# EM algorithm for bivariate phase distributions

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**Abstract**

In this paper, we descibe the general construction of multivariate phase-type distributions of discret or continuous type and study their commun distributions and densities on lower dimensional subspaces. Finally, we adopt the known EM algorithm for approximating bivariate phase distributions to a set of two-dimensional observations.

*Key words* :Multivariate phase distributions, copula, EM algorithm, simulation of dependent random varibles

## 1 Introduction

It seems that phase-type distributions, here simply called phase distributions, are becoming more and more important in actuarial applications. The reasons for this impact are easily found out. Phase distributions are nice to handle, they allow in many cases explicit expressions, for examples for higher moments, for ruin probabilities and so on, and they are extremely flexible, since (i) the class of phase distributions is closed under convexe combinations, convolutions and random mixtures, if tzhe counting variable is itself a discret phase variable, and (ii) phase distribution are dense in the set of all distributions on $[0, \infty)$ under the weak topology (see [7] for an overview). On the contrary, there are two major drawbacks: phase variables can not be used for large or extreme value problemes, since their tail distributions are all exponentially decreasing. However, the second drawback, which lies in the complexitiy of their calculations, becomes less important by the use of modern computers and very rapid programs. Since the simulation of trajectories of the underlying Markov chain is easily done by a number of independent exponential and discrete variables, it is not difficult to simulate multivartiate phase varibles.

So it is nice to see that phase varibles also open the way to handle dependent multivariate variables ([4], [8]). In this paper, we first present the general setup for discret and continuous multivariate phase variables. We calculate their commun distributions and their "densities" which in general are situated on lower dimensional subspaces. In a simple example, we calculate the copula of a bivariate phase variable. Finally, we extend the known EM algorithm to fit parameters of bivariate phase variables to a set of two-dimensional observations.

## 2 Definition of Multivariate Phase Variables

Phase-type distributions, in the sequel simply called phase distrbutions, are defined as the distribution of the survival time of a time-homogeneous Markov chain with a finite state space having an attracting and absorbing state. The phase-type distribution may be discrete, having values in $\mathbb{N}_0 = \{0, 1, ...\}$ or continuous with values in $[0, \infty)$.

In the discrete case, we have a homogenuous Markov chain $(X(t))_{t \in \mathbb{N}_0}$ with state space $\widetilde{D} = \{0, 1, 2, ..., d\}$, a transition matrix $\widetilde{P} = \left(\widetilde{P}_{jk}\right)_{0 \leq j,k \leq d}$ and a starting distribution $\widetilde{\pi} = (\pi_0, \pi_1, ..., \pi_d) = (\pi_0, \pi)$, where we assume that the state $0 \in \widetilde{D}$ is as well absorbing as attracting: i.e.

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(i) $\widetilde{P}_{0j} = 0$ for all $j \in D := \{1, 2, ..., d\}$ (absorption at 0 ) ,

(ii) there exists a power $t$ of the transition matrix $\widetilde{P}$ such that $\widetilde{P}^t_{j0} > 0$ for all $j \in D$ (attraction of 0 ).

According to the Perron-Frobenius theorem, these conditions imply that the maximal real positive eigenvalue $\lambda_0 = 1$ is simple and the submatrice $P = (P_{jk})_{1\leq j,k\leq d}$ is submarkovian having a real maximal eigenvalue $\lambda_1 < 1$: $|\lambda| \leq \lambda_1$ for all eigenvalues of $P$ , which are also the eigenvalues of $\widetilde{P}$.

The **phase variable** $T$ is defined as the 'life time' of the Markov process $X(t)$ , interpreting the attracting and absorbing state 0 as 'cemetery' of the process

$$T = \min\{t \in \mathbb{N}_0,\ X(t) = 0\} \ . \tag{1}$$

Since $X(t) = 0$ for $t \to \infty$,the phase variable $T$ is almost surely finite. The distribution of $T$ is the discrete phase distribution

$$\begin{aligned} \mathbb{P}_T(t) &= \mathbb{P}(T = t) \ , t \in \mathbb{N}_0 \ , \text{ with} \\ \mathbb{P}(T > t) &= \pi \cdot P^{t-1} \cdot \eta \end{aligned} \tag{2}$$

where

$$\eta = (1, .., 1) \in \mathbb{R}^d \ . \tag{3}$$

Obviously,we can generalize the definition by replacing the absorbing and attracting state 0 by a subset $A$ of $\widetilde{D}$ with the same properties: (i) $A$ is absorbing if $\widetilde{P}_{jk} = 0$ for all $j \in A$ and $k \in \widetilde{D} \setminus A$, and (ii) $A$ is attracting if $\sum_{j\in A} \widetilde{P}^t_{kj} > 0$ for all $k \in \widetilde{D} \setminus A$ and some power $t$ . The complement $C = \widetilde{D} \setminus A$ of an absorbing and attracting subset $A$ we call an **inaccessible** set. The lifetime $T_C$ within an inaccessible set $C$ is a phase variable:

$$T_C = \min\{t \in \mathbb{N}_0,\ X(t) \notin C\} \ .$$

Let us introduce the matrices

$$\Delta(j, k) \in \mathbb{R}^{d\times d}$$

defined by

$$\Delta(j, k)_{pq} = \begin{cases} 1 & \text{si } j = p \text{ et } k = q \\ 0 & \text{sinon} \end{cases} \quad , \tag{4}$$

$1 \leq j, k, p, q \leq d$ , as well as

$$\Delta^C = \sum_{j\in C} \Delta(j, j) \ . \tag{5}$$

Then, (2) can be generalized to

$$\mathbb{P}(T_C > t) = \pi \cdot P^{t-1} \cdot \Delta^C \cdot \eta \tag{6}$$

If we regard a sequence $\mathcal{C} = \{C_1, ..., C_m\}$ of inaccessible sets $C_\nu$ , $\nu = 1, .., m$, we get a multivariate phase variable

$$\mathbf{T} = (T_1, .., T_m) = (T_{C_1}, .., T_{C_m}) \ . \tag{7}$$

Notice that the intersection and union of inaccessible sets are again inaccessible, which means that the minimum and maximum of phase variables, defined on the same Marov process, are also phase variables.

In the time continuous case, the Markov process is characterized by its stationary generator $\widetilde{L} = (l_{jk})_{0\leq j,k\leq d}$ where for $j \neq k$ , $l_{jk} \geq 0$ is the intensity of the transition from state $j$ to $k$ and

$$l_{jj} = -\sum_{k\neq j} l_{jk} \ . \tag{8}$$

Here, a subset $A$ of $\widetilde{D}$ is (i) absorbing, if.and olny if $l_{jk} = 0$ for all $j \in A$ and $k \notin A$ , and (ii) attracting if and only if for all $j \in \widetilde{D}$ there exists a sequence of states: $j_0 = j,\ j_1, .., j_n \in A$ with $l_{j_{\nu-1}j_\nu} > 0$ for $\nu = 1, .., n$. Again, we call the complement $C$ of an absorbing and attracting set inaccessible.

If $D$ is inacessible, then applying Perron-Frobenius theorem to the matrix $\widetilde{P} = e^{\widetilde{L}}$ and then the logarithmic function to the eigenvalues of $\widetilde{P}$, shows that 0 is a simple eigenvalue of $\widetilde{L}$, that $\widetilde{L}$ has a second real negative eigenvalue $\mu_1 < 0$ and the real part of all other eigenvalues $\mu$ is les or equal to $\mu_1$: $\mathcal{R}e(\mu) \leq \mu_1 < 0$ .

For a sequence $\mathcal{C} = \{C_1, ..., C_m\}$ of inaccessible sets, we have again $\mathbf{T} = (T_1, .., T_m) = (T_{C_1}, .., T_{C_m})$ as a multivariate phase variable with values in $[0, \infty)^m$.

Without loss of generality, we may assume lateron that

$$\widetilde{D} = \bigcup_{\nu=1}^{m} C_\nu \,\dot{\cup}\, \{0\} \ . \tag{9}$$

# 3 Commun Distributions of Multivariate Phase Variables

Lets start with the discret case based on a discret homogeneous Markov chain $X(t)$ with characteristics $\left(\widetilde{D},\ \widetilde{P},\ \widetilde{\pi}\right)$ and a set $\mathcal{C} = \{C_1, ..., C_m\}$ of inaccessible sets satisfying (9). The commun distribution of the multivariate phase variable $\mathbf{T}_{\mathcal{C}}$ is given by the following result.

**Theorem 1**

(i)

$$\mathbb{P}\left(T_1 = .. = T_m = 0\right) = \pi_0 \tag{10}$$

(ii) Let $1 \leq p \leq m$ , $0 \leq \mu_0 < .. < \mu_p = m$ , $0 = t_0 < t_1 < ... < t_p$ and $\kappa$ a permutation of $\{1, .., m\}$ . Then

$$\begin{aligned} & \mathbb{P}\left(T_{\kappa(1)} = .. = T_{\kappa(\mu_0)} = 0 \text{ and for } i = 1, .., p : T_{\kappa(\mu_{i-1}+1)} = .. = T_{\kappa(\mu_i)} = t_i\right) \\ = \ & \pi \cdot \Delta^{A(\mu_0)} \cdot \prod_{i=1}^{p-1} \left(P^{(t_i - t_{i-1} - 1)} \cdot \Delta^{C(\mu_i)} P \Delta^{A(\mu_i)}\right) \cdot P^{(t_p - t_{p-1} - 1)} \Delta^{C(\mu_p)} \left(Id - P\right) \cdot \eta \end{aligned} \tag{11}$$

where $A(\mu_0) = D \setminus \left(\bigcup_{j=1}^{\mu_0} C_{\kappa(j)}\right)$ and $C(\mu_i) = \bigcap_{j=\mu_{i-1}+1}^{\mu_i} C_{\kappa(j)}$ , $A(\mu_i) = D \setminus \left(\bigcup_{j=\mu_{i-1}+1}^{\mu_i} C_{\kappa(j)}\right)$ , $i = 1, .., p$ .

**Proof:**

Let $\kappa$ be the identity. Then

$$\begin{aligned} & \mathbb{P}\left(T_1 = .. = T_{\mu_0} = 0 \text{ and for } i = 1, .., p : T_{\mu_{i-1}+1} = .. = T_{\mu_i} = t_i\right) \\ = & \sum_{\substack{j_0 \in A(\mu_0) \\ k_\nu \in C(\mu_\nu)\ ,\ j_\nu \in A(\mu_\nu),\ 1 \leq \nu \leq p}} \mathbb{P}\left(X(0) = j_0\right) \cdot \\ & \cdot \prod_{\nu=1}^{p} \left[\mathbb{P}\left(X(t_\nu) = k_\nu \,|\, X(t_{\nu-1}+1) = j_{\nu-1}\right) \mathbb{P}\left(X(t_\nu+1) = j_\nu \,|\, X(t_\nu) = k_\nu\right)\right] \\ = & \ \widetilde{\pi} \widetilde{\Delta^{A(\mu_0)}} \prod_{\nu=1}^{p} \left(\widetilde{P}^{(t_\nu - t_{\nu-1} - 1)} \widetilde{\Delta^{C(\mu_\nu)}} \cdot \widetilde{P} \cdot \widetilde{\Delta^{A(\mu_\nu)}}\right) \cdot \widetilde{\eta} \\ = & \ \widetilde{\pi} \widetilde{\Delta^{A(\mu_0)}} \prod_{\nu=1}^{p-1} \left(\widetilde{P}^{(t_\nu - t_{\nu-1} - 1)} \widetilde{\Delta^{C(\mu_\nu)}} \cdot \widetilde{P} \cdot \widetilde{\Delta^{A(\mu_\nu)}}\right) \cdot \widetilde{P}^{(t_p - t_{p-1} - 1)} \widetilde{\Delta^{C(\mu_p)}} \cdot \widetilde{P} \cdot \widetilde{\Delta^{A(\mu_p)}} \cdot \widetilde{\eta} \end{aligned}$$

By the iterated multiplications of $\widetilde{\Delta^{A(\mu_\nu)}}$ where $\widetilde{A}(\mu_i)$ are absorbing sets et $\{0\} = \bigcap_{\nu=1}^{p} \widetilde{A}(\mu_i)$ by (9) and by the fact that $\widetilde{P} \cdot \widetilde{\Delta^{\{0\}}} \cdot \widetilde{\eta} = \begin{pmatrix} 0 \\ (Id-P)\cdot\eta \end{pmatrix}$, we find

$$\begin{aligned} & \mathbb{P}\left(T_1 = .. = T_{\mu_0} = 0 \text{ and for } i = 1,..,p : T_{\mu_{i-1}+1} = .. = T_{\mu_i} = t_i\right) \\ = \quad & \pi\Delta^{A(\mu_0)} \prod_{\nu=1}^{p-1} \left(P^{(t_\nu - t_{\nu-1}-1)}\Delta^{C(\mu_\nu)} \cdot P \cdot \Delta^{A(\mu_\nu)}\right) \cdot P^{(t_p - t_{p-1}-1)}\Delta^{C(\mu_p)} \cdot (Id-P)\cdot\eta \,. \end{aligned}$$

■

In the continuous case, we get the following result.

**Theorem 2**

(i)

$$\mathbb{P}\left(T_1 = .. = T_m = 0\right) = \pi_0 \tag{12}$$

(ii) Let $\kappa$ be a permutation of $\{1,..,m\}$ , $0 \le \mu < m$ and $0 \le t_1 \le ... \le t_{m-\mu}$ . Setting $A(\mu) = D \setminus \left(\bigcup_{\nu=1}^{\mu} C_{\kappa(\nu)}\right)$, we get for the multivariate varible $\mathbf{T}$

$$\begin{aligned} & \mathbb{P}\left(T_{\kappa(1)} = .. = T_{\kappa(\mu)} = 0 \text{ and } T_{\kappa(\mu+\nu)} > t_\nu \text{ for } \nu = 1,..,m-\mu\right) \\ = \quad & \pi \cdot \Delta^{A(\mu)} \cdot e^{L\cdot t_1} \cdot \Delta^{C_{\kappa(\mu+1)}} \cdot e^{L\cdot(t_2-t_1)} \cdot \Delta^{C_{\kappa(\mu+2)}} \cdot e^{L\cdot(t_m - t_{m-1})} \cdot \Delta^{C_{\kappa(m)}} \cdot \eta \,. \end{aligned} \tag{13}$$

**Proof:**

For simplicity, lets take $\kappa$ as identity. We know $\{T_1 = .. = T_\mu = 0\} = \{X(0) \notin \bigcup_{\nu=1}^{\mu} C_\nu\} = \{X(0) \in A(\mu)\}$ , and with.$t_0 = 0$ , we get

$$\begin{aligned} & \mathbb{P}\left(T_1 = .. = T_\mu = 0 \text{ and } T_{\mu+\nu} > t_\nu \text{ for } \nu = 1,..,m-\mu\right) \\ = \quad & \sum_{\substack{j_0 \in A(\mu) \\ j_\nu \in C_{\mu+\nu}\,,\ 1 \le \nu \le m-\mu}} \left(\mathbb{P}\left(X(0) = j_0\right) \prod_{\nu=1}^{m-\mu} \mathbb{P}\left(X(t_\nu) = j_\nu \middle| X(t_{\nu-1}) = j_{\nu-1}\right)\right) \\ = \quad & \pi \cdot \Delta^{A(\mu)} \cdot \left(\prod_{\nu=1}^{m-\mu} e^{L\cdot(t_\nu - t_{\nu-1})} \cdot \Delta^{C_{\mu+\nu}}\right) \cdot \eta \end{aligned}$$

■

Now, we would like to find a more refined result of the distribution $\mathbb{P}_{\mathbf{T}}$ on $[0,\infty)^m$. In the preceeding theorem, we saw that the events $\left\{T_{\kappa(1)} = .. = T_{\kappa(\mu)} = 0\right\}$ may have a positive probability measure such that $\mathbb{P}_{\mathbf{T}}$ is in general not absolutely continuous with respect to Lebeque's measure. More preciely, if the process $X(t)$ jumps in $(t, t+\varepsilon)$ from the set $\cap C = \bigcap_{\nu=1}^{\mu} C_{\kappa(\nu)}$ to the set $A = \widetilde{D} \setminus \left(\bigcup_{\nu=1}^{\mu} C_{\kappa(\nu)}\right)$, we get using $\widetilde{L} \cdot \widetilde{\eta} = 0$ and $\cup C = \bigcup_{\nu=1}^{\mu} C_{\kappa(\nu)}$

$$\begin{aligned} \mathbb{P}\left(T_{\kappa(1)} = .. = T_{\kappa(\mu)} \in (t, t+\varepsilon)\right) &= \varepsilon \cdot \widetilde{\pi} e^{\widetilde{L}\cdot t} \cdot \widetilde{\Delta^{\cap C}} \cdot \widetilde{L} \cdot \widetilde{\Delta^{A}} \cdot \widetilde{\eta} + \mathfrak{o}(\varepsilon) \\ &= -\varepsilon \cdot \pi e^{L\cdot t} \cdot \Delta^{\cap C} \cdot L \cdot \Delta^{\cup C} \cdot \eta + \mathfrak{o}(\varepsilon) \,. \end{aligned} \tag{14}$$

Here, $\widetilde{\eta}$ and $\widetilde{\Delta^{A}}$ are the natural generalizations to $\mathbb{R}^{d+1}$ and $\mathbb{R}^{(d+1)\times(d+1)}$of $\eta$ and $\Delta^A$, respectively. $\mathfrak{o}(\varepsilon)$ is Landau's symbol.

**Theorem 3**

Let $1 \le p \le m$ , $0 \le \mu_0 < .. < \mu_p = m$ , $0 = t_0 < t_1 < ... < t_p$ and $\kappa$ a permutation of $\{1,..,m\}$ . Then we have

$$\begin{aligned} & \mathbb{P}\left(T_{\kappa(1)} = .. = T_{\kappa(\mu_0)} = 0 \text{ and for } i = 1,..,p : T_{\kappa(\mu_{i-1}+1)} = .. = T_{\kappa(\mu_i)} \in dt_i\right) \\ = \;& -\pi \cdot \Delta^{A(\mu_0)} \cdot \prod_{i=1}^{p-1} \left(e^{L\cdot(t_i - t_{i-1})} \cdot \Delta^{C(\mu_i)} \cdot L \cdot \Delta^{A(\mu_i)} dt_i\right) \left(e^{L\cdot(t_p - t_{p-1})} \cdot \Delta^{C(\mu_p)} \cdot L \cdot dt_p\right) \eta \; . \end{aligned} \tag{15}$$

where $A(\mu_0) = D \setminus \left(\bigcup_{j=1}^{\mu_0} C_{\kappa(j)}\right)$ and $C(\mu_i) = \bigcap_{j=\mu_{i-1}+1}^{\mu_i} C_{\kappa(j)}$ , $A(\mu_i) = D \setminus \left(\bigcup_{j=\mu_{i-1}+1}^{\mu_i} C_{\kappa(j)}\right)$, $i = 1,..,p$ .

**Proof:**

Again, let $\kappa$ be the identity. With $\varepsilon_0 = 0$ , $\varepsilon_\nu > 0$ and $\widetilde{A}(\mu_i) = \widetilde{D} \setminus \left(\bigcup_{j=\mu_{i-1}+1}^{\mu_i} C_{\kappa(j)}\right)$, we get

$$\begin{aligned} & \mathbb{P}\left(T_1 = .. = T_{\mu_0} = 0 \text{ and for } \nu = 1,..,p : T_{\mu_{\nu-1}+1} = .. = T_{\mu_\nu} \in (t_\nu, t_\nu + \varepsilon_\nu)\right) \\ = \;& \sum_{\substack{j_0 \in A(\mu_0) \\ k_\nu \in C(\mu_\nu)\ ,\ j_\nu \in \widetilde{A}(\mu_\nu),\ 1 \le \nu \le p}} \mathbb{P}\left(X(0) = j_0\right) \cdot \\ & \cdot \prod_{\nu=1}^{p} \left[\mathbb{P}\left( X(t_\nu) = k_\nu \middle| X(t_{\nu-1} + \varepsilon_{\nu-1}) = j_{\nu-1}\right) \mathbb{P}\left( X(t_\nu + \varepsilon_\nu) = j_\nu \middle| X(t_\nu) = k_\nu\right)\right] \\ = \;& \widetilde{\pi} \widetilde{\Delta^{A(\mu_0)}} \prod_{\nu=1}^{p} \left(e^{\widetilde{L}\cdot(t_\nu - t_{\nu-1} - \varepsilon_{\nu-1})} \widetilde{\Delta^{C(\mu_\nu)}} \cdot \widetilde{L} \cdot \widetilde{\Delta^{\widetilde{A}(\mu_\nu)}} \cdot \varepsilon_\nu + \mathfrak{o}(\varepsilon_\nu)\right) \cdot \widetilde{\eta} \\ = \;& \widetilde{\pi} \widetilde{\Delta^{A(\mu_0)}} \prod_{\nu=1}^{p-1} \left(e^{\widetilde{L}\cdot(t_\nu - t_{\nu-1} - \varepsilon_{\nu-1})} \widetilde{\Delta^{C(\mu_\nu)}} \cdot \widetilde{L} \cdot \widetilde{\Delta^{\widetilde{A}(\mu_\nu)}} \cdot \varepsilon_\nu + \mathfrak{o}(\varepsilon_\nu)\right) \cdot \\ & \left(e^{\widetilde{L}\cdot(t_p - t_{p-1} - \varepsilon_{i-1})} \widetilde{\Delta^{C(\mu_p)}} \cdot \widetilde{L} \cdot \widetilde{\Delta^{\widetilde{A}(\mu_p)}} \cdot \varepsilon_p + \mathfrak{o}(\varepsilon_p)\right) \cdot \widetilde{\eta} \end{aligned}$$

Again, we have $\{0\} = \bigcap_{\nu=1}^{p} \widetilde{A}(\mu_i)$ by (9) and by the fact that $\widetilde{L} \cdot \widetilde{\Delta^{\{0\}}} \cdot \widetilde{\eta} = \begin{pmatrix} 0 \\ -L \cdot \eta \end{pmatrix}$, we find

$$\begin{aligned} & \mathbb{P}\left(T_1 = .. = T_{\mu_0} = 0 \text{ and for } \nu = 1,..,p : T_{\mu_{\nu-1}+1} = .. = T_{\mu_\nu} \in (t_\nu, t_\nu + \varepsilon_\nu)\right) \\ = \;& -\pi \Delta^{A(\mu_0)} \prod_{\nu=1}^{p-1} \left(e^{L\cdot(t_\nu - t_{\nu-1} - \varepsilon_{\nu-1})} \Delta^{C(\mu_\nu)} \cdot L \cdot \Delta^{A(\mu_\nu)} \cdot \varepsilon_\nu + \mathfrak{o}(\varepsilon_\nu)\right) \\ & \left(e^{L\cdot(t_p - t_{p-1} - \varepsilon_{p-1})} \Delta^{C(\mu_p)} \cdot L \cdot \varepsilon_p + \mathfrak{o}(\varepsilon_p)\right) \cdot \eta \end{aligned}$$

Taking the limit $\varepsilon_\nu \to 0$ for $\nu = 1,..,p$ we get (15). ■

In the bivariate case, we find the following result:

**Corollary 4**

For $m = 2$ , $\mathcal{C} = \{C_1, C_2\}$ and $\nu = 1, 2$ , we have

(i)

$$\mathbb{P}\left(T_1 = T_2 = 0\right) = \pi_0 \; ; \tag{16}$$

(ii) for $0 \leq t$

$$\begin{aligned} \mathbb{P}(T_\nu = 0\ ,\ T_{3-\nu} > t) &= \pi \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot \eta \text{ and} \\ \mathbb{P}(T_\nu = 0\ ,\ T_{3-\nu} \in dt) &= -\pi \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot L \cdot \eta\ ; \end{aligned} \tag{17}$$

(iii) for $0 < t_1 < t_2$

$$\begin{aligned} \mathbb{P}(T_\nu > t_1\ ,\ T_{3-\nu} > t_2) &= \pi \cdot e^{L\cdot t_1} \cdot \Delta^{C_\nu} \cdot e^{L\cdot(t_2-t_1)} \cdot \Delta^{C_{3-\nu}} \cdot \eta \text{ and} \\ \mathbb{P}(T_\nu \in dt_1\ ,\ T_{3-\nu} \in dt_2) &= \pi \cdot e^{L\cdot t_1} \cdot \left(L \cdot \Delta^{C_\nu} - \Delta^{C_\nu} \cdot L\right) \cdot \\ &\quad \cdot e^{L\cdot(t_2-t_1)} \cdot L \cdot \eta \cdot dt_1 \cdot dt_2\ ; \end{aligned} \tag{18}$$

(iv) for $0 < t$

$$\begin{aligned} \mathbb{P}(T_1 = T_2 \in dt) &= -\pi \cdot e^{L\cdot t} \cdot \Delta^{C_1\cap C_2} \cdot L \cdot \eta \cdot dt \text{ and} \\ \mathbb{P}(T_1 = T_2 > 0) &= \pi \cdot L^{-1} \cdot \Delta^{C_1\cap C_2} \cdot L \cdot \eta\ . \end{aligned} \tag{19}$$

**Proof:**

(i) is evident. For the first equation in (ii), we have with (**??**)

$$\mathbb{P}(T_\nu = 0\ ,\ T_{3-\nu} > t) = \pi \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot \Delta^{C_{3-\nu}} \cdot \eta = \pi \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot \eta\ .$$

since

$$\Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} = \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot \Delta^{D\setminus C_\nu} = \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot \Delta^{C_{3-\nu}}\ . \tag{20}$$

For the second equation of (ii), (15) and (20) give us

$$\mathbb{P}(T_\nu = 0\ ,\ T_{3-\nu} \in dt) = -\pi \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot \Delta^{C_{3-\nu}} \cdot L \cdot \eta = -\pi \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot t} \cdot L \cdot \eta\ .$$

The first equality of (iii) is indentical with (13). By (15) we get

$$\begin{aligned} \mathbb{P}(T_\nu \in dt_1, T_{3-\nu} \in dt_2) &= -\pi e^{L\cdot t_1} \cdot \Delta^{C_\nu} L \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot(t_2-t_1)} \cdot \Delta^{C_{3-\nu}} L \cdot \eta \cdot dt_1 dt_2 \\ &\underset{(20)}{=} -\pi e^{L\cdot t_1} \cdot \Delta^{C_\nu} L \cdot \Delta^{D\setminus C_\nu} \cdot e^{L\cdot(t_2-t_1)} \cdot L \cdot \eta \cdot dt_1 dt_2 \\ &= -\pi e^{L\cdot t_1} \cdot \left(\Delta^{C_\nu} L - L \Delta^{C_\nu}\right) \cdot e^{L\cdot(t_2-t_1)} \cdot L \cdot \eta \cdot dt_1 dt,. \end{aligned}$$

since for an inaccessible set $C$

$$\begin{aligned} \Delta^C \cdot L \cdot \Delta^C &= L \cdot \Delta^C \text{ and therefore} \\ \Delta^C \cdot L \cdot \Delta^{D\setminus C} &= \Delta^C \cdot L - \Delta^C \cdot L \cdot \Delta^C = \Delta^C \cdot L - L \cdot \Delta^C\ . \end{aligned} \tag{21}$$

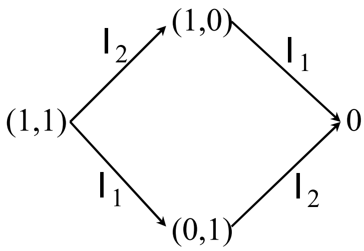
(iv) is an immediate consequence of (15). ■

**Example 1**

Let $Z_1$ and $Z_2$ be two exponential independent variables with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, abd define

$$T_1 = Z_1 \text{ and } T_2 = \max(Z_1, Z_2)\ . \tag{22}$$

Then, $(T_1, T_2)$ are two phase varibles. To see that we set $\widetilde{D} = \{0, (1,1), (1,0), (0,1)\}$ with transition intensities according to the diagram




which gives us the matrix

$$L = \begin{pmatrix} -\lambda_1 - \lambda_2 & \lambda_2 & \lambda_1 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{pmatrix} \tag{23}$$

and its exponentiel

$$e^{Lt} = \begin{pmatrix} e^{-(\lambda_1+\lambda_2)\cdot t} & e^{-\lambda_1 \cdot t} \cdot \left(1 - e^{-\lambda_2 \cdot t}\right) & e^{-\lambda_2 \cdot t} \cdot \left(1 - e^{-\lambda_1 \cdot t}\right) \\ 0 & e^{-\lambda_1 \cdot t} & 0 \\ 0 & 0 & e^{-\lambda_2 \cdot t} \end{pmatrix} , \tag{24}$$

We see that $C_1 = \{(1,1),(1,0)\}$ and $C_2 = D = \{(1,1),(1,0),(0,1)\}$ are inaccessible subsets of $\widetilde{D}$. For initial distribution, we choose $\widetilde{\pi} = (0,1,0,0)$. Now (18) gives us

$$\mathbb{P}(T_1 > t_1, T_2 > t_2) = \begin{cases} e^{-\lambda_1 t_2} + e^{-\lambda_2 t_2}\left(e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2}\right) & \text{if } 0 < t_1 < t_2 \\ e^{-\lambda_1 t_1} & \text{if } 0 < t_2 < t_1 \end{cases} \tag{25}$$

with marginal distributions for $T_1$ and $T_2$

$$\begin{aligned} \mathbb{P}(T_1 > t_1,) &= e^{-\lambda_1 t_1} \text{ and} \\ \mathbb{P}(T_2 > t_2) &= e^{-\lambda_1 t_2} + e^{-\lambda_2 t_2} - e^{-(\lambda_1+\lambda_2)t_2} \end{aligned}$$

or

$$\begin{aligned} F_1(t_1) &= \mathbb{P}(T_1 \le t_1) = 1 - e^{-\lambda_1 t_1} \text{ and} \\ F_2(t_2) &= \mathbb{P}(T_2 \le t_2\ ) = \left(1 - e^{-\lambda_1 t_2}\right) \cdot \left(1 - e^{-\lambda_2 t_2}\right) . \end{aligned}$$

For the densities, we get

$$\begin{aligned} \mathbb{P}(T_1 \in dt_1, T_2 \in dt_2) &= \begin{cases} \lambda_1 \cdot e^{-\lambda_1 t_1} \cdot \lambda_2 \cdot e^{-\lambda_2 t_2} \cdot dt_1 dt_2 & \text{if } 0 < t_1 < t_2 \\ 0 & \text{if } 0 < t_2 < t_1 \end{cases} \quad \text{and} \\ \mathbb{P}(T_1 = T_2 \in dt) &= \lambda_1 \cdot e^{-\lambda_1 t} \cdot \left(1 - e^{-\lambda_2 t}\right) \cdot dt . \end{aligned} \tag{26}$$

So we can calculate the commun distribution fonction

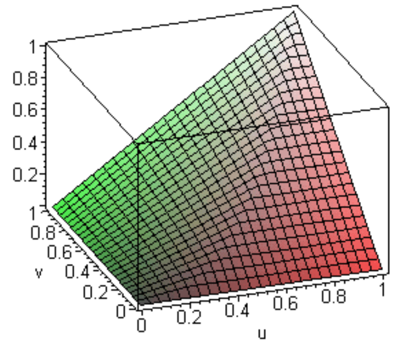
$$\begin{aligned} F(t_1,t_2) &= \mathbb{P}(T_1 \le t_1, T_2 \le t_2) = 1 - \mathbb{P}(T_1 > t_1) - \mathbb{P}(T_2 > t_2) + \mathbb{P}(T_1 > t_1, T_2 > t_2) \\ &= 1 - e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2} - e^{-\lambda_2 t_2} + e^{-(\lambda_1+\lambda_2)t_2} \\ &\quad + \left\{ \begin{array}{ll} e^{-\lambda_1 t_2} + e^{-\lambda_2 t_2}\left(e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2}\right) & \text{if } 0 < t_1 < t_2 \\ e^{-\lambda_1 t_1} & \text{if } 0 < t_2 < t_1 \end{array} \right\} \\ &= 1 - e^{-\lambda_1 (t_1 \wedge t_2)} - e^{-\lambda_2 t_2} + e^{-\lambda_1 (t_1 \wedge t_2) - \lambda_2 t_2} = \left(1 - e^{-\lambda_1 (t_1 \wedge t_2)}\right) \cdot \left(1 - e^{-\lambda_2 t_2}\right) \\ &= \left(\left(1 - e^{-\lambda_1 t_1}\right) \wedge \left(1 - e^{-\lambda_1 t_2}\right)\right) \cdot \left(1 - e^{-\lambda_2 t_2}\right) = \left(F_1(t_1) \cdot \left(1 - e^{-\lambda_2 t_2}\right)\right) \wedge F_2(t_2) . \end{aligned}$$

$F_2(t_2) = v$ is a strictly increasing function in $t_2$ with inverse $t_2 = F_2^{\leftarrow}(v)$. Thus we get the copula

$$C(u,v) = \left(u \cdot \left(1 - e^{-\lambda_2 \cdot F_2^{\leftarrow}(v)}\right)\right) \wedge v \ ,$$

which in the special case $\lambda_1 = \lambda_2$, gives us $\left(1 - e^{-\lambda_2 t_2}\right) = F_2(t_2)^{1/2} = v^{1/2}$ , hence

$$\begin{aligned} C(u,v) &= \left(u \cdot v^{1/2}\right) \wedge v \\ &= \begin{cases} u \cdot v^{1/2} & \text{if } u^2 \leq v \\ v & \text{else} \end{cases} \ . \end{aligned} \quad (27)$$




Copula $C(u,v) = \left(u \cdot v^{1/2}\right) \wedge v$

**Remark 1**

An other example is given in [8], where we find tail copulas of Marshall-Olkin type for bivariate phase distributions.

## 4 The EM Algorithm for Continuous Bivariate Phase Distributions

The EM algorithm is based on the following fact:

If $\xi$ is a random variable having with respect to some $\sigma$-finite measure $\mu(d\xi)$ a density $f_\chi(\xi)$ which depends on some parameter set $\chi$, and if $\widehat{T}$ is the realization of a statistic $T(\xi)$ , then the marginal density $g_\chi(T)$ given by

$$g_\chi(T) = \int f_\chi(\xi) \cdot \mu\left(d\xi \,|T(\xi) = T\right)$$

is increasing in the parameter set $\chi$ if we choose $\chi$ as the argument of the optimization problem of the conditional expectation of the log-likelihhood function:

$$\mathbb{E}_{\chi_0}\left(\ln f_\chi(\xi)|\, T(\xi) = \widehat{T}\right) \underset{\chi}{\longrightarrow} \ \max \ ! \quad (28)$$

This means that starting with some $\chi_0$ and taking $\chi_1$ as the argument of (28)

$$\chi_1 = \arg \ \max_{\chi} \ \mathbb{E}_{\chi_0}\left(\log f_\chi(\xi)|\, T(\xi) = \widehat{T}\right) \ . \quad (29)$$

then

$$g_{\chi_1}(\widehat{T}) \geq g_{\chi_0}(\widehat{T}) \ . \quad (30)$$

This is a consequence of an entropy inequality.

We apply this result to the Markov process $X(t)$ given the characteristics

$$\left(\widetilde{D}=\{0,..,d\},\widetilde{L},\widetilde{\pi},\mathcal{C}=\{C_1,C_2\}\right)\ . \tag{31}$$

where $C_\nu$ are inaccessible sets: $\emptyset\neq C_\nu\subseteq D,\ \nu=1,2$ , defining the bivariate phase variable $\mathbf{T}=(T_1,T_2)$. We may assume $\widetilde{D}=\{0\}\cup C_1\cup C_2$ with 0 as an absorbing and attracting state.

A trajectory of $X(t)$ may be given by $\xi=(j_1,t_1,j_2,t_2,..,j_m,t_m)$, where $j_1$ is the initial state, $t_1$ the time of $X(t)$ spent in $j_1$, then $X(t)$ jumps to $j_2$ and stays there a time $t_2$, and so on, until its last sejourn time $t_m$ in $j_m$ it finally jumps to 0. Of course, $j_1$, ..,$j_m\in D$. Under the parameter set $\chi=(L,\pi)$, the density of such a trajectory is

$$\begin{aligned} f_\chi(\xi) &= \pi_{j_1}\cdot(-l_{j_1j_1})\cdot\exp(t_1l_{j_1j_1})\cdot p(j_1,j_2)\cdot(-l_{j_2j_2})\cdot\exp(t_2l_{j_2j_2})\cdot. \\ &\quad ..\cdot(-l_{j_mj_m})\cdot\exp(t_ml_{j_mj_m})\cdot p(j_m,j_{m+1})\ . \end{aligned}$$

where we set $j_{m+1}=0$ and for $j\neq k$ , $p(j,k)=l_{jk}/\left(-l_{jj}\right)$ the conditional probability of a jump from $j$ to $k$ , given that a jump takes place. Therefore, the log-density function is

$$\begin{aligned} \ln\left(f_\chi(\xi)\right) &= \ln\left(\pi_{j_1}\right)+\sum_{\nu=1}^{m}t_\nu\cdot l_{j_\nu j_\nu}+\sum_{\nu=1}^{m}\ln\left(l_{j_\nu j_{\nu+1}}\right). \\ &= =\sum_{j=1}^{d}\ln\pi_j\cdot N_j+\sum_{j=1}^{d}l_{jj}\cdot Z_j+\sum_{j=1}^{d}\sum_{k=0\ (!),k\neq j}^{d}\ln l_{jk}\cdot N_{jk} \end{aligned}$$

where

$$N_j = \mathbf{1}_{\{j\}}(j_1), \tag{32}$$

$$N_{jk} = \#\{(j_\nu,j_{\nu+1})=(j,k),\ \text{avec}\ \nu\leq m\ \},\ \text{the number of jumps from } j \text{ to } k, \tag{33}$$

$$Z_j = \sum_{\nu=1}^{m}t_\nu\cdot\mathbf{1}_{\{j\}}(j_\nu)\ ,\ \text{the local time of } j\ . \tag{34}$$

The EM algorithm implies the maximization problem

$$\mathbb{E}_{\chi_0}\left(\sum_{j=1}^{d}\ln\pi_j\cdot N_j+\sum_{j=1}^{d}l_{jj}\cdot Z_j+\sum_{j=1}^{d}\sum_{k=0\ (!),k\neq j}^{d}\ln l_{jk}\cdot N_{jk}\,\middle|\,T=\widehat{T}\right)\underset{\chi=(L,\pi)}{\longrightarrow}\max\ ! \tag{35}$$

and the restrictions:

$$\sum_{j=1}^{d}\pi_j=1-\pi_0\ \text{and}\ l_{jj}=-\sum_{\substack{k=0\\k\neq j}}^{d}l_{jk}\ . \tag{36}$$

Now, let $\widehat{\mathbf{T}}_1,...,\widehat{\mathbf{T}}_n$ , where

$$\widehat{\mathbf{T}}_i=\left(\widehat{T}_{i1},\widehat{T}_{i2}\right)\ i\leq n\ ,$$

be realizations of independent bivariate phase varibles. We want to give a model of bivariate phase varibles of type (31) to these realizations $\left(\widehat{\mathbf{T}}_i\right)_{i\leq n}$ . First, we assume that the necessary condition

$$\bigcap_{i=1}^{n}C_1^{\lceil\widehat{T}_{i1}\rceil}\cap C_2^{\lceil\widehat{T}_{i2}\rceil}\neq\emptyset\ \text{where}\ C^{\lceil t\rceil}=\begin{cases}C & \text{if } t>0\\ \widetilde{D}\setminus C & \text{if } t=0\end{cases} \tag{37}$$

is realized and moreover

$$\widetilde{D}=\{0\}\ \dot{\cup}\ (C_1\cup C_2)\ . \tag{38}$$

Having fixed $\widetilde{D}$ and $C_\nu$, we want to optimize the parameters

$$\begin{aligned}\widetilde{\pi} &= (\pi_0, .., \pi_d) \ , \ \sum_{j=0}^{d} \pi_j = 1 \text{ and} \\ \widetilde{L} &= (l_{jk})_{j,k\in\widetilde{D}} \ , \ \sum_{k=0}^{d} l_{jk} = 0 \text{ for } j \in \widetilde{D} \end{aligned} \tag{39}$$

in such a way to get the optimal fit to our observations $\left(\widehat{\mathbf{T}}_i\right)_{i\leq n}$. Obviously, we should start by setting

$$\pi_0 = \frac{1}{n}\#\left\{i \leq n \ , \widehat{\mathbf{T}}_i = (0,0)\right\} \ , \tag{40}$$

such that we may assume lateron $\widehat{\mathbf{T}}_i \neq (0,0)$ , $i \leq n$ ( where for simplicity we do not change the number $n$). Starting with some initial parameters $\chi^{(0)} = (L^{(0)}, \pi^{(0)})$ satisfying (39), the EM algorithm boils down to (35) and (36) where the condition has to be replaced by $T_i = \widehat{T}_i$ , $i \leq n$. A simple calculation shows that the solution to this problem gives the following recursion procedure of the EM algorithm

$$\begin{aligned}\pi_j^{(\nu+1)} &= \pi_j^{(\nu)} \cdot \frac{1-\pi_0}{n} \sum_{i=1}^{n} \mathbb{E}_\chi\left(N_j | \left(\widehat{T}_{i1}, \widehat{T}_{i2}\right)\right) \text{ for } 1 \leq j \leq d \text{ and} \\ l_{jk}^{(\nu+1)} &= \frac{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)}, L^{(\nu)}\right)}\left(N_{jk} | \left(\widehat{T}_{i1}, \widehat{T}_{i2}\right)\right)}{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)}, L^{(\nu)}\right)}\left(Z_j | \left(\widehat{T}_{i1}, \widehat{T}_{i2}\right)\right)} \ , \\ l_{j0}^{(\nu+1)} &= \frac{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)}, L^{(\nu)}\right)}\left(N_{j0} | \left(\widehat{T}_{i1}, \widehat{T}_{i2}\right)\right)}{\sum_{i=1}^{n} \mathbb{E}_{\left(\pi^{(\nu)}, L^{(\nu)}\right)}\left(Z_j | \left(\widehat{T}_{i1}, \widehat{T}_{i2}\right)\right)} \ , \\ l_{jj}^{(\nu+1)} &= -\sum_{k=0\ ,k\neq j}^{d} l_{jk}^{(\nu)} \ . \end{aligned} \tag{41}$$

which means that we have to calculate the conditional expectations in (41). With the results (18) and (19) of corollary 4, we get the following series of equation according to the different cases.

(i) Case: $0 = \widehat{T}_\nu < \widehat{T}_{3-\nu}$ for $\nu = 1$ or 2 and $1 \leq j,k \leq d$

$$\begin{aligned}\mathbb{E}_\chi\left(N_j | \left(\widehat{T}_1, \widehat{T}_2\right)\right) &= \pi_j \begin{cases} \frac{\eta_j \cdot e^{L\cdot T_{3-\nu}} \cdot L \cdot \eta}{\pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot T_{3-\nu}}\cdot L\cdot\eta} & \text{if } j \in D\setminus C_\nu \\ 0 & \text{else} \end{cases} \ , \\ \mathbb{E}_\chi\left(N_{jk} | \left(\widehat{T}_1, \widehat{T}_2\right)\right) &= l_{jk} \begin{cases} \frac{\int_0^{\widehat{T}_{3-\nu}} \pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot u}\cdot\Delta(j,k)\cdot e^{L\cdot(\widehat{T}_{3-\nu}-u)}\cdot L\cdot\eta\cdot du}{\pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot\widehat{T}_{3-\nu}}\cdot L\cdot\eta} & \text{if } j,k \in C_{3-\nu} \\ 0 & \text{else} \end{cases} \ , \\ \mathbb{E}_\chi\left(N_{j0} | \left(\widehat{T}_1, \widehat{T}_2\right)\right) &= l_{j0} \begin{cases} \frac{\pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot\widehat{T}_{3-\nu}}\cdot L\cdot\eta_j}{\pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot\widehat{T}_{3-\nu}}\cdot L\cdot\eta} & \text{if } j \in C_{3-\nu} \\ 0 & \text{else} \end{cases} \ , \\ \mathbb{E}_\chi\left(Z_j | \left(\widehat{T}_1, \widehat{T}_2\right)\right) &= \begin{cases} \frac{\int_0^{\widehat{T}_{3-\nu}} \pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot u}\cdot\Delta(j,j)\cdot e^{L\cdot(\widehat{T}_{3-\nu}-u)}\cdot L\cdot\eta_j\cdot du}{\pi\cdot\Delta^{D\setminus C_\nu}\cdot e^{L\cdot\widehat{T}_{3-\nu}}\cdot L\cdot\eta} & \text{if } j \in C_{3-\nu} \\ 0 & \text{else} \end{cases} \ ; \end{aligned} \tag{42}$$

(ii) Case: $0 < \widehat{T}_\nu < \widehat{T}_{3-\nu}$ for $\nu = 1$ or 2 and $1 \le j,k \le d$

$$\mathbb{E}_\chi\left(N_j|\left(\widehat{T}_1,\widehat{T}_2\right)\right) = \pi_j \begin{cases} \frac{\eta_j\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_1\cap C_2 \\ 0 & \text{else} \end{cases},$$

$$\mathbb{E}_\chi\left(N_{jk}|\left(\widehat{T}_1,\widehat{T}_2\right)\right)$$

$$= l_{jk}\begin{cases} \frac{\int_0^{\widehat{T}_\nu}\pi\cdot e^{L\cdot u}\cdot\Delta(j,k)\cdot e^{L\cdot\left(\widehat{T}_\nu-u\right)}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j,k\in C_1\cap C_2 \\ \frac{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\Delta(j,k)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_\nu, k\in C_{3-\nu}\setminus C_\nu \\ \frac{\int_{\widehat{T}_\nu}^{\widehat{T}_{3-\nu}}\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(u-\widehat{T}_\nu\right)}\cdot\Delta(j,k)\cdot e^{L\cdot\left(\widehat{T}_{\nu-3}-u\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j,k\in C_{3-\nu}\setminus C_\nu \\ 0 & \text{else} \end{cases},$$

$$\mathbb{E}_\chi\left(N_{j0}|\left(\widehat{T}_1,\widehat{T}_2\right)\right) = l_{j0}\begin{cases} \frac{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta_j}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_{3-\nu}\setminus C_\nu \\ 0 & \text{else} \end{cases},$$

$$\mathbb{E}_\chi\left(Z_j|\left(\widehat{T}_1,\widehat{T}_2\right)\right)$$

$$= \begin{cases} \frac{\int_0^{\widehat{T}_\nu}\pi\cdot e^{L\cdot u}\cdot\Delta(j,j)\cdot e^{L\cdot\left(\widehat{T}_\nu-u\right)}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_1\cap C_2 \\ \frac{\int_{\widehat{T}_\nu}^{\widehat{T}_{3-\nu}}\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(u-\widehat{T}_\nu\right)}\cdot\Delta(j,j)\cdot e^{L\cdot\left(\widehat{T}_{\nu-3}-u\right)}\cdot L\cdot\eta\cdot du}{\pi\cdot e^{L\cdot\widehat{T}_\nu}\cdot\left(L\cdot\Delta^{C_\nu}-\Delta^{C_\nu}\cdot L\right)\cdot e^{L\cdot\left(\widehat{T}_{3-\nu}-\widehat{T}_\nu\right)}\cdot L\cdot\eta} & \text{if } j\in C_{3-\nu}\setminus C_\nu \\ 0 & \text{else} \end{cases}; \qquad (43)$$

(ii) Case: $0 < \widehat{T}_1 = \widehat{T}_2$ and $1 \le j,k \le d$

$$\mathbb{E}_\chi\left(N_j|\left(\widehat{T}_1,\widehat{T}_2\right)\right) = \pi_j\begin{cases} \frac{\eta_j\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot\widehat{T}_1}\cdot\eta}{\pi\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot\widehat{T}_1}\cdot\eta} & \text{if } j\in C_1\cap C_2 \\ 0 & \text{else} \end{cases},$$

$$\mathbb{E}_\chi\left(N_{jk}|\left(\widehat{T}_1,\widehat{T}_2\right)\right) = l_{jk}\begin{cases} \frac{\int_0^{\widehat{T}_1}\pi\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot u}\cdot\Delta(j,k)\cdot e^{L\cdot\left(\widehat{T}_1-u\right)}\cdot\eta\cdot du}{\pi\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot\widehat{T}_1}\cdot\eta} & \text{if } j,k\in C_1\cap C_2 \\ 0 & \text{else} \end{cases},$$

$$\mathbb{E}_\chi\left(N_{j0}|\left(\widehat{T}_1,\widehat{T}_2\right)\right) = l_{j0}\begin{cases} \frac{\pi\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot\widehat{T}_1}\cdot\eta_j}{\pi\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot\widehat{T}_1}\cdot\eta} & \text{if } j,k\in C_1\cap C_2 \\ 0 & \text{else} \end{cases},$$

$$\mathbb{E}_\chi\left(Z_j|\left(\widehat{T}_1,\widehat{T}_2\right)\right) = \begin{cases} \frac{\int_0^{\widehat{T}_1}\pi\cdot\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot u}\cdot\Delta(j,j)\cdot e^{L\cdot\left(\widehat{T}_1-u\right)}\cdot\eta\cdot du}{\pi\cdot\Delta^{C_1\cap C_2}\cdot e^{L\cdot\widehat{T}_1}\cdot\eta} & \text{if } j\in C_{3-\nu} \\ 0 & \text{else} \end{cases}. \qquad (44)$$

**Remark 2**

(i) In [2], it has been shown how the integrals in (42) to (44) can be calculated by a system of matrix-valued differential equations and a good numerical procedure such as Runge-Kutta to find the solutions.

(ii) Once the characteristics of a multivariate phase variable are estabished, it is easy to simulate such a varible, since it suffices to simulate the trajectory of the underlying Makov chain $X(t)$.

But for this, just a sequence of independent exponential variables, the waiting times, and discret transition varibles $Y_j$ with probabilities $\mathbb{P}(Y_j = k) = -l_{jk}/l_{jj}$ are needed.

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