On the financial risk factor in fair valuation of the mathematical provision*

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Abstract

The paper focuses on the financial variable for the provision evaluation and analyses the sensitivity of the fair valuation to interest rate parameters. In a determinist scenario the evaluation risk is studied in a market value perspective and its impact is measured through the sensitivity of the net value of the intermediation portfolio to a modification of the financial risk driver. In a stochastic scenario the sensitivity of the current values of projected liability cash-flow is investigated by means of a numerical implementation.

Key words and phrases: *Life insurance, financial risk, insolvency risk, mathematical provisions, financial regulation, Lee-Carter model.*
JEL classification: G22, G28, G13

1 Introduction

Life insurance business is traditionally characterised by a complex system of risks that, as known, can be split, at a minimal level, into two main types of drivers: actuarial and financial; they refer, within the pricing process, to the insurance company aptitude to select the "right" mortality table while the other concerns the ability to apply the right discounting process, where precision encompasses forecasting proficiency. Both the aspects can be regarded at the same time as risk drivers and value drivers, since they can give rise to a loss or a profit if the ex ante expected values prove to be higher or lower than the ex post actual realizations. These effects, with special reference to the intermediation portfolio (contingent claims versus correspondent asset), can be enhanced according to the specific accounting standard applied for the financial statement, that is to say that a fair valuation system can substantially modify the disclosure of the economic result and the solvency appraisal.

At the end of March 2004 the International Accounting Standards Board (IASB) issued International Financial Reporting Standard 4 Insurance Contracts (IFRS 4), providing, for the first time, guidance on accounting for insurance contracts, and marking the first step in the IASBs project to achieve the convergence of widely varying insurance industry accounting practices around the world, without imposing significant costs that could prove to be wasted when the more comprehensive project will be completed. More specifically, on the one hand the IFRS permits an insurer to change its accounting policies for insurance contracts only if, as a result, its financial statements present *information that is more relevant and no less reliable, or more reliable and no less relevant*; while on the other hand it permits the introduction of an accounting policy that involves *remeasuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions)*, thus giving rise to a potential reclassification of some or all financial assets as at fair value through profit or loss, when an insurer changes its accounting policies for insurance liabilities. The compromising solution derives on the one hand from the recognition of a fair value disclosure requirement, also to comply with a more general tendency concerning financial statements, and on the other

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hand from the equally widespread and deep worries concerning the lack of agreement upon a definition of fair value as well as of any guidance from the Board on how the fair value has to be calculated. According to the majority of commentators including the National Association of Insurance Commissioners, the International Association of Insurance Supervisors and the Basle Committee on Banking Supervision, this uncertainty may lead to fair value disclosures that are unreliable and inconsistently measured among insurance entities. As also the American Academy of Actuaries (2003) clearly states, market valuations do not exist for many items on the insurance balance sheet and this would lead to the reliance on entity specific measurement for determining insurance contract and asset fair values. However, such values would be unreasonably subject to wide ranges of judgment, be subject to significant abuse, and may provide information that is not at all comparable among companies. The cause for this concern is the risk margin component of the fair value. Risk margins are clearly a part of market values for uncertain assets and liabilities, but with respect to many insurance contracts their value cannot be reliably calibrated to the market. Hence, a market-based valuation basis for them would produce irrelevant information.

This statement gives rise to a wider trouble. If there is a change in the evaluation criteria for the reserve from one year to another according to current market yields or even to current mortality tables, there is the possibility of a modification of change in the value of the reserve according to the application of a more stringent or, at the opposite, a more flexible criterion.

In the accounting perspective, the introduction of an accounting policy involving remeasuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions) implies that the fair value of the mathematical provision is properly a *current value or a net present value*. Consistently, the fair value of the mathematical provision could be properly defined as the net present value of the residual debt towards the policyholders evaluated at current interest rates. As a consequence, an insurer is permitted, but not required, to change its accounting policies so that it remeasures designated insurance liabilities to reflect current market interest rates and recognises changes in those liabilities in profit or loss. At that time, it may also introduce accounting policies that require other current estimates and assumptions for the designated liabilities. Therefore, there the fair value of the pure mathematical provision could be more properly defined as net present value of the residual debt towards the policyholders evaluated at *current interest rates and mortality rates*. In a sense, this is marking to market. However, this is the crucial point. The International Accounting Standards Committee (IASC) in the Insurance Issues Paper (November 1999) defines the fair value as *the amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arms-length transaction*, while the Financial Accounting Standard Board (FASB) in a Preliminary Views document (December, 1999) defines the fair value as *an estimate of the price an entity would have realized if it had sold an asset or paid if it had been relieved of a liability on the reporting date in an arms-length exchange motivated by normal business considerations. That is, it is an estimate of an exit price determined by market interactions*. Hence, the CAS Fair Value Task Force defined fair value as *the market value, if a sufficiently active market exists, or an estimated market value, otherwise.*

In a "mathematical" perspective, marking to market implies the discovery of a price and more specifically of an equilibrium price, while in the accounting perspective the current value is not necessarily an equilibrium price, but merely a market price. Many problems and misunderstanding arise from this "cultural" discrepancy: the majority of the actuarial models are very difficult to implement on a large scale in a fair value accounting and give rise to a high space/time variability, while the issue is the comparability of the results across different companies. In a sense, there is a proper *fair valuation risk*, which can be defined as the change in the economic results due to a change in the evaluation criteria adopted for insurance liabilities. This *evaluation risk* will produce a twofold impact on the portfolio performance: a current earning impact, that can be measured trough the variability of the net income from year to year according to changes in relevant variables ([Cocozza et al., 2004b]), and a market value impact, that can be measured trough the sensitivity of the net value of the intermediation portfolio to a modification of specific risk drivers ([Cocozza et al., 2004c]). As a consequence, there is a need for a "measure of variability". The paper focuses on such aspect and tries to model the effect of such variations in the case of a non homogeneous portfolio. Therefore, it focuses on financial variables for the provision evaluation and tries to address to question of the sensitivity of the fair valuation to interest rates parameters, in both a deterministic and stochastic scenario.

2 The basic model

As showed elsewhere ([Cocozza et al., 2004a]), the net value of the intermediation portfolio in any t-year (K_t) can essentially be represented as the algebraic sum of: insurer capital at the end of preceding year (plus); premiums written during the year (plus); return accrued on the final reserve of preceding year and on the initial capital (plus/minus); the claims paid during the year (minus); the provision at the end of the year (minus).

For an immediate unitary annuity the following relation holds:

$$K_t = \left[K_{t-1} + N^x(t-1)\sum_{r=1}^{n-t-1} {}_rp_{x+t-1}e^{-r\delta_{L_{t-1}}}\right]e^{\delta_{A_{t-1}}} - N^x(t)\left[1+\sum_{r=1}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}}\right] \tag{1}$$

where

n = duration of the policy

x = initial age of the insured

c = number of policies sold (cohort constituents)

$N^x(r)$ = actual number of survivors at age $x+r$

${}_rp_x$ = mortality table applied in the reserve evaluation at the end of the r-year

δ_{L_r} = interest rate applied in the reserve evaluation at the end of the r-year

δ_{A_r} = instantaneous rate of return on assets earned during the r-year

Similarly, using the same symbols, for a deferred unitary annuity with periodical premiums (P) the following formulas hold respectively before and after the maturity of the deferral period (d):

$$K_{t|t<d} = \left[K_{t-1} + N^x(t-1)\left(\sum_{r=d-t-1}^{n-t-1} {}_rp_{x+t-1}e^{-r\delta_{L_{t-1}}} - P\sum_{r=0}^{d-t-1} {}_rp_{x+t}e^{-r\delta_{L_{t-1}}}\right)\right]e^{\delta_{A_{t-1}}} + \tag{2}$$

$$-N^x(t)\left[\sum_{r=d-t}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}} - P\sum_{r=0}^{d-t} {}_rp_{x+t}e^{-r\delta_{L_t}}\right]$$

$$K_{t|t\geq d} = \left[K_{t-1} + N^x(t-1)\left(\sum_{r=0}^{n-t-1} {}_rp_{x+t-1}e^{-r\delta_{L_{t-1}}} - 1\right)\right]e^{\delta_{A_{t-1}}} + \tag{3}$$

$$-N^x(t)\left[\sum_{r=0}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}}\right]$$

from which it can be easily inferred that the net value of the intermediation portfolio depends not only on the insurer initial capital K_{t-1}, on the actual number of contracts existing at the end of the t-year ($N^x(t)$) and on the return on assets earned in the year (δ_{A_t}), but also on the table adopted for provision evaluation (${}_rp_{x+t}$)), as well as on the rate selected for the evaluation δ_{L_r}. The net value, in all cases, is a growing function of the two interest rates while is inversely affected by an increase in the actual number of survivors at the end of the year. Consistently, the choice at the end of the year of a mortality table different from that applied in any previous evaluation (both for pricing and reserving) produces a difference of the net value which is opposite with respect to the "sign" to the variant of the table itself. As a matter of fact, the choice for example of a more prudential table, showing higher probability of surviving, will give rise to an increase in the present value of future net outflows and therefore a decrease in the net value, while the selection of a table showing lower probability will end up in an increase in the same value. As far as the intensity of this impact is concerned, it is easy to verify that is filtered by the size of the portfolio under observation and by its implicit financial discounting process.

This modelling gives the opportunity to measure directly the impact of a change in the evaluation rate (evaluation rate risk). In this context, the measurement can be easily obtained through the first derivative with respect to the rate applied for the evaluation of the mathematical provision (R_t), that is

$$R_t = N^x(t)\sum_{r=1}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}} \tag{4}$$

$$\frac{\partial R_t}{\partial \delta_{L_t}} = -N^x(t)\sum_{r=1}^{n-t} r\,{}_rp_{x+t}e^{-r\delta_{L_t}} \tag{5}$$

$$R_{t|t<d} = N^x(t)\left[\sum_{r=d-t}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}} - P\sum_{r=0}^{d-t} {}_rp_{x+t}e^{-r\delta_{L_t}}\right] \tag{6}$$

$$\frac{\partial R_{t|t<d}}{\partial \delta_{L_t}} = -N^x(t)\left[\sum_{r=d-t}^{n-t} r\,{}_rp_{x+t}e^{-r\delta_{L_t}} + P\sum_{r=0}^{d-t} r\,{}_rp_{x+t}e^{-r\delta_{L_t}}\right] \tag{7}$$

$$R_{t|t\geq d} = N^x(t)\sum_{r=0}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}} \tag{8}$$

$$\frac{\partial R_{t|t\geq d}}{\partial \delta_{L_t}} = -N^x(t)\sum_{r=0}^{n-t} r\,{}_rp_{x+t}e^{-r\delta_{L_t}} \tag{9}$$

It is easy to demonstrate that the dollar change in the value of the reserve is directly proportional to the Macaulay duration of the mathematical provision D_{Rt} since

$$\frac{\partial R_t}{\partial \delta_{L_t}}\frac{1}{R_t} = -\frac{N^x(t)\sum_{r=1}^{n-t} r\,{}_rp_{x+t}e^{-r\delta_{L_t}}}{N^x(t)\sum_{r=1}^{n-t} {}_rp_{x+t}e^{-r\delta_{L_t}}} = -D_{R_t} \quad \Rightarrow \quad \Delta R_t \simeq -R_t D_{R_t}\Delta\delta_{L_t} \tag{10}$$

Likewise, the impact of a change in the evaluation rate for a mix of provisions, that can be also a non homogeneous, can be measured through weighted average of the duration of the single portfolio components. For example, in the case of a portfolio with two homogeneous components, the reserve for the whole portfolio (PR) will exhibit the following sensitivity

$$\frac{\partial PR_t}{\partial \delta_{L_t}}\frac{1}{PR_t} = -\left(D_{R_t}\frac{R_t}{PR_t} + D_{R_t|t<d}\frac{R_{t|t<d}}{PR_t}\right) \quad \Rightarrow \quad \Delta PR_t \simeq -(R_tD_{R_t} + R_{t|t<d}D_{R_t|t<d})\Delta\delta_{L_t} \tag{11}$$

Formulations (10) and (11) give the opportunity to state that the evaluation rate risk can be effectively indexed to the Macaulay duration of the reserve and to the value of the reserve (cf. table 1). Moreover, it can be measured through the value-at-risk of the reserve, after the selection of a measure of interest rates volatility. Therefore, any change in the interest rates parameters is able to alter more steadily those reserves which exhibit higher absolute values and higher duration values (sensitivity indexes), as can be immediately appreciated by the last row of Table 1 the VaR at issue time for the whole portfolio and the individual components.

3 Numerical evidence for the basic model

We refer, as an exemplification, to a non-homogeneous life annuity portfolio composed as follows:

•100 immediate 10-year temporary life annuities, with unitary annual payments, each one issued to a male insured aged 40;

•100 6-year temporary life annuities, 3-year deferred, each one issued to a male insured aged 40, with periodic level premiums, paid at the beginning of the first three years;

• 80 immediate 8-year temporary unitary life annuities, each one issued to a male insured aged 50.

For the sake of clarity it is supposed that all premiums are earned exactly at the beginning of the year as well as the payments concerning the deferred annuity, while the claims on the immediate annuity are paid exactly at the end of the period. Calculations have been performed with both a fixed rate of 4% and with a term structure (CIR Model Table 2). The survival probabilities are deduced by the Italian Mortality Table called RG48. The results are contained in Table 1.

Table 1 – Value and Duration

	Fixed 4% RG48		10 years annuity		8 years annuity		deferred annuity	
year	Portfolio	Duration	Value	Duration	Value	Duration	Value	Duration
0	1.506,46	5,69	806,08	5,17	533,33	4,28	167,05	12,71
1	1.386,97	5,30	738,41	4,73	474,82	3,83	173,73	11,71
2	1.436,53	4,19	668,14	4,29	414,14	3,38	354,25	4,97
3	1.488,19	3,09	595,16	3,84	351,20	2,92	541,82	2,38
4	1.265,12	2,64	519,39	3,38	285,93	2,45	459,81	1,92
5	1.033,59	2,19	440,70	2,92	218,25	1,97	374,63	1,45
6	793,29	1,74	359,02	2,45	148,09	1,49	286,18	0,97
7	543,93	1,31	274,22	1,97	75,37	1,00	194,34	0,49
8	285,19	0,97	186,20	1,49	0,00	0,00	98,99	0,00
9	94,84	1,00	94,84	1,00	0,00	0,00	0,00	0,00
10	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00

	CIR RG48		10 years annuity		8 years annuity		deferred annuity	
year	Portfolio	Duration	Value	Duration	Value	Duration	Value	Duration
0	1.500,82	5,63	804,13	5,11	535,46	4,25	161,23	13,03
1	1.358,30	5,27	724,17	4,68	468,89	3,80	165,23	12,03
2	1.400,42	4,17	650,33	4,24	406,04	3,35	344,04	4,99
3	1.454,45	3,06	578,23	3,80	343,76	2,90	532,45	2,36
4	1.238,33	2,62	505,31	3,36	280,30	2,44	452,71	1,90
5	1.014,84	2,17	430,12	2,90	214,65	1,97	370,08	1,44
6	781,85	1,73	351,79	2,44	146,23	1,49	283,83	0,97
7	538,96	1,30	270,28	1,97	74,97	1,00	193,71	0,49
8	283,67	0,97	184,68	1,49	0,00	0,00	98,99	0,00
9	94,12	1,00	94,12	1,00	0,00	0,00	0,00	0,00
10	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00

VaR of the mathematical provision at issue time (c.i. 99%)

	Portfolio	Duration	Value	Duration	Value	Duration	Value	Duration
VaR	51,47	5,63	27,41	5,11	12,79	4,25	10,42	13,03

Table 2 CIR Model

r_0 *starting value*	γ *long term mean*	ϕ *initial value*	σ_α *volatiliy coeff. (discrete case)*	κ *drift coff. (discrete case)*	σ *diffusion coeff.*
1,72%	4,52%	97,40%	0,52%	2,63%	0,53%

Time	0	1	2	3	4	5	6	7	8	9	10
$E[r_t]$	1,72%	2,48%	3,03%	3,43%	3,73%	3,94%	4,10%	4,21%	4,29%	4,35%	4,40%
$\sigma[r_t]$	0,00%	0,23%	0,31%	0,37%	0,40%	0,43%	0,44%	0,46%	0,47%	0,47%	0,48%

4 The cash-flow analysis

Within a fair valuation framework, we take into account the financial risk and the demographic risk, where the last one refers to the unsystematic phenomenon of mortality (i.e. the random deviations of the mortality rates from their anticipated values).

To this aim, we introduce two probability spaces $(\Omega, \mathfrak{F}', P')$, $(\Omega, \mathfrak{F}'', P'')$, denoting by $\mathfrak{F}$' and $\mathfrak{F}$" the σ-algebras containing, respectively, the *financial events* and the *life duration events*. Considering the randomness in mortality independent on the fluctuations of interest rates, let us introduce the probability space $(\Omega, \mathfrak{F}, P)$ generated by the preceding two by means of canonical procedures; $\mathfrak{F}$ contains the information flow about mortality and financial history, represented by the filtration $\{\mathfrak{F}_k\} \subset \mathfrak{F}$, with $\mathfrak{F}_k = \mathfrak{F}'_k \cup \mathfrak{F}''_k$ and $\{\mathfrak{F}'_k\} \subset \mathfrak{F}'$, $\{\mathfrak{F}_k\}'' \subset \mathfrak{F}''$.

As well known, in a *fair valuation* framework, the hypothesis on the market consist in frictionless, continuous trading , no restrictions on borrowing or short-sales, the zero-bonds and the stocks are both infinitely divisible. Considering the cash-flow $\{X_j\}$ connected to a portofolio of c policies, in a risk-neutral valuation, the fair value at time t ([Coppola et al, 2005]) is given by

$$\mathfrak{V}_t = \mathbb{E}\left[\sum_{j>t} c\mathbf{1}_{\{K_{x,t}>j\}} X_j v(t,j) | \mathfrak{F}_t\right] \tag{12}$$

where $\mathbb{E}$ represents the expectation under the risk-neutral probability measure, whose existence derives by well known results based on the completeness of the market, and the indicator function $\mathbf{1}_{\{K_{x,t}>j\}}$ is connected to death or survival, according to the specific kind of contract (i.e. for a life annuity contract it takes the value 1 if the curtate future lifetime of the insured, aged x at issue, takes values greater than $t+j$ $(j = 1, 2, \dots)$, that is if the insured aged $x+t$ survives up to the time $t+j$, 0 otherwise. On the contrary, it takes the complementary values in the case of life insurance, such as whole life and term insurance). In particular for a portfolio of identical immediate life annuities, formula (12) simply provides (cf. [Coppola *et al.*, 2005])

$$\mathfrak{V}_t = \sum_{j>t} cX_j \; {}_tp_x \; {}_jp_{x+t} \mathbb{E}[v(t,j)|\mathfrak{F}_t] \tag{13}$$

Considering a non-homogeneous portfolio of life annuities, the fair valuation procedure can be easily extended dividing the porfolio in homogeneous sub-portfolios, say m their number, identified by common characteristics, such as age at issue, policy duration, and so on. We introduce the following notations:

- n_i = policy duration for the i-th group
- c_i= number of policies in the i-th group $(\sum_{i=1}^{n} c_i = c)$
- x_i = age at issue of the insureds of the i-th group
- T_i = deferment period of the annuities in the i-th group $(0 \le T_i < n_i)$
- $B_{i,s}$=benefit payable to each insured of the i-th group at time s,
- $P_{i,s}$=premium payable by each insured of the i-th group at time s,
- n =$\max_i n_i$
- $X_{i,s}$= the flow at time s related to each insured of the i-th group, with

$$X_{i,s} = \begin{cases} -P_{i,s} & \text{if } s < T_i \\ B_{i,s} & \text{if } s \ge T_i \end{cases}$$

Taking into account formula (12), we get

$$\begin{aligned} \mathfrak{V}_t =& \mathbb{E}\left[\sum_{i=1}^{m}\sum_{j>t} X_{i,j} c_i \mathbf{1}_{\{K_{x_i,t}>j\}} v(t,j)|\mathfrak{F}_t\right] = \\ &= \sum_{i=1}^{m}\sum_{j>t} c_i X_{i,j} \; {}_tp_{x_i} \; {}_jp_{x_i+t} \mathbb{E}[v(t,j)|\mathfrak{F}_t] \end{aligned} \tag{14}$$

with obvious meaning of the symbol $\mathfrak{F}_t$ in this context (cf. also [Coppola *et al.*, 2005]).

5 Background hypotheses for the fair valuation

In this section we introduce a valuation basis consisting in demographic and financial scenarios, where the calculus can be framed.

5.1 The demographic framework: recalls and empyrical testing on the Lee-Carter model

We will describe the mortality phenomenon by means of the Lee- Carter model (cf [Lee and Carter, 1992]): this model, not computationally difficult in implementing, represents a further step in the betterment of the survival representation and forecasting, compared with the traditional models commonly used in the portfolio valuation framework. It presents important features, both for practitioners and for who, in different fields, develops models in order to describe and to quantify the longevity phenomenon in the long and in the medium/short run.

The mechanism for generating the parameters of the model is inside the model itself, so reducing the influence of personal judgments. It generates a stream of parameters different for every age and year included in the analysis, this flexibility allowing to follow, age by age and year by year, every mortality trend taking place. The model has the important skill to correct itself by introducing in it the new data as they come, in this way being capable to catch the changes in the trend. The model furnishes excellent representations of the mortality phenomenon for time intervals of 8-10 years. It must be noted that for longer time horizons it shows both a light tendency to overestimate the mortality, in particular for the elderly population, and that the spread between the actual mortality rate and the estimated one tends to increase when the time horizon increases (cf. [De Feo, 2004], page 69-91).

As known, the Lee-Carter model is a two dimensional model in which each parameter brings different information about the phenomenon; the base equation is (cf. [Lee and Carter,1992]):

$$m_{x,t} = e^{a_x + k_t b_x + e_{x,t}} \tag{15}$$

where:

- $m_{x,t}$ is the central death rate calculated for an insured with age x at time t
- a_x is the simple average of $lnm_{x,t}$ along the whole observation period. It describes the averaged behaviour of the central death rate for every age x
- k_t is the mortality index. It shows for all ages how the mortality phenomenon has evolved over the past death rate for each age x
- b_x is the sensibility parameter. For every age it explains how $lnm_{x,t}$ reacts when time passes
- $e_{x,t}$ represents that part of the mortality which is not caught by the model, with mean equal to zero and variance $\sigma_e < \infty$.

As showed in [Lee and Carter, 1992], [Renshaw and Haberman, 2000], and in Appendix, the model ARIMA (0,1,0) well describes the behaviour of the time index k_t expressed by the following:

$$k_t = k_{t-1} - c + e_t \tag{16}$$

where:

- c is the ratio between the overall decrement of k_t over the whole observation period and the number of periods where the decrement happened
- e_t is the error factor at time t

As known, the Lee-Carter method can be used for generating families of reduction factors, in this way having the advantage to increase the model usability and the accuracy of the forecasts. Fixing an appropriate year 0 (consistent with recent data), being RF the reduction factor, by definition we have:

$$\frac{m_{x,t}}{m_{x,0}} = RF(x,t). \tag{17}$$

Knowing the parameters of the model, it is easy and quick to obtain forecasts of $m_{x,t}$ for every age at every time simply multiplying the value at time zero by the appropriate factor. In order to make a correct choice of the year 0, we could use the last available table or what we think is the most relevant one or we could work out as an average over a certain group of available tables (cf [Renshaw and Haberman, 2000]). Concerning the latter, we can write:

$$a_{x,0} = \ln \prod_{t=t_m}^{t_n} m_{x,t}^{\frac{1}{g}} \tag{18}$$

where $t_n - t_m = g$, the year zero is $t_m + \left[\frac{g}{2}\right]$ and $\left[\frac{g}{2}\right]$ is the integer part of the ratio.

On the basis of what said before and the definition of the central death rate, the survival probabilities of an insured aged x at issue and alive in $x+t$ survives after j years can be written as follows:

$$\{{}_jp_{x,t}\}_{t=0,1,\dots,j-1} = \prod_{g=0}^{j-1} \left(1 - \frac{2e^{a_{x+g,0}+k'_{t+g}b_{x+g}}}{2+e^{a_{x+g,0}+k'_{t+g}b_{x+g}}}\right) \tag{19}$$

5.2 The financial framework: notation and calculus basis

The term structure we will use in the following, in order to frame the fair valuation, is expressed by means of the Cox-Ingersoll-Ross square root model; it is possible to estimate its parameters on the basis of a simple discretisation (cf. [Chan *et al.*, 1992] and [Deelstra *et al.*, 1995]), in which the continuous centred interest rate involves the stochastic differential equation

$$dr_t = -kr_t dt + s\sqrt{r_t+\gamma}\, dB_t$$

where k and s are positive constants, γ is the long term mean and B_t is a Brownian motion. The time series related to the interest rates has been extracted from Bank of Italy official statistics and consists of annualised net interest rate of Government 3-month T-Bill rate covering the period from January 1996 until January 2004.

For the specific discretization and estimation procedure here, one can refer to [Cocozza *et al.*, 2004].

For sake of clarity, we stress that, following [Deelstra *et al.*, 1995], we use a proxy of the expected value of the stochastic discounted factor involving a CIR process given by

$$E\left[e^{\int_0^t r_u du}\right] = \frac{e^{-\frac{x}{s^2}w\frac{1+\frac{k}{w}Coth(\frac{wt}{2})}{coth(\frac{wt}{2})+\frac{k}{w}}+\frac{xk}{s^2}+\frac{k^2\gamma t}{s^2}}}{\left[\cosh(\frac{wt}{2})+\frac{k}{w}\sinh(\frac{wt}{2})\right]^{\frac{2K\gamma}{s^2}}}$$

where

$$x = r_0, \qquad w = \sqrt{k^2+2s^2}.$$

6 Numerical evidence for the continuous case

We refer to the same portfolio considered in §3, framed into the financial and demographic scenario depicted in §5.

In Table 1 we report the fair values of the reserve at time t ($t = 0, 1, 2, \dots, 10$) for the entire portfolio.

We can observe that the fair values increase with t in the interval (0,3), and then decrease, therefore strongly showing the effect of the deferred life annuity sub-portfolio, whose behavioural pattern is kept in the reserve temporal profile.

For sake of completeness, we remark that in $t = 0$ the reserves of the first and the third sub-portfolios are calculated as "initial reserves", i.e. considered in $t = 0^+$, whilst for the deferred life annuities the premium paid at the beginning of the first year is considered.

6.1 Current values of projected liability cash-flow: a sensitivity analysis

In a solvency perspective, it is interesting to analyse the reserve fair values for a contract characterised by a long duration, such as an immediate whole life annuity. At first we consider this kind of policy for a male insured aged 40 at issue: in Figure 1 the reserve fair value is represented as function of the valuation time t and the interest rate volatility s. We note that the fair value decreases when t increases, as it is obvious to suppose, whilst it increases with s; this last phenomenon is more emphasized as the

Table 3
Non-homogeneous portfolio

year	Current values of reserves
0	1277,821
1	1374,9655
2	1482,6912
3	1599,154
4	1405,206
5	1200,666
6	985,018
7	757,719
8	518,194
9	190,1108
10	0

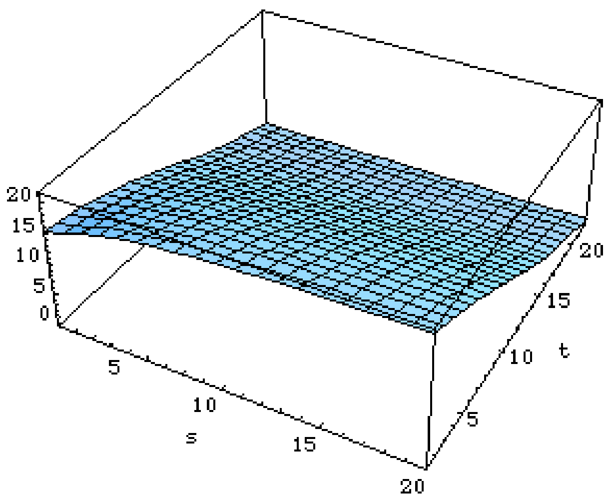

Figure 1: Whole life annuity: reserve current values

valuation time is far from the policy end. Surely this is a consequence of the prepondence of the interest rate volatility on the reserve itself in long periods, and it is confirmed by Figure 2, which illustrates the behaviour of the "elasticity" of the reserve fair value with respect to s. It is also evident that for large values of t the function value smoothly tends to 0.

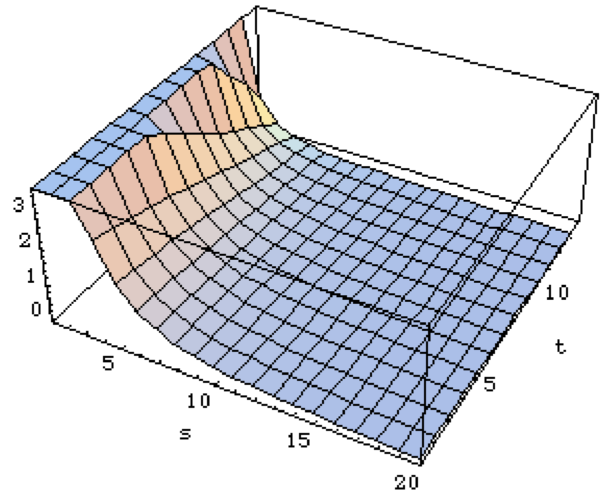

Figure 2: Whole life annuity: elasticity with respect to σ

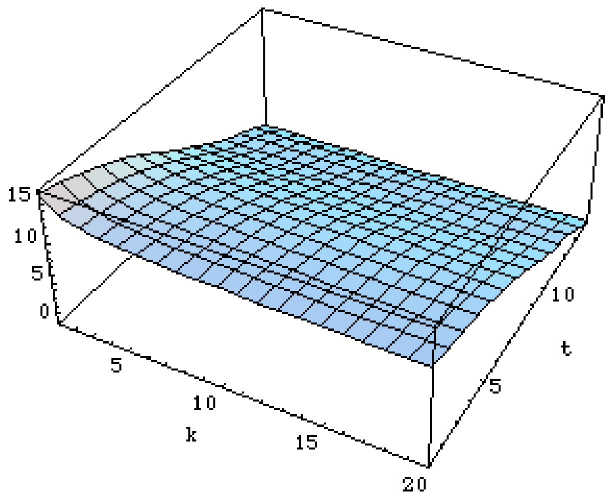

Figure 3: Whole life annuity: dependence on k

Moreover Figure 3 shows the reserve fair value as function of the time valuation t and the drift parameter k of the interest rate process. The fair value decreases when k increases, since for high value of k the interest rate is forced towards the long term mean $\gamma = 4,50\%$, and for this reason the present values of the future cash-flows decrease.

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7 Appendix: Estimation of the parameters of the Lee-Carter model

On the basis of the data available on the website www.mortality.org (Berkeley University of California), we refer to the overall Italian population from 1947 to 1999 and from age 0 to 109, where 109 is the limiting age.

Data are ordered year by year and age by age. In order to make forecasts, in the Table A.1 the estimated parameters of the L-C model are reported.

The parameter a_x was calculated following (18), having chosen the last four tables available and 1997 as the year zero. Following [Lee and Carter, 1992], we assume that b_x is the same from age 80 to 109.

Finally,k_t' is the difference between the time index forecasted at time t, with $2000 \leq t \leq 2065$, and the value of the time index on the year 1997. The Graph A.1 shows the value of the time index for Italian data from 1947 to 1999 and forecasts from 2000 to 2065 with the relative 95% confidence interval. In this case, too, they concern the overall population from age 0 to 109. The data are divided age by age and year by year.

Table A.1

Year	k'$_t$		Age	$a_{x,0}$	b_x
2000	-10,6703		0	-5,20272	0,026152
2001	-13,0598		1	-7,9378	0,036513
2002	-15,4492		2	-8,39096	0,029475
2003	-17,8387		3	-8,48265	0,025706
2004	-20,2281		4	-8,70828	0,024845
2005	-22,6176		5	-8,81623	0,022741
2006	-25,007		6	-8,82286	0,021298
2007	-27,3965		7	-8,86963	0,02065
2008	-29,786		8	-9,08837	0,020129
2009	-32,1754		9	-8,97217	0,019306
2010	-34,5649		10	-9,02635	0,019121
2011	-36,9543		11	-8,82312	0,01783
2012	-39,3438		12	-8,79847	0,017596
2013	-41,7332		13	-8,65398	0,016409
2014	-44,1227		14	-8,27029	0,013618
2015	-46,5121		15	-8,05735	0,012482
2016	-48,9016		16	-7,90558	0,010566
2017	-51,2911		17	-7,74092	0,009815
2018	-53,6805		18	-7,49726	0,009065
2019	-56,07		19	-7,4743	0,009658
2020	-58,4594		20	-7,43381	0,009617
2021	-60,8489		21	-7,39562	0,009334
2022	-63,2383		22	-7,35037	0,009636
2023	-65,6278		23	-7,32706	0,010168
2024	-68,0172		24	-7,33277	0,010267
2025	-70,4067		25	-7,32943	0,010342
2026	-72,7962		26	-7,34281	0,010403
2027	-75,1856		27	-7,30698	0,010329
2028	-77,5751		28	-7,24483	0,009925
2029	-79,9645		29	-7,24761	0,009723
2030	-82,354		30	-7,16389	0,009578
2031	-84,7434		31	-7,11425	0,009535
2032	-87,1329		32	-7,07822	0,009618
2033	-89,5223		33	-6,99265	0,00943
2034	-91,9118		34	-6,95989	0,009612
2035	-94,3013		35	-6,93174	0,009684
2036	-96,6907		36	-6,89803	0,00998
2037	-99,0802		37	-6,82642	0,009966
2038	-101,47		38	-6,77799	0,009765
2039	-103,859		39	-6,77999	0,009973
2040	-106,249		40	-6,68956	0,00988
2041	-108,638		41	-6,61333	0,009634
2042	-111,027		42	-6,57635	0,009762
2043	-113,417		43	-6,48516	0,009508
2044	-115,806		44	-6,37885	0,009225
2045	-118,196		45	-6,31081	0,00928
2046	-120,585		46	-6,20363	0,00892
2047	-122,975		47	-6,11346	0,008638
2048	-125,364		48	-6,02841	0,008503
2049	-127,754		49	-5,93101	0,008327
2050	-130,143		50	-5,82165	0,00812

2051	-132,533
2052	-134,922
2053	-137,311
2054	-139,701
2055	-142,09
2056	-144,48
2057	-146,869
2058	-149,259
2059	-151,648
2060	-154,038
2061	-156,427
2062	-158,817
2063	-161,206
2064	-163,595
2065	-165,985

51	-5,73052	0,007909
52	-5,60388	0,007645
53	-5,51063	0,007389
54	-5,42392	0,007274
55	-5,31822	0,006894
56	-5,24418	0,006838
57	-5,16785	0,00671
58	-5,06167	0,006559
59	-4,96177	0,006429
60	-4,8623	0,006372
61	-4,75284	0,006003
62	-4,66424	0,006249
63	-4,56428	0,006168
64	-4,45691	0,006073
65	-4,348	0,006179
66	-4,25998	0,006118
67	-4,15313	0,006162
68	-4,04814	0,006339
69	-3,94831	0,006465
70	-3,85342	0,006748
71	-3,74823	0,006771
72	-3,6496	0,006825
73	-3,55761	0,006954
74	-3,45824	0,006985
75	-3,36106	0,007129
76	-3,22969	0,007153
77	-3,14135	0,007072
78	-3,03543	0,007108
79	-2,93488	0,007035
80	-2,83754	0,007105
81	-2,70766	0,007105
82	-2,60584	0,007105
83	-2,48123	0,007105
84	-2,37723	0,007105
85	-2,26327	0,007105
86	-2,14018	0,007105
87	-2,03259	0,007105
88	-1,93079	0,007105
89	-1,81465	0,007105
90	-1,71502	0,007105
91	-1,6195	0,007105
92	-1,52105	0,007105
93	-1,43076	0,007105
94	-1,34638	0,007105
95	-1,24416	0,007105
96	-1,15731	0,007105
97	-1,07387	0,007105
98	-0,99315	0,007105
99	-0,91765	0,007105
100	-0,84401	0,007105
101	-0,77493	0,007105
102	-0,71003	0,007105
103	-0,6501	0,007105
104	-0,5917	0,007105

105	-0,54423	0,007105
106	-0,50158	0,007105
107	-0,46301	0,007105
108	-0,40504	0,007105
109	-0,40547	0,007105

Figure A.1