarrow{A\widetilde{A}} + \overrightarrow{\widetilde{A}\widetilde{B}} - \overrightarrow{AB} - \overrightarrow{B\widetilde{B}} = 0. \tag{22}$$

In the equation of the state variation (22), $\overrightarrow{A\widetilde{A}}$ and $\overrightarrow{B\widetilde{B}}$ represent the variation of the state of the business, $\overrightarrow{\widetilde{A}\widetilde{B}}$ is one of the six vectors (20) that defines the new state of the business and $\overrightarrow{AB}$ is one of the six vectors (14) that defines the original state.

3.4. The classification and the ordering of X. The classification and the ordering of X are based on the states of the business $B(Y)$ (for which Assumption 3.1 holds) and $B(\widetilde{Y})$, and on the operative (but not realistic) assumption that X is completely retained by the i.c.

The classification is developed in two steps: a first distinction is based on the fact that $\widetilde{Y}$ is authorized or not and divide the *normal* from the *dangerous risks* (Definition 3.4). Further distinctions are made inside these two classes: the comparison between $h(\widetilde{\alpha}, \widetilde{\eta})$ and $h(\widetilde{\alpha}^*, \widetilde{\eta}^*)$ divides the normal risks between *capacity generator risks* and *capacity risks* (Definition 3.5); the distinction of the dangerous risks in *great*, *catastrophic* and *mega–catastrophic risks* (Definition 3.6) depends on the belongings of $h(\widetilde{\alpha}^*, \widetilde{\eta}^*)$ to the intervals $]h(\widetilde{\alpha}, \widetilde{\eta}), h(M_\alpha, \widetilde{M_\eta})]$, $]h(M_\alpha, \widetilde{M_\eta}), h(M_M, \widetilde{M_\eta})]$ and $]h(M_M, \widetilde{M_\eta}) + \infty[$, respectively.

The calculation of M_M in a operative framework would require an additional study which is beyond the scope of this paper. Hence, we justify the existence of the upper bound M_M with the following (not realistic but) simplifying assumption.

Assumption 3.3. *The insurance market is constituted by $k+1$ i.c.'s, the first one our i.c. We suppose that each of the remaining k i.c.'s is running a portfolio Y_i endowed by a potential increment $\Delta h_{Y_i} = (M_{\alpha_i} - \alpha_i)(1+\eta_i)$, $i = 1, \ldots, k$, that is at the i.c.'s disposal. Thus, we assume that our i.c. could have a global potential increment at disposal given by $\sum_{i=1}^{k} \Delta h_{Y_i} E[S_{Y_i}]/E[S_Y]$, and we suppose that such amount can be expressed in the following way:*

$$\sum_{i=1}^{k} \Delta h_{Y_i} \frac{E[S_{Y_i}]}{E[S_Y]} = (M_M - M_\alpha)(1 + \widetilde{M}_\eta), \qquad M_M > M_\alpha. \tag{23}$$

Adopting also the terminology used in [6] and [16], we provide the following classification.

Definition 3.4. *X is a normal risk if and only if the i.c. is authorized to manage portfolio $\widetilde{Y}$ (in the sense of Definition 2.8), that is, $h(\widetilde{\alpha}, \widetilde{\eta}) \geq h(\widetilde{\alpha}^*, \widetilde{\eta}^*)$, $(\widetilde{\alpha}, \widetilde{\eta}) \in [\widetilde{m}_\alpha, M_\alpha] \times [\widetilde{m}_\eta, \widetilde{M}_\eta]$. In this case the risk can be entirely retained by the i.c. X is a dangerous risk if and only if the reverse holds, that is, $h(\widetilde{\alpha}, \widetilde{\eta}) < h(\widetilde{\alpha}^*, \widetilde{\eta}^*)$. In this case a part of the risk must be necessarily ceded in reinsurance.*

We make a further distinction inside the classes of normal and dangerous risks.

Definition 3.5. *Let X be a normal risk.*

(i) *X is a capacity generator risk if and only if*

$$h(\widetilde{\alpha}, \widetilde{\eta}) > h(\widetilde{\alpha}^*, \widetilde{\eta}^*) + \tau(h(\alpha, \eta) - h(\alpha^*, \eta^*)), \qquad \tau = \frac{E[S_Y]}{E[S_{\widetilde{Y}}]}. \tag{24}$$

(ii) *Let assume $h(\alpha, \eta) > h(\alpha^*, \eta^*)$. X is a capacity risk if and only if*

$$h(\widetilde{\alpha}^*, \widetilde{\eta}^*) \leq h(\widetilde{\alpha}, \widetilde{\eta}) < h(\widetilde{\alpha}^*, \widetilde{\eta}^*) + \tau(h(\alpha, \eta) - h(\alpha^*, \eta^*)). \tag{25}$$

Definition 3.6. *Let X be a dangerous risk.*

(i) *X is a great risk if and only if*

$$h(\widetilde{\alpha}, \widetilde{\eta}) < h(\widetilde{\alpha}^*, \widetilde{\eta}^*) \leq h(M_\alpha, \widetilde{M}_\eta), \tag{26}$$

(ii) *X is a catastrophic risk if and only if*

$$h(M_\alpha, \widetilde{M}_\eta) < h(\widetilde{\alpha}^*, \widetilde{\eta}^*) \leq h(M_M, \widetilde{M}_\eta), \tag{27}$$

(iii) *X is a mega–catastrophic risk if and only if*

$$h(M_M, \widetilde{M}_\eta) < h(\widetilde{\alpha}^*, \widetilde{\eta}^*). \tag{28}$$

Remark 3.7. Eq. (24) and (25) yield $C_{\widetilde{Y}} > C_Y$ and $0 \leq C_{\widetilde{Y}} < C_Y$, respectively. It is also possible to establish a link between the capacity $C_{\widetilde{Y}}$ and the hypothetical premiums $P(X)$ which should be paid to get that a normal risk X is either a capacity generator risk or a capacity risk (Proposition 4.2) and how great $P(X)$ should have been in order to get $C_{\widetilde{Y}} \geq 0$ when X is a great risk or a catastrophic risk (Proposition 4.3).

■

The total orderings on the set of risks X contained in the previous classes depend on a fixed business $B(Y) = (Y; \Theta(Y))$. More precisely, under Assumption 3.1, we suppose that, at time $t = 0$, the i.c. takes over either the new risk X_1, or X_2, so that it manages either the new portfolio $\widetilde{Y}_1 = \{Y; X_1\}$, or $\widetilde{Y}_2 = \{Y; X_2\}$. The order between $\widetilde{Y}_1$ and $\widetilde{Y}_2$, defined similarly to Definitions 2.9 and 2.10, yields the order between X_1 and X_2.

Definition 3.8. *We say that X_1 precedes X_2 in the "$oc_i(B(Y))$–order", and we write $X_1 \leq_{oc_i(B(Y))} X_2$, if and only if $\widetilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \widetilde{Y}_2$, $i = 1, \ldots, 6$.*

We say that X_1 precedes X_2 in the "$B(Y)$–order", and we write $X_1 \leq_{B(Y)} X_2$, if and only if $\widetilde{Y}_1 \leq_{\Theta(\cdot)} \widetilde{Y}_2$.

The following Proposition establishes a link between the vectors $\overrightarrow{A\widetilde{A}}$, $\overrightarrow{BG}$ and $\overrightarrow{BF}$, as defined in (20), and the $oc_i(B(Y))$–order (for its proof see the Appendix). For this purpose, we need to introduce the measure of such vectors.

$$\begin{cases} \rho(\overrightarrow{A\widetilde{A}}) = \operatorname{sgn}(\overrightarrow{A\widetilde{A}}) \cdot \mid \overrightarrow{A\widetilde{A}} \mid, \\ \rho(\overrightarrow{BG}) = \operatorname{sgn}(\overrightarrow{BG}) \cdot \mid \overrightarrow{BG} \mid, \\ \rho(\overrightarrow{BF}) = \operatorname{sgn}(\overrightarrow{BF}) \cdot \mid \overrightarrow{BF} \mid, \\ \rho(\overrightarrow{BF} - \overrightarrow{A\widetilde{A}}) = \operatorname{sgn}(\overrightarrow{BF} - \overrightarrow{A\widetilde{A}}) \cdot \mid \overrightarrow{BF} - \overrightarrow{A\widetilde{A}} \mid, \end{cases} \tag{29}$$

where $\operatorname{sgn}(\overrightarrow{A\widetilde{A}}) > [<] 0$ when $\widetilde{h}^* > [<] h^*$, $\operatorname{sgn}(\overrightarrow{BG}) > [<] 0$ when $\widetilde{\varepsilon} > [<] \varepsilon$, $\operatorname{sgn}(\overrightarrow{BF}) > [<] 0$ when $\widetilde{h} > [<] h$, and $\operatorname{sgn}(\overrightarrow{BF} - \overrightarrow{A\widetilde{A}}) > [<] 0$ when $\widetilde{h} - h > [<] \widetilde{h}^* - h^*$.

Proposition 3.9. **(i)** *$\widetilde{Y}_1 \leq_{oc_1(\Theta(\cdot))} \widetilde{Y}_2$ if and only if $\rho(\overrightarrow{A\widetilde{A}_1}) \leq \rho(\overrightarrow{A\widetilde{A}_2})$.*

(ii) *$\widetilde{Y}_1 \leq_{oc_4(\Theta(\cdot))} \widetilde{Y}_2$ if and only if $\rho(\overrightarrow{BG_1}) \geq \rho(\overrightarrow{BG_2})$.*

(iii) *$\widetilde{Y}_1 \leq_{oc_5(\Theta(\cdot))} \widetilde{Y}_2$ if and only if $\rho(\overrightarrow{BF_1} - \overrightarrow{A\widetilde{A}_1}) \leq \rho(\overrightarrow{BF_2} - \overrightarrow{A\widetilde{A}_2})$.*

(iv) *$\widetilde{Y}_1 \leq_{oc_6(\Theta(\cdot))} \widetilde{Y}_2$ if and only if $\rho(\overrightarrow{BF_1}) \leq \rho(\overrightarrow{BF_2})$.*

Furthermore, if $\widetilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \widetilde{Y}_2$ for $i = 1, 4, 5, 6$, then $X_1 \leq_{B(Y)} X_2$.

Remark 3.10. A sufficient condition for $\widetilde{Y}_1 \leq_{oc_5(\Theta(\cdot))} \widetilde{Y}_2$ is given by $\rho(\overrightarrow{BF_1}) \leq \rho(\overrightarrow{BF_2})$ and $\rho(\overrightarrow{A\widetilde{A}_1}) \geq \rho(\overrightarrow{A\widetilde{A}_2})$. Indeed, by eq. (16) and (29), we see that if either $\operatorname{sgn}(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_i)}) > 0$, or $\operatorname{sgn}(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_i)}) < 0$, then $\rho(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_i)}) = \rho(\overrightarrow{\widetilde{v}_6(\widetilde{Y}_i)}) - \rho(\overrightarrow{\widetilde{v}_1(\widetilde{Y}_i)})$. Then we use the results of item

(i) and item (iv). In the case $\text{sgn}(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_1)}) < 0$ and $\text{sgn}(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_2)}) > 0$, we have immediately that $\rho(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_1)}) \leq \rho(\overrightarrow{\widetilde{v}_5(\widetilde{Y}_2)})$.

∎

4 Appendix

Proposition 4.1. *A system of basic state equations related to $B(Y)$ is given by the equations (15).*

Proof. The state equations related to $B(Y)$, namely

$$\begin{cases} \overrightarrow{v}_1 + \overrightarrow{v}_2 - \overrightarrow{v}_3 = 0, \\ \overrightarrow{v}_2 + \overrightarrow{v}_4 - \overrightarrow{v}_5 = 0, \\ \overrightarrow{v}_3 + \overrightarrow{v}_4 - \overrightarrow{v}_6 = 0, \\ \overrightarrow{v}_1 + \overrightarrow{v}_5 - \overrightarrow{v}_6 = 0, \end{cases}$$

define the associated matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

It is easy to see that $\text{Rank}(A) = 3$ and one of the minor different from zero is given by the first three elements of the first three rows which correspond to the basic state equations (15).

∎

Proposition 4.2. (i) $C_{\widetilde{Y}} \geq 0$ [<] *if and only if*

$$P(X) \geq [<] \frac{CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y})}{(1 + \widetilde{\alpha})} - P(Y). \tag{30}$$

(ii) *Now let X be a normal risk and $(\alpha, \eta) \in [m_\alpha, M_\alpha] \times [m_\eta, M_\eta]$, $(\widetilde{\alpha}, \widetilde{\eta}) \in [\widetilde{m}_\alpha, M_\alpha] \times [\widetilde{m}_\eta, \widetilde{M}_\eta]$. There results:*

$$C_{\widetilde{Y}} \begin{cases} \leq C_Y & \text{if } 0 < P(X) \leq \{CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}) - CS(\alpha^*, \eta^*; Y) - (\widetilde{\alpha} - \alpha)P(Y)\}(1 + \widetilde{\alpha})^{-1}, \\ > C_Y & \text{if } P(X) > \{CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}) - CS(\alpha^*, \eta^*; Y) - (\widetilde{\alpha} - \alpha)P(Y)\}(1 + \widetilde{\alpha})^{-1}. \end{cases} \tag{31}$$

In particular, if $CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}) < CS(\alpha^*, \eta^*; Y)$, then we have

$$C_{\widetilde{Y}} > C_Y + P(X)(1 + \widetilde{\alpha}) + (\widetilde{\alpha} - \alpha)P(Y), \tag{32}$$

so that, for any choice of $P(X) > 0$ and $\widetilde{\alpha} \geq \alpha$, X is a normal capacity generator risk.

Proof. We write $P^*(Y) = (1+\eta^*)E[S_Y]$, $u^*(Y) = \alpha^* P^*(Y)$, $\Delta P(Y) = P(Y) - P^*(Y)$, $\Delta u(Y) = u(Y) - u^*(Y)$ and we use similar symbols for portfolio $\widetilde{Y}$. Then $CS(\alpha, \eta; Y) = P(Y) + u(Y)$, $CS(\alpha^*, \eta^*; Y) = P^*(Y) + u^*(Y)$ and we have

$$CS(\widetilde{\alpha}, \widetilde{\eta}; \widetilde{Y}) = P(Y) + P(X) + u(Y) + \Delta u, \qquad CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}) = P^*(Y) + \Delta P^* + \Delta u^* + u^*(Y),$$

where $\Delta u = u(\widetilde{Y}) - u(Y)$, $\Delta u^* = u^*(\widetilde{Y}) - u^*(Y)$ and $\Delta P^* = P^*(\widetilde{Y}) - P^*(Y)$. Hence, we can write $C_Y = \Delta P(Y) + \Delta u(Y)$, $C_{\widetilde{Y}} = \Delta P(Y) + P(X) - \Delta P^* + \Delta u(Y) + \Delta u - \Delta u^*$ and with few algebra we obtain

$$C_{\widetilde{Y}} - C_Y = P(X) + \Delta u - (\Delta P^* + \Delta u^*),$$

which is equivalent to

$$C_{\widetilde{Y}} - C_Y = P(X)(1 + \widetilde{\alpha}) + (\widetilde{\alpha} - \alpha)P(Y) - (\Delta P^* + \Delta u^*), \tag{33}$$

where $\Delta P^* + \Delta u^* = CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}) - CS(\alpha^*, \eta^*; Y)$. After substituting the expression of $C_Y = (1 + \alpha)P(Y) - CS(\alpha^*, \eta^*; Y)$ into (33), we also get

$$C_{\widetilde{Y}} = P(X)(1 + \widetilde{\alpha}) + (1 + \widetilde{\alpha})P(Y) - CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}). \tag{34}$$

Item (i) of Proposition 4.2 follows from eq. (34). Item (ii) simply says under which choice of $P(X)$ there results either $C_{\widetilde{Y}} \leq C_Y$, or the reverse, if inequality $\Delta P^* + \Delta u^* \geq 0$ is inserted in (33). In particular, using again (33) when $\Delta P^* + \Delta u^* < 0$, that is, $CS(\widetilde{\alpha}^*, \widetilde{\eta}^*; \widetilde{Y}) < CS(\alpha^*, \eta^*; Y)$, there immediately results eq. (32).

∎

Proposition 4.3. (i) *If X is a great risk then $C_{\widetilde{Y}} \geq 0$ if*

$$P(X) \geq \frac{h(M_\alpha, \widetilde{M}_\eta)}{(1 + \widetilde{\alpha})} E[S_{\widetilde{Y}}] - P(Y). \tag{35}$$

(ii) *If X is a catastrophic risk then $C_{\widetilde{Y}} \geq 0$ if*

$$P(X) \geq \frac{h(M_M, \widetilde{M}_\eta)}{(1 + \widetilde{\alpha})} E[S_{\widetilde{Y}}] - P(Y). \tag{36}$$

Proof. If X is a great risk there results

$$C_{\widetilde{Y}} \geq (1 + \widetilde{\alpha})P(X) + (1 + \widetilde{\alpha})P(Y) - h(M_\alpha, \widetilde{M}_\eta)E[S_{\widetilde{Y}}],$$

so that $C_{\widetilde{Y}} \geq 0$ when (35) is fulfilled. Similarly, if X is a catastrophic risk then

$$C_{\widetilde{Y}} \geq (1+\widetilde{\alpha})P(X) + (1+\widetilde{\alpha})P(Y) - h(M_M, \widetilde{M}_\eta)E[S_{\widetilde{Y}}],$$

so that $C_{\widetilde{Y}} \geq 0$ if (36) holds.

■

Remark 4.4. Obviously, Proposition 4.3 excludes the mega–catastrophic risk since, in this case, the only information available concerning the capacity $C_{\widetilde{Y}}$ is that $C_{\widetilde{Y}} < (1+\widetilde{\alpha})P(Y) + (1+\widetilde{\alpha})P(X) - h(M_M, \widetilde{M}_\eta)E[S_{\widetilde{Y}}]$ (see Definition 3.6).

■

Proof of Proposition 3.9.
item (i)

When $\mathrm{sgn}(\overrightarrow{A\widetilde{A}_i}) > 0$, $i = 1, 2$, there results $\mid \overrightarrow{\widetilde{v}_1(\widetilde{Y}_1)} \mid \leq \mid \overrightarrow{\widetilde{v}_1(\widetilde{Y}_2)} \mid$ if and only if $\mid \overrightarrow{A\widetilde{A}_1} \mid \leq \mid \overrightarrow{A\widetilde{A}_2} \mid$. On the contrary, in the case $\mathrm{sgn}(\overrightarrow{A\widetilde{A}_i}) < 0$, $i = 1, 2$, the same results holds if and only if $\mid \overrightarrow{A\widetilde{A}_1} \mid \geq \mid \overrightarrow{A\widetilde{A}_2} \mid$. When $\mathrm{sgn}(\overrightarrow{A\widetilde{A}_1}) < 0$ and $\mathrm{sgn}(\overrightarrow{A\widetilde{A}_2}) > 0$, it is immediate that $\mid \overrightarrow{\widetilde{v}_1(\widetilde{Y}_1)} \mid \leq \mid \overrightarrow{\widetilde{v}_1(\widetilde{Y}_2)} \mid$.
item (ii)

When $\mathrm{sgn}(\overrightarrow{BG_i}) > 0$, $i = 1, 2$, there results $\mid \overrightarrow{\widetilde{v}_4(\widetilde{Y}_1)} \mid \leq \mid \overrightarrow{\widetilde{v}_4(\widetilde{Y}_2)} \mid$ if and only if $\mid \overrightarrow{BG_1} \mid \geq \mid \overrightarrow{BG_2} \mid$. For the other cases, we argue as made in item (i).
item (iii)

We follow the arguments of item (i), with $\rho(\overrightarrow{BF_i} - \overrightarrow{A\widetilde{A}_i})$ instead of $\rho(\overrightarrow{A\widetilde{A}_i})$, $i = 1, 2$.
item (iv)

The proof of this item is the same of that one of item (i), having substituted $\rho(\overrightarrow{A\widetilde{A}_i})$ with $\rho(\overrightarrow{BF_i})$, $i = 1, 2$.

Now, by Proposition 2.11, applied to the present case, we get that if $\widetilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \widetilde{Y}_2$ for $i = 1, 4, 5, 6$, then $\widetilde{Y}_1 \leq_{oc_i(\Theta(\cdot))} \widetilde{Y}_2$ for $i = 1, \ldots, 6$, hence, by Definition 3.8, $X_1 \leq_{B(Y)} X_2$.

■

References

[1] American Academy of Actuaries, Comittee on Risk Classification, 1981. Risk Classification. Statement of Principle.

[2] Artzner, P., Delbaen, F., Eber, J.M., Heath, D., 1999. Coherent Measures of Risk. Mathematical Finance, Vol. 9, n. 3, 203–228.

[3] Dhaene, J., Goovaerts, M.J., Kaas, R., 2004. Economic Capital Alloscation Derived from Risk Measures. North American Actuarial Journal 7, n. 2, pp. 44-59.

[4] Dhaene, J., Laeven, R.J.A., Vanduffel, S., Darkiewicz, G., Goovaerts, M.J., 2004. Can a Coherent Risk Measure be too Subadditive?. http://afir2004.soa.org/afir_papers.htm.

[5] Dong, W., Shah, H., Wong, F., 1996. A Rational Approach to Pricing of Catastrophe Insurance. Journal of Risk and Uncertainty, n. 12, pp. 201–218.

[6] Freddi, A., Sargenti, G., 2004. A Model for the Insurance & Reinsurance of a Mega-Catastrophic Risk. http://www.ime2004rome.com/full_test.htm.

[7] Frittelli, M., Rosazza Gianin, E., 2004. Dynamic Convex Risk Measures. In: Risk Measures for the 21st Century. (G. Szegö ed.), John Wiley & Sons, pp. 227–248.

[8] Garrido, J., 1993. Discussion on the paper by Ramsay [15].

[9] Goovaerts, M. J., Kaas, R., Dhaene, J., Tang, Q., 2004. Some New Classes of Consisten Risk Measures. Insurance: Mathematics and Economics, n. 34, pp. 505–516.

[10] Hürlimann, W., 1995. Trasforming, Ordering and Rating Risks. Mitteilungen–Bulletin, n. 2, pp. 213–234.

[11] International Actuarial Association, 2004. A Global Framework for Insurer Solvency Assessment. A Report by the Insurer Solvency Assessment Working Party of the International Actuarial Association. http://www.actuaires.org/members/en/documents/papers/Global_Framework_Insurer_Solvency_Assessment-public.pdf.

[12] Kaas, R., Heerwaarden, A.E. van, Goovaerts, M.J., 1994. Ordering of Actuarial Risks, CAIRE, Brussels (Educational Series).

[13] Luenberger, D.G., 1998. Investment Science. Oxford University Press.

[14] Property–Casualty Risk–Based Capital Requirement. A Conceptual Framework. Actuarial Advisory Committee to the NAIC Property & Casualty Risk–Based Capital Working Group.

[15] Ramsay, C.M., 1993. Loading Gross Premiums for Risk without Using Utility Theory. Transactions of the Society of Actuaries, n. 45, pp. 305–336. Discussion, pp. 337–349.

[16] Stone, J.M., 1973. A Theory of Capacity and the Insurance of Catastrophe Risks. The Journal of Risk and Insurance 40, n. 2, pp. 231–244, pp. 339–356.

[17] Walters, M.A., 1981. Risk Classification Standards. Proceedings of the Casualty Actuarial Society, Vol. XVIII, Part I, n. 129, pp. 1–81.