Optimal Dividends in the Brownian Motion Model with Credit and Debit Interest

Jun Cai
Department of Statistics and Actuarial Science
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
e-mail: jcai@uwaterloo.ca.
Hans U. Gerber
Ecole des hautes études commerciales
Université de Lausanne
CH-1015 Lausanne, Switzerland
e-mail: hgerber@unil.ch.
Hailiang Yang
Department of Statistics and Actuarial Science
University of Hong Kong
Pokfulam Road, Hong Kong
e-mail: hlyang@hkusua.hku.hk.

Abstract: The income process of a company is modeled by a Brownian motion, and in addition, the surplus earns investment income at a constant rate of credit interest. Dividends are paid to the shareholders according to a barrier strategy. It is shown how the expected discounted value of the dividends and the optimal dividend barrier can be calculated; Kummer's confluent hypergeometric differential equation plays a key role in this context. An alternative assumption is that business can go on after ruin, as long as it is profitable. When the surplus is negative, a higher rate of debit interest is applied. Several numerical examples document the influence of the parameters on the optimal dividend strategy.

Keywords: Brownian motion, barrier strategy, optimal dividend, ruin time, confluent hypergeometric function, Kummer's confluent hypergeometric equation.

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1 Introduction

How should the dividends be paid to the shareholders, if the aim is to maximize the expectation of the discounted dividends until possible ruin of the company? This problem goes back to De Finetti (1957), who solved it in the discrete time model with gains of +1 and -1. A history and a set of references can be found in Gerber and Shiu (2004). In their paper, the surplus process of the company before dividends is modeled by a Brownian motion with constant drift $\mu > 0$ and variance per unit time σ^2. Like most other authors, they do not assume that the surplus earns interest. The purpose of this paper is to examine the problem under the assumption that the surplus does earn interest at the constant force $\rho > 0$. It should be distinguished from the constant force of interest $\delta > 0$, which is used to discount the dividends. In fact, we must assume

$$\rho < \delta, \tag{1.1}$$

for the optimization problem. If (1.1) does not hold, the expected discounted dividends before ruin can be made as large as possible.

In our search for an optimal dividend strategy we restrict ourselves to barrier strategies. In the classical case ($\rho = 0$), the value of a barrier strategy satisfies a linear differential equation with constant coefficients. In this paper, ρ is positive, which leads to a linear differential equation with variable coefficients. It is shown that this equation belongs to the class of *Kummer's confluent hypergeometric differential equations*, which enables us to calculate the value of a barrier strategy and to determine the optimal barrier in numerical examples. *Mathematica* turns out to be useful tool in this context.

In Section 6, we bring the model a step closer to reality. We allow for the possibility that business can go on after "ruin." When the surplus is negative, interest is charged at a constant force τ. It is assumed that

$$\delta < \tau. \tag{1.2}$$

Thus ρ is the *rate of credit interest*, and τ is the *rate of debit interest*. Now the business has to stop when the company is not profitable anymore (due to the interest payments on the debt). This model is more general than the first model, which can be considered as the limit $\tau \to \infty$. Luckily, the mathematics are essentially the same.

The interesting paper of Paulsen and Gjessing (1997) deserves some credit, in particular the two examples. In the first example, the surplus process is modeled by a shifted compound Poisson process with exponential claims, and a constant rate of credit interest is assumed. It is shown that the evaluation of a barrier strategy also leads to Kummer's confluent hypergeometric differential equation (note that in formula (2.8) of Paulsen and Gjessing (1997), z must be replaced by $-z$). In the second example, the surplus process is modeled by a Brownian motion, and the stochastic rate of credit interest is modeled by another Brownian motion. But in their paper, the numerical implementation and the search for the optimal barrier seem to have a low priority.

2 The Model

Let $X(t)$ denote the surplus of the company, if no dividends are paid. Then the surplus process follows the following dynamics:

$$dX(t) = (\mu + \rho X(t))dt + \sigma dW(t), \quad t \geq 0, \tag{2.1}$$

where $\{W(t), t \geq 0\}$ is a standard Wiener process. This means that the surplus process is a *Brownian motion*, with variance σ^2 per unit time and drift $\mu + \rho x$, where x is the current surplus.

As in Gerber and Shiu (2004), we assume that dividends are paid to the shareholders according to a *barrier strategy*, say with parameter $b > 0$. Whenever the (modified) surplus is about to go above the level b, the excess (or "overflow") will be paid as dividends. This scheme can be formalized as follows. Let

$$M(t) = \max_{0 \leq s \leq t} X(s) \tag{2.2}$$

be the observed maximum of the surplus process, and let $D(t)$ denote the aggregate dividends by time t. Then, according to the barrier strategy with parameter b,

$$D(t) = \max(M(t) - b, 0). \tag{2.3}$$

The modified surplus at time t is $X(t) - D(t)$, and

$$T = \inf\{t : X(t) - D(t) = 0\} \tag{2.4}$$

is the *time of ruin*, which is certain. Now

$$D = \int_0^T e^{-\delta t} dD(t), \tag{2.5}$$

is the discounted value of the dividends until ruin. We are interested in $V(x;b)$, $0 \le x \le b$, the expectation of D, considered a function of the initial surplus x.

By virtually the same arguments as in Gerber and Shiu (2004), it can be seen that $V(x;b)$ satisfies the following second-order differential equation,

$$\frac{\sigma^2}{2}V''(x;b) + (\mu + \rho x)V'(x;b) - \delta V(x;b) = 0, \quad 0 < x < b, \tag{2.6}$$

in combination with the boundary conditions

$$\begin{cases} V(0;b) = 0, \\ V'(b;b) = 1. \end{cases} \tag{2.7}$$

It is possible to write $V(x;b)$ in a more transparent form. Let the function $g(x)$ be a solution of the differential equation

$$\frac{\sigma^2}{2}g''(x) + (\mu + \rho x)g'(x) - \delta g(x) = 0, \quad x > 0, \tag{2.8}$$

combined with the boundary condition

$$g(0) = 0. \tag{2.9}$$

The function $g(x)$ is unique apart from a constant factor. From (2.6) - (2.7) we obtain the *factorization formula*,

$$V(x;b) = \frac{g(x)}{g'(b)}, \quad 0 \le x \le b. \tag{2.10}$$

Thus the determination of the function $g(x)$ is a central problem.

From (2.3) it follows that

$$V(x;b) = x - b + V(b;b),\ x > b, \tag{2.11}$$

which has the following interpretation: if the initial surplus x exceeds the dividend barrier b, the difference is immediately paid out as dividends.

Remark 2.1. For given $x > 0$, $b > 0$,

$$\lim_{\sigma \to \infty} V(x;b) = x. \tag{2.12}$$

To see this, we introduce *operational time*, such that with respect to the operational time units, the variance per unit time of the surplus process is 1. Then the drift parameter becomes μ/σ^2, and the interest rates become δ/σ^2 and ρ/σ^2. All three vanish in the limit. It follows from (2.6) that $V''(x;b) = 0$ in the limit. (Another way to see that $V''(x;b)$ vanishes in the limit is to divide (2.6) by σ^2 and then to let σ tend to ∞.) From this and (2.7) we see that $V(x;b) = x$ for $0 \le x \le b$ in the limit. Because of (2.11), it follows that this limiting equation is also true for $x > b$.

3 Kummer's confluent hypergeometric equation

The differential equation for the function $g(x)$ can be converted into Kummer's confluent hypergeometric equation. We show this by two steps. In each step, a new variable is introduced.

In the first step, we define $z = \frac{1}{\sigma}\sqrt{\frac{2}{\rho}}\,(\mu + \rho x)$ and define the function $f(z)$ such that $g(x) = f(z)$. Hence,

$$\begin{aligned} g'(x) &= f'(z)\frac{dz}{dx} = \frac{\sqrt{2\rho}}{\sigma} f'(z), \\ g''(x) &= \frac{2\rho}{\sigma^2} f''(z). \end{aligned}$$

Substitution in (2.8) yields the differential equation for $f(z)$:

$$f''(z) + z f'(z) - \left(\frac{\delta}{\rho}\right) f(z) = 0, \quad z \; > \; \frac{\mu\sqrt{2}}{\sigma\sqrt{\rho}}. \tag{3.1}$$

In the second step, we define $t = -\frac{1}{2}z^2$ and the function $h(t)$ by the requirement that $f(z) = h(t)$. Hence,

$$\begin{aligned} f'(z) &= h'(t)\frac{dt}{dz} = -z h'(t), \\ f''(z) &= z^2 h''(t) - h'(t) = -2t h''(t) - h'(t). \end{aligned}$$

Substitution in (3.1) yields the differential equation for the function $h(t)$:

$$t\, h''(t) + \left(\frac{1}{2} - t\right) h'(t) + \frac{\delta}{2\rho}\, h(t) = 0, \quad t \; < \; -\frac{\mu^2}{\rho\sigma^2}. \tag{3.2}$$

Thus the function $h(t)$ satisfies indeed equation (A.1) of the appendix, with parameters

$$c = \frac{1}{2}, \qquad a = -\frac{\delta}{2\rho}. \tag{3.3}$$

We conclude that

$$g(x) \; = \; h(t) = \alpha\,(-t)^{1-c}\, e^t\, M\,(1-a, 2-c; -t) + \beta\, e^t\, U(c-a, c; -t), \tag{3.4}$$

for certain coefficients α and β, with

$$t = -\frac{1}{\rho\sigma^2}(\mu + \rho x)^2 \tag{3.5}$$

and a and c given by (3.3).

The ratio between α and β follows from condition (2.9). For example, we may set

$$\begin{aligned}\alpha &= U\left(c-a,\, c;\, \frac{\mu^2}{\rho\sigma^2}\right),\\ \beta &= -\left(\frac{\mu^2}{\rho\sigma^2}\right)^{1-c} M\left(1-a,\, 2-c;\, \frac{\mu^2}{\rho\sigma^2}\right).\end{aligned}$$

Furthermore, taking derivatives in (3.4), we obtain

$$g'(x) = h'(t)\frac{dt}{dx} = -\frac{2}{\sigma^2}(\mu+\rho x)h'(t). \tag{3.6}$$

Using the product rule and (A.2), (A.3), we see that

$$\begin{aligned}h'(t) &= h(t) - \alpha(1-c)(-t)^{-c}\, e^t\, M\left(1-a, 2-c; -t\right)\\ &- \alpha\left(\frac{1-a}{2-c}\right)(-t)^{1-c}\, e^t\, M\left(2-a, 3-c; -t\right)\\ &+ \beta(c-a)\, e^t\, U\left(c-a+1, c+1; -t\right).\end{aligned} \tag{3.7}$$

Finally, we set $x = b$ in (3.6) and substitute it and (3.4) in (2.10) to obtain $V(x;b)$.

Remark 3.1 In the limiting situation $\sigma = 0$, we have

$$X(t) = xe^{\rho t} + \mu \bar{s}_{\overline{t}|},$$

where $\bar{s}_{\overline{t}|}$ is calculated at the force of interest ρ. Now the determination of $V(x;b)$ is an exercise of compound interest. Let t_0 be the time when $X(t_0) = b$. Then we find that

$$V(x;b) = e^{-\delta t_0}V(b;b) = \left(\frac{\mu+\rho x}{\mu+\rho b}\right)^{\delta/\rho} \frac{\mu+\rho b}{\delta} \tag{3.8}$$

after simplification.

Remark 3.2 When $\rho = 0$, by (2.11) - (2.15) of Gerber and Shiu (2004),

$$V(x;b) = \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}}, \tag{3.9}$$

where $r = (-\mu + \sqrt{\mu^2 + 2\delta\sigma^2})/\sigma^2$ and $s = (-\mu - \sqrt{\mu^2 + 2\delta\sigma^2})/\sigma^2$.

Now we are prepared to calculate $V(x;b)$. We present two examples to illustrate the dependence of $V(x;b)$ on ρ and σ.

Table 1: The influence of ρ and x on $V(x;10)$ with $\delta = 4\%$

	$V(x;10)$ with $\sigma = 0.5$ ($\sigma = 5$)				
x	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.2	13.63 (0.36)	14.44 (0.37)	15.25 (0.38)	16.90 (0.39)	18.57 (0.41)
0.4	16.47 (0.72)	17.44 (0.73)	18.42 (0.75)	20.40 (0.77)	22.41 (0.80)
0.6	17.15 (1.07)	18.16 (1.09)	19.17 (1.11)	21.23 (1.15)	23.31 (1.20)
0.8	17.39 (1.42)	18.42 (1.44)	19.44 (1.47)	21.53 (1.53)	23.63 (1.58)
1.0	17.55 (1.76)	18.58 (1.79)	19.62 (1.82)	21.72 (1.89)	23.85 (1.96)
2.0	18.27 (3.38)	19.34 (3.45)	20.41 (3.51)	22.59 (3.64)	24.78 (3.78)
4.0	19.79 (6.30)	20.92 (6.42)	22.06 (6.53)	24.35 (6.77)	26.67 (7.01)
6.0	21.43 (8.87)	22.61 (9.02)	23.80 (9.17)	26.19 (9.47)	28.59 (9.79)
8.0	23.20 (11.16)	24.42 (11.33)	25.64 (11.50)	28.09 (11.85)	30.54 (12.21)
10	25.12 (13.24)	26.35 (13.42)	27.59 (13.60)	30.05 (13.96)	32.52 (14.34)

Example 3.1. Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 0.5$. We calculate $V(x;10)$ for selected values of x and ρ.

The results are exhibited in Table 1. Of course, $V(x;10)$ is an increasing function of both x and ρ. Table 1 shows that the influence of ρ is substantial. In the same table, results for $\sigma = 5$ are shown in parentheses.

Example 3.2. Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\rho = 2\%$. We calculate $V(x;10)$ for selected values of x and σ. The results are displayed in Table 2. We note that for $x \leq 0.06$, $V(x;10)$ appears to be a decreasing function of σ. But for $x \geq 0.08$, $V(x;10)$ first increases with σ, attains a maximum and decreases thereafter. The following is a tentative to explain this phenomenon. If σ is large, ruin occurs soon. In particular, the probability that the dividend barrier is attained before ruin is small. In this sense, a high value of σ has a negative impact on $V(x;10)$. On the other hand, if σ is large, dividends are paid at a high rate when the surplus is on the barrier. In this sense, a high value of σ has a positive impact on $V(x;10)$. Then Table 2 shows that for $x \geq 0.08$, the second factor prevails up to a certain value of σ, where $V(x;10)$ attains its maximum. Thereafter, the first factor prevails.

We calculate $V(x;10)$ also for $\rho = 6\%$; the results are shown in parentheses. In this case $V(x;10)$ appears to be a decreasing function of σ for any x.

Table 2: The influence of σ and x on $V(x;10)$ with $\delta = 4\%$

	$V(x;10)$ with $\rho = 2\%$ ($\rho = 6\%$)				
x	$\sigma = 0$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
0.2	21.00 (29.47)	16.90 (23.70)	7.28 (10.22)	0.98 (1.34)	0.39 (0.45)
0.4	21.17 (29.71)	20.40 (28.56)	12.17 (17.07)	1.91 (2.61)	0.77 (0.90)
0.6	21.34 (29.94)	21.23 (29.69)	15.47 (21.66)	2.81 (3.83)	1.15 (1.34)
0.8	21.51 (30.17)	21.53 (30.09)	17.71 (24.75)	3.67 (5.00)	1.52 (1.77)
1.0	21.68 (30.40)	21.72 (30.35)	19.25 (26.84)	4.49 (6.12)	1.89 (2.19)
2.0	22.53 (31.53)	22.59 (31.49)	22.42 (31.02)	8.10 (11.00)	3.64 (4.21)
4.0	24.30 (33.75)	24.35 (33.71)	24.50 (33.58)	13.44 (18.01)	6.76 (7.78)
6.0	26.13 (35.89)	26.19 (35.85)	26.34 (35.73)	17.12 (22.55)	9.47 (10.80)
8.0	28.03 (37.97)	28.09 (37.93)	28.24 (37.81)	19.84 (25.63)	11.85 (13.36)
10	30.00 (40.00)	30.05 (39.96)	30.21 (39.84)	22.02 (27.90)	13.96 (15.53)

4 The optimal barrier

For given $x > 0$, we want to find b which maximizes $V(x;b)$. From (2.10) and (2.11) we obtain

$$\frac{\partial}{\partial b}V(x;b) = \begin{cases} -V(b;b)\,\dfrac{g''(b)}{g'(b)} & \text{if} \quad 0 < b < x, \\ \\ -V(x;b)\,\dfrac{g''(b)}{g'(b)} & \text{if} \quad b \geq x. \end{cases} \tag{4.1}$$

From (2.8) and (2.9) it follows that

$$\frac{g''(0)}{g'(0)} = -\frac{2\mu}{\sigma^2} < 0.$$

Hence $\frac{\partial}{\partial b}V(x;b)$ is positive for small values of b. Because $V(x;b) \to 0$ for $b \to \infty$, we gather that $V(x;b)$ attains its maximum for a finite and positive value of b.

According to (4.1), the first order condition is that

$$g''(b) = 0. \tag{4.2}$$

It turns out that this equation has a unique solution $b = b^*$. Therefore, this is the barrier which maximizes $V(x; b)$, independently of x. This is illustrated by Figure 1.

We note that

$$V''(x; b) = \frac{g''(x)}{g'(b)} \quad \text{for} \quad 0 < x < b$$

and hence

$$V''(b^*;\, b^*) = 0. \tag{4.3}$$

Thus, if we set $x = b = b^*$ in (2.6) and use the second condition in (2.7), we see that

$$\mu + \rho\, b^* - \delta V(b^*;\, b^*) = 0,$$

from which it follows that

$$V(b^*;\, b^*) = \frac{\mu + \rho\, b^*}{\delta}. \tag{4.4}$$

Thus, $V(b^*; b^*)$ is identical to the present value of a *perpetuity*, where the payment rate is the sum of the drift and the interest on the initial capital. For $\rho = 0$, formula (4.4) can be found as formula (7.1) in Gerber and Shiu (2004). In this case, there is a closed form expression for the optimal barrier:

$$b^* = \frac{2}{r - s} \log\left(-\frac{s}{r}\right), \tag{4.5}$$

where r and s are the same as in (3.9).

The difference $V(x; b^*) - x$ is the maximal return on the investment x. Formula (2.12) shows that in the limit $\sigma \to \infty$ (*extreme volatility business*), the maximal return becomes zero. In particular, we have $V(b^*; b^*) = b^*$ in the limit. From this and (4.4) we obtain a linear equation for the limiting value of b^*. We find that

$$\lim_{\sigma \to \infty} b^* = \frac{\mu}{\delta - \rho}. \tag{4.6}$$

This result generalizes formula (7.3) in Gerber and Shiu (2004).

Example 4.1. Assume that $\mu = 1$, $\delta = 4\%$. We calculate the optimal barrier b^* for selected values of ρ and σ. The results are shown in Table 3. We note that b^* is an increasing function of ρ. As ρ tends to δ, b^* tends to ∞. Moreover, b^* increases from 0 to the value given by formula (4.6) as σ varies between 0 and ∞. Also, note that Table 3 tells us $b^* = 26.1876$ in Figure 1.

Table 3: The influence of ρ and σ on b^* with $\delta = 4\%$

	Optimal Barrier b^*				
σ	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.05	0.02476	0.02492	0.02511	0.02562	0.02648
0.10	0.08514	0.08580	0.08656	0.08855	0.09198
0.20	0.28484	0.28739	0.29033	0.29814	0.31161
0.50	1.31399	1.32847	1.34534	1.39034	1.46887
5	19.00860	20.49930	22.17000	26.18760	31.74960
50	24.91700	28.44770	33.13750	49.34760	95.14190
500	24.99920	28.57020	33.33130	49.99330	99.94670
∞	25	28.57140	33.33333	50	100

Example 4.2. As in Example 3.1, we assume that $\mu = 1$, $\delta = 4\%$, $\sigma = 0.5\,(\sigma = 5)$. Now we calculate $V(x;b^*)$ for the same combinations of x and ρ as in Table 1. The results are shown in Table 4. We observe that $V(x;b^*)$ exceeds $V(x;b)$ significantly. Note that the optimal barrier values can be found in the rows $\sigma = 0.5$ and $\sigma = 5$ in Table 3. In particular, we see that $b^* < 2$ if $\sigma = 0.5$. Therefore, in accordance with (2.11) with $b = b^*$, $V(x;b^*)$ is linear for $x \geq 2$.

5 The distribution of the time of ruin under a barrier strategy

In this section we assume that dividends are paid according to a barrier strategy with parameter b, which may be different from the optimal value b^*. Consider the expected present value of a payment of 1, due at the time of ruin. We use the notation

$$L(x;b) = E[e^{-\delta T}] \tag{5.1}$$

to indicate that this expression will be treated as a function of the initial capital x. Hence, $L'(x;b)$ denotes the derivative with respect to x. As a function of δ, $L(x;b)$ is the *Laplace transform* of the probability density function of the time of ruin T.

Table 4: The influence of ρ and x on $V(x;b^*)$ with $\delta = 4\%$

	$V(x;b^*)$ with $\sigma = 0.5$ ($\sigma = 5$)				
x	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.2	19.16 (0.42)	19.29 (0.45)	19.42 (0.48)	19.68 (0.56)	19.96 (0.67)
0.4	23.16 (0.84)	23.30 (0.89)	23.45 (0.95)	23.76 (1.11)	24.08 (1.32)
0.6	24.11 (1.25)	24.26 (1.33)	24.41 (1.42)	24.73 (1.65)	25.05 (1.97)
0.8	24.46 (1.66)	24.61 (1.76)	24.76 (1.88)	25.07 (2.18)	25.40 (2.56)
1.0	24.68 (2.06)	24.83 (2.18)	24.99 (2.33)	25.30 (2.70)	25.63 (3.22)
2	25.69 (3.96)	25.84 (4.20)	25.99 (4.48)	26.30 (5.21)	26.63 (6.20)
4	27.69 (7.39)	27.84 (7.82)	27.99 (8.34)	28.30 (9.67)	28.63 (11.51)
6	29.69 (10.39)	29.84 (10.99)	29.99 (11.71)	30.30 (13.55)	30.63 (16.09)
8	31.69 (13.07)	31.84 (13.81)	31.99 (14.69)	32.30 (16.94)	32.63 (20.06)
10	33.69 (15.51)	33.84 (16.36)	33.99 (17.37)	34.30 (19.96)	34.63 (23.55)

By analogy with (3.4) in Gerber and Shiu (2004), $L(x;b)$ can be characterized as the solution of second order differential equation

$$\frac{\sigma^2}{2}L''(x;b) + (\mu + \rho x)L'(x;b) - \delta L(x;b) = 0, \quad 0 < x < b, \tag{5.2}$$

in conjunction with the boundary conditions

$$\begin{cases} L(0;b) = 1, \\ L'(b;b) = 0. \end{cases} \tag{5.3}$$

Let the function $g(x)$ be a solution of the differential equation (2.8), subject to the boundary condition

$$g'(b) = 0. \tag{5.4}$$

Then it follows from (5.2) and (5.3) that

$$L(x;b) = \frac{g(x)}{g(0)}, \qquad 0 \le x \le b. \tag{5.5}$$

The function $g(x)$ is obtained by the same method as in (3.4), except that the coefficients α and β in (3.4) have different values. Their ratio now follows from condition (5.4), which is equivalent to the condition

$$h'\left(-\frac{(\mu + \rho b)^2}{\rho\sigma^2}\right) = 0.$$

From this and (3.7) with (3.4), we see that we may set

$$\alpha = (c-a)U\left(c-a+1,\, c+1;\, \frac{(\mu+\rho b)^2}{\rho\sigma^2}\right) + U\left(c-a, c;\, \frac{(\mu+\rho b)^2}{\rho\sigma^2}\right), \quad (5.6)$$

$$\begin{aligned}\beta = {} & \frac{1-a}{2-c}\left(\frac{(\mu+\rho b)^2}{\rho\sigma^2}\right)^{1-c} M\left(2-a,\, 3-c;\, \frac{(\mu+\rho b)^2}{\rho\sigma^2}\right) \\ & - \left(\frac{(\mu+\rho b)^2}{\rho\sigma^2}\right)^{-c}\left(\frac{(\mu+\rho b)^2}{\rho\sigma^2} - (1-c)\right) M\left(1-a, 2-c;\, \frac{(\mu+\rho b)^2}{\rho\sigma^2}\right). \quad (5.7)\end{aligned}$$

Then (3.4) with this choice of α and β must be substituted in (5.5) to obtain $L(x;b)$.

As an application, we can calculate the expectation of the time of ruin by the formula

$$E[T] = -\frac{dL(x,b)}{d\delta}\Big|_{\delta=0},$$

if $\rho > 0$. If $\rho = 0$, we have

$$E[T] = \frac{\sigma^2}{2\mu^2}\left(e^{2\mu b/\sigma^2} - e^{2\mu(b-x)/\sigma^2} - \frac{2\mu x}{\sigma^2}\right), \quad 0 \le x \le b,$$

see, for example, formula (3.8) in Gerber and Shiu (2004).

Example 5.1. Assume $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 3$. We calculate the expected time of ruin $E[T]$ for different values of ρ and x. The results are displayed in Table 5. Not surprisingly, $E[T]$ is an increasing function of both x and ρ.

6 Life after ruin

As before, we assume that dividends are paid according to a barrier strategy with parameter b. But now we allow for the possibility that the business goes on after "ruin", and hence that dividends can be paid after ruin. Whenever the surplus is negative, interest is debited at a force $\tau > \delta$. Now we assume that the company has to go out of business, when the surplus is at the critical level

$$\lambda = -\frac{\mu}{\tau}. \quad (6.1)$$

If the surplus is at this level, the company is not profitable any more in that the instantaneous drift of the surplus process is 0.

Table 5: The influence of ρ and x on $E[T]$ with $\delta = 4\%$ and $\sigma = 3$

	$E[T]$					
x	$\rho = 0$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 4\%$	$\rho = 6\%$	$\rho = 8\%$
0.2	1.605	1.701	1.805	2.039	2.314	2.637
0.4	3.132	3.320	3.523	3.981	4.517	5.148
0.6	4.584	4.859	5.157	5.827	6.614	7.538
0.8	5.963	6.322	6.710	7.583	8.608	9.811
1.0	7.274	7.713	8.166	9.252	10.502	11.970
2.0	12.900	13.676	14.514	16.398	18.604	21.193
4.0	20.454	21.656	22.952	25.857	29.243	33.199
6.0	24.579	25.973	27.473	30.823	34.711	39.234
8.0	26.507	27.962	29.525	30.010	37.045	41.728
10	27.025	28.488	30.058	33.559	37.611	42.311

Let $V(x;b)$, $\lambda \leq x \leq b$, denote the expectation of the discounted dividends until the company has to go out of business. The differential equation (2.6) and the second boundary condition in (2.7) remain valid. The first boundary condition in (2.7) is replaced by the condition

$$V(\lambda;b) = 0, \tag{6.2}$$

and for $\lambda < x < 0$, $V(x;b)$ satisfies

$$\frac{\sigma^2}{2}V''(x;b) + (\mu + \tau x)V'(x;b) - \delta V(x;b) = 0, \quad \lambda < x < 0. \tag{6.3}$$

Furthermore, it can be shown that $V(x;b)$ and $V'(x;b)$ are continuous at the junction $x = 0$.

From these considerations we obtain the factorization formula,

$$V(x;b) = \frac{g(x)}{g'(b)}, \quad \lambda \leq x \leq b. \tag{6.4}$$

Here $g(x)$ is a continuously differentiable function, satisfying (2.8),

$$\frac{\sigma^2}{2}g''(x) + (\mu + \tau x)g'(x) - \delta g(x) = 0, \quad \lambda < x < 0, \tag{6.5}$$

and

$$g(\lambda) = 0. \tag{6.6}$$

Again, the function $g(x)$ is unique apart from a constant factor.

By introducing the new variable

$$\tilde{t} = -\frac{1}{\tau\sigma^2}(\mu + \tau x)^2, \tag{6.7}$$

equation (6.5) can be converted into Kummer's confluent hypergeometric equation for the function $\tilde{h}(\tilde{t}) = g(x)$. The parameters are now

$$c = \frac{1}{2}, \qquad \tilde{a} = -\frac{\delta}{2\tau}. \tag{6.8}$$

We conclude that $g(x)$, $\lambda \le x \le 0$, must be a linear combination of the two functions $(-\tilde{t})^{1-c}\, e^{\tilde{t}}\, M\left(1 - \tilde{a},\, 2 - c;\, -\tilde{t}\right)$ and $e^{\tilde{t}}\, U(c - \tilde{a},\, c;\, -\tilde{t})$. Because $U\left(c - \tilde{a},\, c;\, 0\right) \neq 0$, it follows from (6.6) that $g(x), \lambda \le x \le 0$, is a multiple of the first function. We may set

$$g(x) = \tilde{h}(\tilde{t}) = (-\tilde{t})^{1-c}\, e^{\tilde{t}}\, M(1 - \tilde{a}, 2 - c; -\tilde{t}),\ \ \lambda \le x \le 0, \tag{6.9}$$

with $\tilde{t}$ given by (6.7).

For $x > 0$, $g(x)$ is of the form (3.4). To determine the coefficients α and β, we use the continuity conditions,

$$g(0+) = g(0-), \quad g'(0+) = g'(0-), \tag{6.10}$$

which lead to the conditions

$$h\left(-\frac{\mu^2}{\rho\,\sigma^2}\right) = \tilde{h}\left(-\frac{\mu^2}{\tau\,\sigma^2}\right), \quad h'\left(-\frac{\mu^2}{\rho\,\sigma^2}\right) = \tilde{h}'\left(-\frac{\mu^2}{\tau\,\sigma^2}\right),$$

resulting in the equations

$$\begin{aligned} &\alpha\left(\frac{\mu^2}{\rho\,\sigma^2}\right)^{1-c} \exp\left\{-\frac{\mu^2}{\rho\,\sigma^2}\right\} M\left(1 - a,\, 2 - c;\, \frac{\mu^2}{\rho\,\sigma^2}\right) \\ &+ \beta\, \exp\left\{-\frac{\mu^2}{\rho\,\sigma^2}\right\} U\left(c - a,\, c;\, \frac{\mu^2}{\rho\,\sigma^2}\right) \\ = \quad &\left(\frac{\mu^2}{\tau\,\sigma^2}\right)^{1-c} \exp\left\{-\frac{\mu^2}{\tau\,\sigma^2}\right\} M\left(1 - \tilde{a},\, 2 - c;\, \frac{\mu^2}{\tau\,\sigma^2}\right), \end{aligned}$$

and

$$
\begin{aligned}
&\alpha(1-c)\left(\frac{\mu^2}{\rho\,\sigma^2}\right)^{-c}\exp\left\{-\frac{\mu^2}{\rho\,\sigma^2}\right\}M\left(1-a,\,2-c;\,\frac{\mu^2}{\rho\,\sigma^2}\right)\\
&+\alpha\left(\frac{\mu^2}{\rho\,\sigma^2}\right)^{1-c}\exp\left\{-\frac{\mu^2}{\rho\,\sigma^2}\right\}\left(\frac{1-a}{2-c}\right)M\left(2-a,\,3-c;\,\frac{\mu^2}{\rho\,\sigma^2}\right)\\
&+\beta\exp\left\{-\frac{\mu^2}{\rho\,\sigma^2}\right\}(a-c)\,U\left(c-a+1,\,c+1;\,\frac{\mu^2}{\rho\,\sigma^2}\right)\\
=\;&(1-c)\left(\frac{\mu^2}{\tau\,\sigma^2}\right)^{-c}\exp\left\{-\frac{\mu^2}{\tau\,\sigma^2}\right\}M\left(1-\tilde{a},\,2-c;\,\frac{\mu^2}{\tau\,\sigma^2}\right)\\
&+\left(\frac{\mu^2}{\tau\,\sigma^2}\right)^{1-c}\exp\left\{-\frac{\mu^2}{\tau\,\sigma^2}\right\}\left(\frac{1-\tilde{a}}{2-c}\right)M\left(2-\tilde{a},\,3-c;\,\frac{\mu^2}{\tau\,\sigma^2}\right).
\end{aligned}
$$

These are two linear equations for α and β.

Remark 6.1. In the limiting situation $\sigma = 0$, equation (3.8) is still valid for $0 \le x \le b$. Furthermore,

$$V(x;b) = \left(\frac{\mu+\tau x}{\mu}\right)^{\delta/\tau} V(0;b) \quad \text{for} \quad \lambda \le x < 0. \tag{6.11}$$

To show this is an exercise of compound interest.

Remark 6.2. The calculations are somewhat simpler when $\rho = 0$. Then (6.4) and (6.9) are still valid. But now $g(x)$ for $x \ge 0$ is of the form

$$g(x) = \alpha e^{rx} + \beta e^{sx},\;\; x \ge 0, \tag{6.12}$$

where r and s are the same as in (3.9). From (6.12) we obtain the conditions

$$\alpha+\beta = \left(\frac{\mu^2}{\tau\,\sigma^2}\right)^{1-c}\exp\left\{-\frac{\mu^2}{\tau\,\sigma^2}\right\}M\left(1-\tilde{a},\,2-c;\,\frac{\mu^2}{\tau\,\sigma^2}\right)$$

and

$$r\alpha + s\beta = -\left(\frac{2\mu}{\sigma^2}\right)\tilde{h}'\left(-\frac{\mu^2}{\tau\,\sigma^2}\right),$$

from which the coefficients α and β can be determined.

Remark 6.3. For $x > \lambda$, $b > 0$,

$$\lim_{\sigma\to\infty} V(x;b) = x + \frac{\mu}{\tau}. \tag{6.13}$$

To see this, we first use the same reasoning as in Remark 2.1 to conclude that $V''(x;b) = 0$ in the limit. In view of the boundary conditions at $x = \lambda$ and $x = b$, we gather that $V(x;b) = x - \lambda$, which is equivalent to (6.13).

Now we are prepared to calculate $V(x;b)$ when dividends can be paid after ruin. We present two examples to illustrate the dependence of $V(x;b)$ on ρ, τ, and x.

Example 6.1. Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 0.5\,(\sigma = 5)$ and $\tau = 6\%$. We calculate $V(x;10)$ for the same combinations of x and ρ as in Table 1.

The results are exhibited in Table 6. Because of the dividends that are paid after ruin, the values of $V(x;10)$ in Table 6 are higher than the corresponding values in Table 1. However, we note that if $\sigma = 0.5$, the two values are practically the same, if x is sufficiently positive. Because the difference between the values in Tables 6 and 1 is the expectation of the discounted dividends after ruin, it can be written as the product $L(x;10)V(0;10)$. Thus, in a given column of Table 6, this difference is proportional to $L(x;10)$.

Table 6: The influence of ρ and x on $V(x;10)$ with $\delta = 4\%$ and $\tau = 6\%$

	$V(x;10)$ with $\sigma = 0.5$ ($\sigma = 5$)				
x	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
-10	9.12 (8.09)	9.65 (8.22)	10.19 (8.35)	11.29 (8.61)	12.39 (8.88)
-8	10.89 (10.44)	11.53 (10.60)	12.17 (10.77)	13.47 (11.11)	14.79 (11.46)
-6	12.52 (12.73)	13.25 (12.93)	13.99 (13.13)	15.49 (13.54)	17.01 (13.97)
-4	14.04 (14.95)	14.87 (15.18)	15.70 (15.42)	17.38 (15.91)	19.09 (16.41)
-2	15.49 (17.09)	16.40 (17.36)	17.32 (17.63)	19.17 (18.19)	21.05 (18.76)
0	16.87 (19.16)	17.87 (19.46)	18.86 (19.77)	20.88 (20.39)	22.93 (21.03)
0.2	17.01 (19.36)	18.01 (19.67)	19.02 (19.98)	21.05 (20.61)	23.11 (21.25)
0.4	17.15 (19.56)	18.15 (19.87)	19.17 (20.18)	21.22 (20.82)	23.30 (21.48)
0.6	17.28 (19.76)	18.30 (20.08)	19.32 (20.39)	21.39 (21.03)	23.48 (21.70)
0.8	17.42 (19.96)	18.44 (20.28)	19.47 (20.60)	21.56 (21.25)	23.67 (21.92)
1.0	17.56 (20.16)	18.59 (20.48)	19.63 (20.80)	21.73 (21.46)	23.85 (22.13)
2.0	18.27 (21.15)	19.34 (21.49)	20.41 (21.82)	22.59 (22.51)	24.78 (23.22)
4.0	19.79 (23.10)	20.92 (23.46)	22.06 (23.83)	24.35 (24.57)	26.67 (25.34)
6.0	21.43 (25.04)	22.61 (25.42)	23.80 (25.81)	26.19 (26.60)	28.59 (27.41)
8.0	23.20 (26.98)	24.42 (27.38)	25.64 (27.78)	28.09 (28.60)	30.54 (29.44)
10	25.12 (28.96)	26.35 (29.36)	27.59 (29.77)	30.05 (30.60)	32.52 (31.45)

Further, to see the influence of τ on $V(x;b)$, we reconsider Example 6.1 as follows.

Example 6.2. Assume that $\mu = 1$, $b = 10, \delta = 4\%$, $\sigma = 0.5\,(\sigma = 5)$ and $\rho = 2\%$. We calculate $V(x; 10)$ for selected values of x and τ.

The results are exhibited in Table 7. Of course, $V(x; 10)$ is an increasing function of x and a decreasing function of τ. Table 7 shows that the influence of τ is substantial for large σ. However, for small σ and x positive, the influence of τ on $V(x; 10)$ is modest. In the limiting situation $\sigma = 0$, $V(x; 10)$ does not depend on τ, provided that x is positive.

Table 7: The influence of ρ and x on $V(x; 10)$ with $\delta = 4\%$ and $\rho = 2\%$

	$V(x; 10)$ with $\sigma = 0.5$ ($\sigma = 5$)				
x	$\tau = 5\%$	$\tau = 6\%$	$\tau = 7\%$	$\tau = 8\%$	$\tau = 10\%$
-10	11.98 (11.36)	11.29 (8.61)	10.37 (6.06)	8.89 (3.77)	0 (0)
-8	13.87 (13.56)	13.47 (11.11)	13.00 (8.82)	12.41 (6.74)	10.19 (3.27)
-6	15.69 (15.72)	15.49 (13.54)	15.27 (11.49)	15.01 (9.63)	14.35 (6.47)
-4	17.47 (17.85)	17.38 (15.91)	17.29 (14.08)	17.20 (12.40)	16.98 (9.56)
-2	19.19 (19.93)	19.17 (18.19)	19.15 (16.55)	19.13 (15.04)	19.08 (12.48)
0	20.89 (21.97)	20.88 (20.39)	20.88 (18.90)	20.88 (17.53)	20.88 (15.20)
0.2	21.05 (22.18)	21.05 (20.61)	21.05 (19.12)	21.05 (17.77)	21.05 (15.46)
0.4	21.22 (22.38)	21.22 (20.82)	21.22 (19.35)	21.22 (18.01)	21.22 (15.71)
0.6	21.39 (22.58)	21.39 (21.03)	21.39 (19.58)	21.39 (18.24)	21.39 (15.97)
0.8	21.56 (22.78)	21.56 (21.25)	21.56 (19.80)	21.56 (18.48)	21.56 (16.22)
1.0	21.73 (22.98)	21.73 (21.46)	21.73 (20.03)	21.73 (18.71)	21.73 (16.47)
2.0	22.59 (23.98)	22.59 (22.51)	22.59 (21.13)	22.59 (19.86)	22.59 (17.70)
4.0	24.35 (25.96)	24.35 (24.57)	24.35 (23.27)	24.35 (22.07)	24.35 (20.04)
6.0	26.19 (27.93)	26.19 (26.60)	26.19 (25.34)	26.19 (24.19)	26.19 (22.23)
8.0	28.09 (29.90)	28.09 (28.60)	28.09 (27.37)	28.09 (26.25)	28.09 (24.33)
10	30.05 (31.89)	30.05 (30.60)	30.05 (29.38)	30.05 (28.26)	30.05 (26.36)

Let b^* denote the optimal barrier. It is the value of b which maximizes $V(x; b)$, independently of x; this is illustrated in Figure 2. As in Section 4, b^* can be obtained from the condition that $g''(b^*) = 0$. Furthermore, equations (4.3) and (4.4) remain valid in this section.

There are no closed form expressions for b^*, with one noteworthy exception. If $\rho = 0$, it follows from Remark 6.2 that

$$b^* = \frac{1}{r-s} \log \left(\frac{-\beta s^2}{\alpha r^2} \right), \tag{6.14}$$

which generalizes (4.5). Furthermore,

$$\lim_{\sigma \to \infty} b^* = \frac{\mu}{\delta - \rho} \left(1 - \frac{\delta}{\tau} \right). \tag{6.15}$$

To see this, we combine (6.13) with $x = b = b^*$ and (4.4) to obtain a linear equation for the limiting value of b^*. Note that (6.15) generalizes (4.6).

We use the following example to illustrate what optimal barriers are when dividends can be paid after ruin.

Example 6.3. As in Example 4.1, we assume that $\mu = 1$ and $\delta = 4\%$. But now business (and dividends) can go on after ruin according to a debit rate of interest of $\tau = 6\%$. In Table 8, the optimal barrier is shown for the same combinations of σ and ρ as in Table 3. We note that for the same pairs of σ and ρ, b^* in Table 8 is smaller than b^* in Table 3. In a given column, b^* increases from 0 to the limiting value in (6.15) as σ varies from 0 to ∞. Also, Table 8 reveals that $b^* = 8.72959$ in Figure 2.

Table 8: The influence of ρ and σ on b^* with $\delta = 4\%$ and $\tau = 6\%$

	Optimal Barrier b^*				
σ	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.05	0.00051	0.00057	0.00064	0.00087	0.00137
0.10	0.00203	0.00226	0.00256	0.00347	0.00549
0.20	0.00812	0.00905	0.01023	0.01388	0.02199
0.50	0.05113	0.05698	0.06439	0.08731	0.13817
5	5.11239	5.70392	6.45109	8.72959	13.49200
50	8.28724	9.46708	11.03840	16.51990	32.75470
500	8.33287	9.52324	11.11030	16.66520	33.32740
∞	8.33333	9.52381	11.11111	16.66667	33.33333

To see influence of τ on b^*, we reconsider Example 6.3 as follows.

Example 6.4. Assume that $\mu = 1$, $\delta = 4\%$, and $\rho = 2\%$. We calculate the optimal barrier b^* for selected values of τ and σ. The results are shown in Table 9. As in Example 6.3, b^* is an increasing function of both τ and σ. If σ is small, b^* is close to 0.

Again, if σ is very large, b^* in this example is close to the limiting value given in formula (6.15).

Table 9: The influence of τ and σ on b^* with $\delta = 4\%$ and $\rho = 2\%$

	Optimal Barrier b^*				
σ	$\tau = 5\%$	$\tau = 6\%$	$\tau = 7\%$	$\tau = 8\%$	$\tau = 10\%$
0.05	0.00051	0.00087	0.00115	0.00137	0.00173
0.10	0.00203	0.00347	0.00458	0.00550	0.00693
0.20	0.00812	0.01388	0.01835	0.02201	0.02778
0.50	0.05101	0.08731	0.11556	0.13872	0.17547
5	5.28134	8.72959	11.17560	13.00690	15.57390
50	9.92057	16.51990	21.22670	24.75300	29.68440
500	9.99920	16.66520	21.42650	24.99750	29.99680
∞	10	16.66667	21.42860	25	30

Further, to see the influence of τ and σ on b^*, we consider the following example.

Example 6.5. Assume that $\mu = 1, \delta = 4\%$ and $\sigma = 5$ and then obtain Table 10 for selected values of τ and ρ. Table 10 shows that b^* is an increasing function of both τ and ρ. Note that the last line is from Table 3.

Table 10: The influence of τ and ρ on b^* with $\delta = 4\%$ and $\sigma = 5$

	Optimal Barrier b^*				
τ	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
5%	2.9176	3.2850	3.7591	5.2813	8.8752
10%	10.0780	11.0680	12.2608	15.5739	21.2945
20%	14.3007	15.5484	17.0031	20.7685	26.5588
50%	17.0589	18.4467	20.0405	23.9767	29.6566
100%	18.0216	19.4630	21.0932	25.0730	30.6977
200%	18.5119	19.9778	21.6284	25.6278	31.2220
500%	18.8092	20.2896	21.9525	25.9631	31.5381
∞	19.0086	20.4993	22.1700	26.1876	31.7496

Appendix

A second order differential equation of the form

$$x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0 \tag{A.1}$$

is called Kummer's confluent hypergeometric equation, see Seaborn (1991), Slater (1960), but also http://mathworld.wolfram.com/ConfluentHypergeometricDifferentialEquation.html. This equation is also called associated Laguerre differential equation, see http://mathworld.wolfram.com/LaguerreDifferentialEquation.html.

For $x > 0$, the general solution of (A.1) can be expressed as a linear combination of the functions $M(a, c; x)$ and $U(a, c; x)$, called the confluent hypergeometric functions of the first and second kinds, respectively.

There are useful formulas for the derivatives of these functions:

$$\frac{d}{dx}M(a, c; x) = \frac{a}{c}\, M(a + 1, c + 1; x), \tag{A.2}$$

$$\frac{d}{dx}U(a, c; x) = -a\, U(a + 1, c + 1; x). \tag{A.3}$$

These formulas and a wealth of other results can be found in Abramowitz and Stegun (1972) or Slater (1960). It is important to know that $M(a, c; 0) = 1$, that $M(a, c; x) \to \infty$ for $x \to \infty$, that $U(a, c; x) \to 0$ for $x \to \infty$, and, for $0 < c < 1$, that $U(a, c; 0)$ is different from 0.

In this paper, we are to solve (A.1) when $x < 0$. Then the general solution is a linear combination of the functions

$$e^x\, U(c-a, c; -x), \quad x < 0, \tag{A.4}$$

and

$$(-x)^{1-c}\, e^x\, M\left(1-a, 2-c; -x\right), \quad x < 0. \tag{A.5}$$

See formulas (13.1.15) and (13.1.18) in Abramowitz and Stegun (1972).

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Figure 1: The curves of $V(x;b)$ with $\mu = 1, \delta = 4\%, \sigma = 5, \rho = 2\%$ and $x = 5,\ 10,\ 15,\ 25,\ 30$.

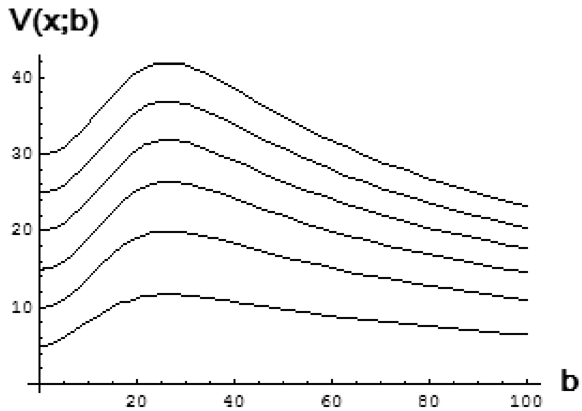

Figure 2: The curves of $V(x;b)$ with $\mu = 1$, $\delta = 4\%$, $\sigma = 5$, $\rho = 2\%$, $\tau = 6\%$ and $x = 5,\ 10,\ 15,\ 25,\ 30$.