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EM Algorithm for Bivariate Phase Distributions

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Phase distributions

- are nice to handle.
- allow for explicit formulas,
  e.g. densities, higher moments , ruin probabilities.
- are closed under convex combinations, convolutions.
- Compound variables are phase variables, if the counting and the size variables are phase variables.
- allow dependent multivariate distributions.
- Matrix exponentials are nowadays easy to calculate.
- Simulations of Phase variables can be done directly by using the underlying homogeneous Markov process.

Phase distributions

- are dense in the class of distributions on [0,∞) in the weak topology

however,

- have exponentially decreasing tail distributions; they can not be used for large- or extreme-value problems.

See e.g.:

- **Encyclopedia of Actuarial Science, 2004, "Phase type distributions" and "Phase method"**
- **Rolski et al.: Stochastic Processes for Insurance and Finance, 2000.**
- **Hipp, Ch.: Lectures in Strasbourg, 2000-2005**

Definitions (continuous case)

Let $X(t)$ be a homogeneous Markov chain with finite state space $D' = D \cup \{0\}$,

$$D = \{ 1, \ldots, d \},$$

having $0 \in D$ as an absorbing and attracting state . The 'life time'

$$T = \inf \{ t \geq 0, X(t) \notin D \}$$

is a phase variable.

We call the <u>complement</u> C of an absorbing subset of D' <u>inaccessible</u> (e.g. D).

If $(C_1, \ldots, C_m)$ are inaccessible subsets, then $(T_1, \ldots, T_m)$ with

$$T_\nu = \inf \{ t \geq 0, X(t) \notin C_\nu \}$$

is a multivariate phase variable.

Common distributions

Let $\widetilde{L} = (l_{jk})_{0 \leq j,k \leq d}$ be the stationary generator of $X(t)$ where $l_{jk} \geq 0$ is the intensity of the transition from state j to $k, j \neq k$,

$$l_{jj} = -\sum_{j \neq k} l_{jk} \ ,$$

$L = (l_{jk})_{1 \leq j,k \leq d}$ the submatrix of $\widetilde{L}$ and $\widetilde{\pi} = (\pi_0, \pi_1, ..., \pi_d) = (\pi_0, \pi)$ be the starting distribution. Then

Theorem 1. *For* $0 \leq \mu < m$ *and* $0 \leq t_1 \leq \ldots \leq t_{m-\mu}$. *we get*

$$\begin{aligned} & \mathbb{P}\,(T_1 = .. = T_\mu = 0 \ \text{and}\ T_{\mu+\nu} > t_\nu \ \text{for}\ \nu = 1, .., m-\mu) \\ = \ & \pi \Delta^{A(\mu)} e^{Lt_1} \Delta^{C_{\mu+1}} e^{L(t_2 - t_1)} \Delta^{C_{\mu+2}} \ \ldots \ e^{L(t_m - t_{m-1})} \,.\, \Delta^{C_m} \eta \end{aligned}$$

where

$$\begin{aligned} \eta &= (1, .., 1) \in \mathbb{R}^d \\ \Delta^C_{pq} &= \begin{cases} 1 & \text{if } p = q \in C \\ 0 & \text{else} \end{cases} \quad \text{and} \\ A(\mu) &= D \setminus \left(\bigcup_{\nu=1}^{\mu} C_\nu \right) . \end{aligned}$$

Corollary 2. *For* $m = 2$, $\mathcal{C} = \{C_1, C_2\}$ *and* $\nu = 1, 2$, *we have*

(i) for $0 < t_1 < t_2$

$$\mathbb{P}(T_\nu > t_1\ ,\ T_{3-\nu} > t_2) = \pi \cdot e^{L \cdot t_1} \cdot \Delta^{C_\nu} \cdot e^{L \cdot (t_2 - t_1)} \cdot \Delta^{C_{3-\nu}} \cdot \eta\ ,$$

(ii) for $0 < t$

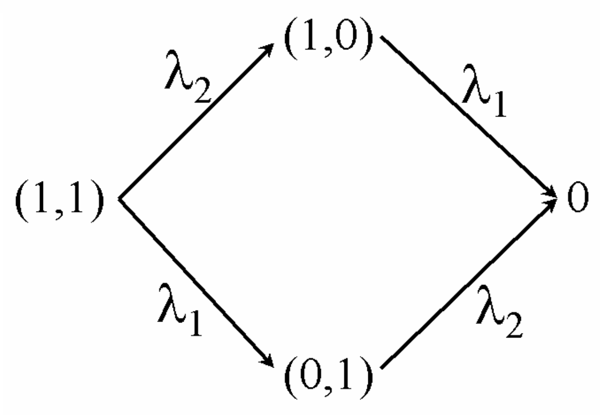
$$\mathbb{P}(T_1 = T_2 \in dt) = -\pi \cdot e^{L \cdot t} \cdot \Delta^{C_1 \cap C_2} \cdot L \cdot \eta \cdot dt\ .$$

Example

Let Z_1 and Z_2 be two independent exponential variables with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, set

$$T_1 = Z_1 \text{ and } T_2 = \max(Z_1, Z_2) \ .$$

(T_1, T_2) are two phase variables with $D' = \{0, (1,1), (1,0), (0,1)\}$ and transition intensities



$C_1 = \{(1,1),(1,0)\}$ and $C_2 = D = \{(1,1),(1,0),(0,1)\}$ are inaccessible subsets of D'.

$$L = \begin{pmatrix} -\lambda_1 - \lambda_2 & \lambda_2 & \lambda_1 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{pmatrix}$$

and

$$\mathbb{P}\left(T_1 > t_1, T_2 > t_2\right) = \begin{cases} e^{-\lambda_1 t_2} + e^{-\lambda_2 t_2}\left(e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2}\right) & \text{if } 0 < t_1 < t_2 \\ e^{-\lambda_1 t_1} & \text{if } 0 < t_2 < t_1 \end{cases}$$

with marginal distributions

$$\begin{aligned} F_1(t_1) &= \mathbb{P}\left(T_1 \le t_1\right) = 1 - e^{-\lambda_1 t_1} \text{ and} \\ F_2(t_2) &= \mathbb{P}\left(T_2 \le t_2\right) = \left(1 - e^{-\lambda_1 t_2}\right) \cdot \left(1 - e^{-\lambda_2 t_2}\right) . \end{aligned}$$

In the special case $\lambda_1 = \lambda_2$, the copula has the explicit form

$$C(u,v) = \left(u \cdot v^{1/2}\right) \wedge v \; .$$

EM algorithm for bivariate phase variables (continuous case)

The EM algorithm is based on the entropy inequality

$$g_{\chi_1}(T) \geq g_{\chi_0}(T)$$

for the marginal density

$$g_{\chi}(T) = \int f_{\chi}(\xi) \cdot \mu\left(d\xi \,|T(\xi) = \boldsymbol{T}\right)$$

of a random variable ξ whose density $f_{\chi}(\xi)$ with respect to some σ-finite measure $\mu(d\xi)$ depends on some parameter set χ, and where

$$\chi_1 = \arg \max_{\chi} \mathbb{E}_{\chi_0}\left(\log f_{\chi}(\xi) \middle| T(\xi) = T\right) \; .$$

$$\text{Let} \ \left(D' = \{0,..,d\}, \mathcal{C} = \{C_1, C_2\}, \widetilde{L}, \widetilde{\pi},\right) \ .$$

be the characteristics of the process $X(t)$ where C_ν are inaccessible sets of D, kept fixed in the sequel, and the parameter set $\chi = (L, \pi)$.
A trajectory is given by $\xi = (j_1, t_1, j_2, t_2, .., j_m, t_m)$, where j_1 is the initial state, t_1 the time $X(t)$ spends in j_1, then $X(t)$ jumps to j_2 and stays there a time t_2, and so on.
The density of such a trajectory is

$$\begin{aligned} f_\chi(\xi) \ &= \ \pi_{j_1} \cdot (-l_{j_1 j_1}) \cdot \exp(t_1 l_{j_1 j_1}) \cdot p(j_1, j_2) \cdot (-l_{j_2 j_2}) \cdot \exp(t_2 l_{j_2 j_2}) \cdot . \\ & \quad .. \cdot (-l_{j_m j_m}) \cdot \exp(t_m l_{j_m j_m}) \cdot p(j_m, j_{m+1}) \ . \end{aligned}$$

where $j_{m+1} = 0$ and for $j \neq k$, $p(j,k) = l_{jk} / (-l_{jj})$ is the conditional probability of a jump from j to k , given that a jump takes place.

The log-density function is

$$
\begin{aligned}
\ln\left(f_\chi(\xi)\right) &= \ln\left(\pi_{j_1}\right) + \sum_{\nu=1}^{m} t_\nu \cdot l_{j_\nu j_\nu} + \sum_{\nu=1}^{m} \ln\left(l_{j_\nu j_{\nu+1}}\right) \\
&= \sum_{j=1}^{d} \ln \pi_j \cdot N_j + \sum_{j=1}^{d} l_{jj} \cdot Z_j + \sum_{j=1}^{d} \sum_{k=0\ (!),k\neq j}^{d} \ln l_{jk} \cdot N_{jk} \\
\text{with} \quad N_j &= \mathbf{1}_{\{j\}}(j_1) \\
N_{jk} &= \#\{(j_\nu, j_{\nu+1}) = (j,k),\ \text{avec } \nu \leq m\ \},\ \text{number of jumps from } j \text{ to } k \\
Z_j &= \sum_{\nu=1}^{m} t_\nu \cdot \mathbf{1}_{\{j\}}(j_\nu)\ ,\ \text{the local time of } j\ .
\end{aligned}
$$

For observations $T_1 = (T_{11}, T_{12}), ..., T_n = (T_{n1}, T_{n2})$, the maximization problem is

$$
\max_{\chi=(L,\pi)} \mathbb{E}_{\chi_0} \left(\sum_{j=1}^{d} \ln \pi_j \cdot N_j + \sum_{j=1}^{d} l_{jj} \cdot Z_j + \sum_{j=1}^{d} \sum_{k=0\ (!),k\neq j}^{d} \ln l_{jk} \cdot N_{jk} \,\middle|\, T_1, ..., T_n \right).
$$

Starting with some initial parameters $\chi^{(0)} = (L^{(0)}, \pi^{(0)})$, the maximization problem boils down to the recursion procedure

$$\pi_j^{(\nu+1)} = \pi_j^{(\nu)} \cdot \frac{1-\pi_0}{n} \sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)},L^{(\nu)}\right)} \left(N_j \middle| (T_i)\right) \text{ for } 1 \leq j \leq d \text{ and}$$

$$l_{jk}^{(\nu+1)} = \frac{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)},L^{(\nu)}\right)} \left(N_{jk} \middle| (T_{i1}, T_{i2})\right)}{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)},L^{(\nu)}\right)} \left(Z_j \middle| (T_i)\right)} ,$$

$$l_{j0}^{(\nu+1)} = \frac{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)},L^{(\nu)}\right)} \left(N_{j0} \middle| (T_i)\right)}{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)},L^{(\nu)}\right)} \left(Z_j \middle| (T_i)\right)} ,$$

$$l_{jj}^{(\nu+1)} = -\sum_{k=0\,,k\neq j}^{d} l_{jk}^{(\nu)} .$$

In the case $0 < T_\nu < T_{3-\nu}$ we get

$$
\begin{aligned}
&\mathbb{E}_\chi\left(N_j \middle| (T_1, T_2)\right) \\
= \;& \pi_j \begin{cases} \dfrac{\eta_j \cdot e^{L\cdot T_\nu} \cdot \left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j \in C_1 \cap C_2 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_\chi\left(N_{jk} \middle| (T_1, T_2)\right) \\
= \;& l_{jk} \begin{cases} \dfrac{\int\limits_0^{T_\nu} \pi\cdot e^{L\cdot u}\cdot\Delta(j,k)\cdot e^{L\cdot(T_\nu - u)}\cdot\left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j,k \in C_1 \cap C_2 \\ \dfrac{\pi\cdot e^{L\cdot T_\nu}\cdot\Delta(j,k)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j \in C_\nu,\, k \in C_{3-\nu} \setminus C_\nu \\ \dfrac{\int\limits_{T_\nu}^{T_{3-\nu}} \pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot(u - T_\nu)}\cdot\Delta(j,k)\cdot e^{L\cdot\left(T_{\nu-3}-u\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu} - \Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j,k \in C_{3-\nu} \setminus C_\nu \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

and

$$\mathbb{E}_\chi\left(N_{j0}\middle|\left(T_1,T_2\right)\right)$$
$$= \quad l_{j0}\begin{cases} \frac{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta_j}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_{3-\nu}\setminus C_\nu \\ 0 & \text{else}\end{cases}$$

$$\mathbb{E}_\chi\left(Z_j\middle|\left(T_1,T_2\right)\right)$$
$$= \begin{cases} \frac{\int_0^{T_\nu}\pi\cdot e^{L\cdot u}\cdot\Delta(j,j)\cdot e^{L\cdot(T_\nu-u)}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_1\cap C_2 \\ \frac{\int_{T_\nu}^{T_{3-\nu}}\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot(u-T_\nu)}\cdot\Delta(j,j)\cdot e^{L\cdot\left(T_{\nu-3}-u\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot T_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(T_{3-\nu}-T_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_{3-\nu}\setminus C_\nu \\ 0 & \text{else}\end{cases}$$