Copulas and dependencies :

practical application for assessing the capital adequacy of a non life insurer

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Agenda

- **Purpose of the study**
- **Copulas : a short introduction**
- **Copulas used**
- **Portfolio overview**
- **Detection of dependencies**
- **Assessment of copula's parameter**
- **Selection of best copulas**
- **Modelling of capital needs**
- **Main results**
- **Conclusion**
- **Q&A**

Purpose of the study

Dependency between risks has often been based on strong assumptions in internal risk models : linear correlations, dependent risks with same margins or independent risks.

An optimal level of capital has become a major goal for insurers. Its modelling should take into consideration dependencies between risks as best as possible.

Copulas enable to model interaction between risks in a realistic way leading to many risk management applications.

This study presents a practical application where copulas are used to assess the capital adequacy of a non life insurer only adjusted for the risk of volatility in insurance losses.

Copulas : a short introduction

- **Definition**

Let $(X_1,\ldots X_n)$ be random variables with continuous distribution functions $F_1,\ldots,F_n$, respectively, and joint distribution F. Then there is a unique function C, called copula, such that

$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$$

- **Simple representation**

$$C(u_1,\ldots,u_n) = Pr(U_1 \leq u_1,\ldots,U_n \leq u_n)$$

- **Interest : separation of margins from multivariate dependence structure**

$C(u_1,\ldots,u_n)$ $\quad$ $F_1(x_1),\ldots,F_n(x_n)$

$F(x_1,\ldots,x_n)$

Copulas : a short introduction

■ Copulas and rank correlation coefficients

Kendall's tau $$\tau(X,Y) = 4 \iint_{[0,1]^2} C(u,v)\,dC(u,v) - 1$$

Spearman's rho $$\rho_S(X,Y) = 12 \iint_{[0,1]^2} C(u,v)\,du\,dv - 3$$

➤ Simple relation linking Kendall's tau and copula's parameter for some specific copulas

➤ Practically, ranks are being used by transforming observations $x_1,\ldots,x_n$ in $u_1,\ldots,u_n$ such that:

$$u_i = \frac{Rank(x_i)}{n+1}$$

Copulas used

- **Archimedean copulas (2-dimension)**

Let $\varphi : [0;1] \rightarrow [0;\infty[$ be a convex strictly decreasing continuous function such that $\varphi(1) = 0$ then

$$C(u_1,u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$$

is an archimedean copula and φ is called the generator of the copula.

➤ **Gumbel, Franck, Clayton, Independent**

- **HRT copula or Heavy Right Tail copula**

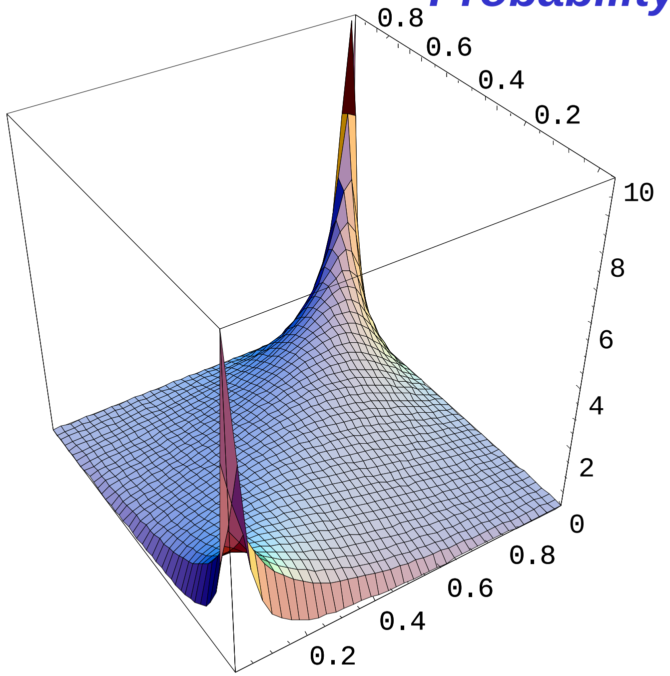
Gumbel

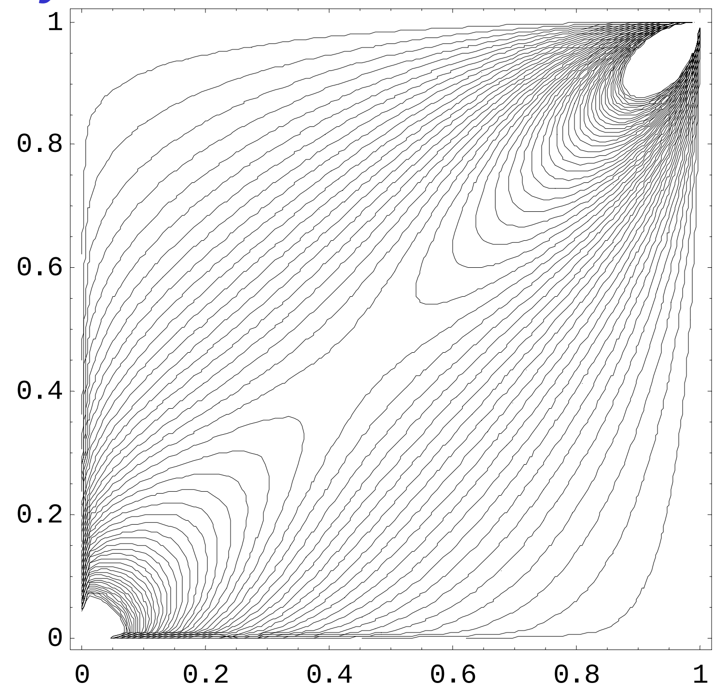
$$C(u,v) = \exp\left(-\left[(-\ln u)^a + (-\ln v)^a \right]^{\frac{1}{a}} \right)$$

$$\varphi(u) = (-\ln u)^a \ , \quad a \geq 1$$

$$\tau_a = 1 - 1/a$$

Probability Density Function





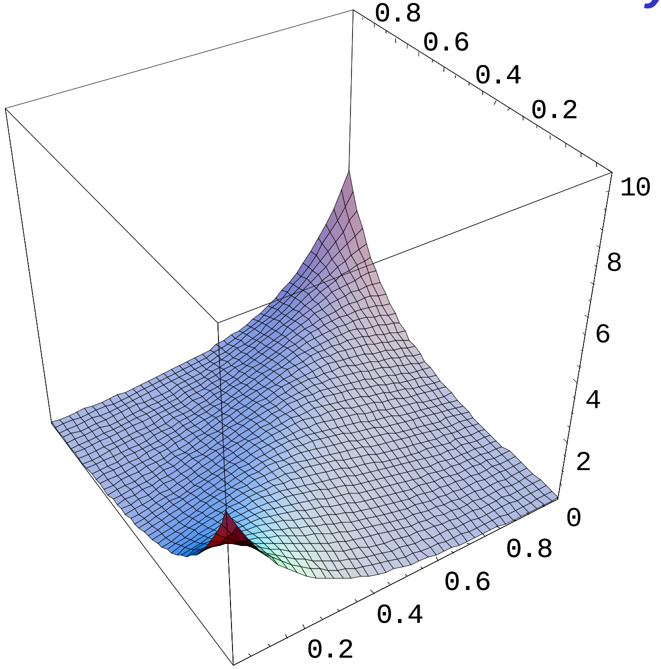
Franck

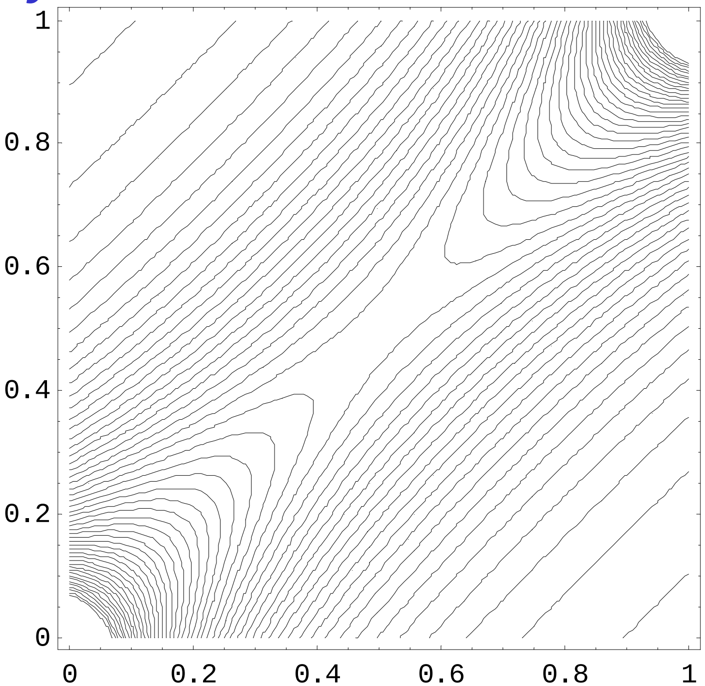
$$C(u,v) = -\frac{1}{a} \ln\left[1 + \frac{(e^{-au}-1)(e^{-av}-1)}{(e^{-a}-1)}\right]$$

$$\varphi(u) = -\ln\left(\frac{e^{-au}-1}{e^{-a}-1}\right), \quad a \neq 0$$

$$\tau_a = 1 - \frac{4}{a} + \frac{4}{a^2}\int_0^a \frac{t}{e^t-1}dt$$

Probability Density Function





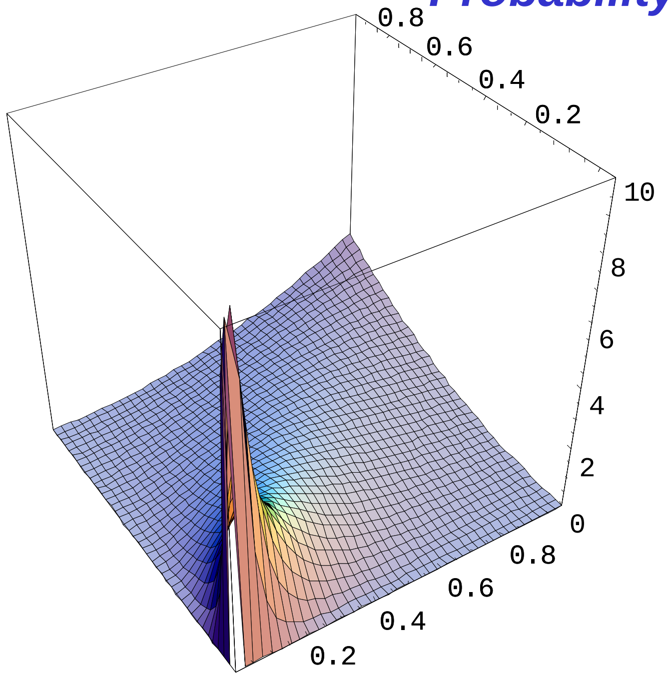
Clayton

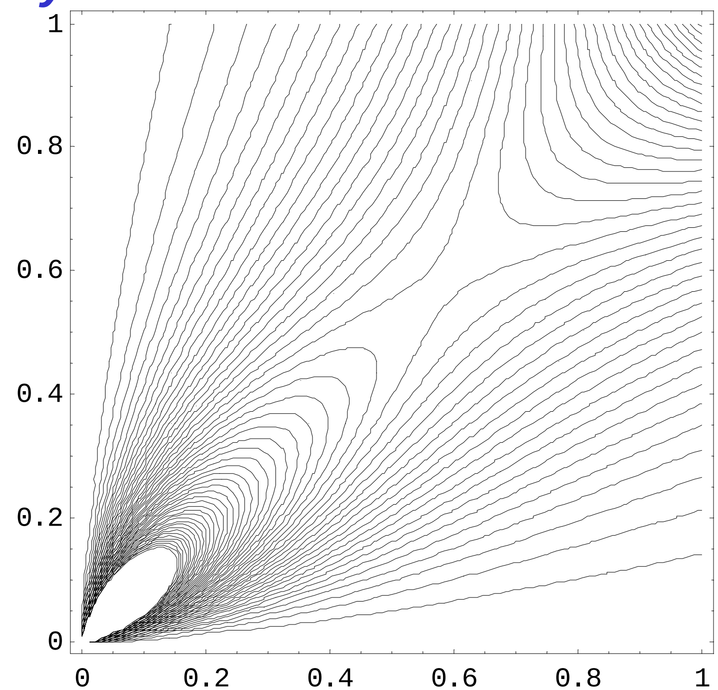
$$C(u,v) = (u^{-a} + v^{-a} - 1)^{-\frac{1}{a}}$$

$$\varphi(u) = \frac{(u^{-a} - 1)}{a}, \quad a > 0$$

$$\tau_a = \frac{a}{a+2}$$

Probability Density Function



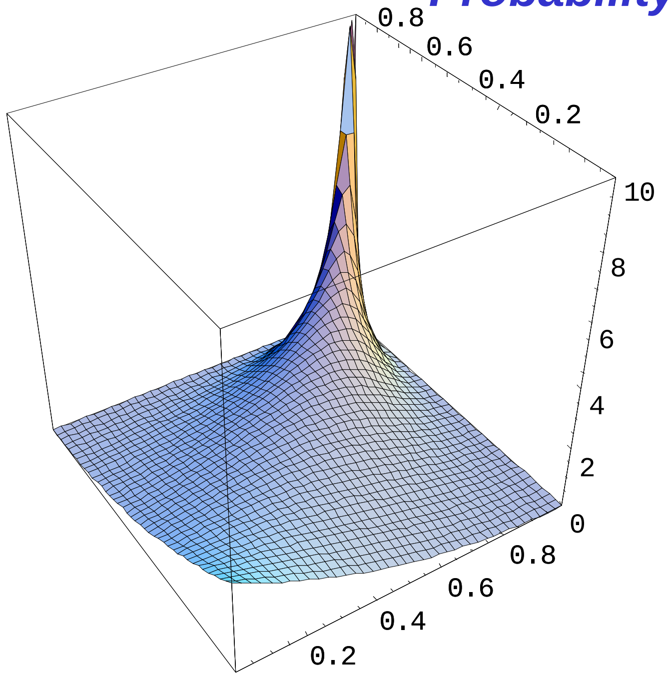


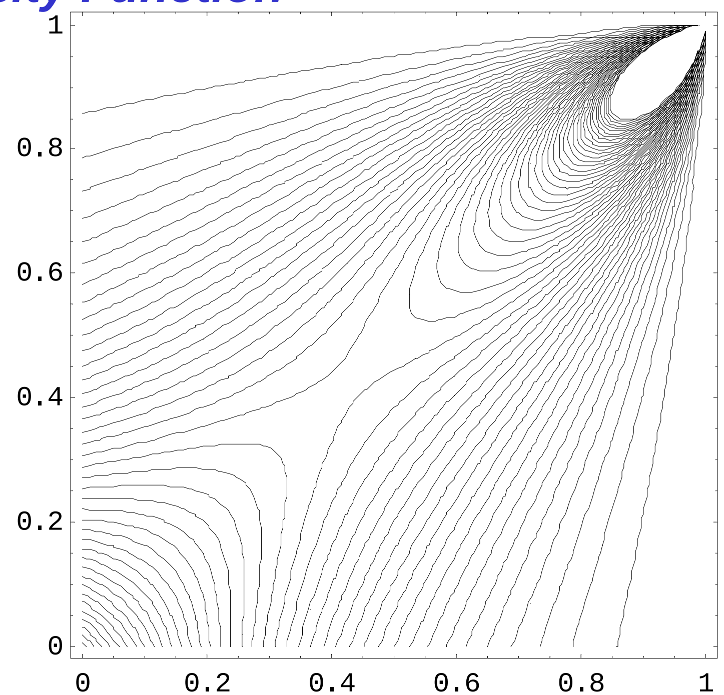
HRT

$$C(u,v) = u + v - 1 + \left((1-u)^{-a} + (1-v)^{-a} - 1\right)^{-\frac{1}{a}} \quad , \quad a > 0$$

$$\tau_a = \frac{a}{a+2}$$

Probability Density Function





Portfolio overview

- **3 lines of business**

Motor, Household, Commercial

- **23 variables and 72 observations**

X_i^j = aggregate loss on month *j* for cover *i*,

j = 1 to 72, *i* =1 to 23

Variables / Observations	MOTOR			HOUSEHOLD			COMMERCIAL		
	Cover 1	…	Cover 8	Cover 9	…	Cover 15	Cover 16	…	Cover 23
Jan. 1996									
Feb. 1996									
Mar. 1996									
⋮									
Oct. 2001									
Nov. 2001									
Dec. 2001									

72 months

Detection of dependencies

- **3 steps approach**

➢ Detection of significant dependencies using Kendall's tau, Spearman's rho and chi-square independence test

➢ Exclusion of negative dependencies

➢ Exclusion of dependencies difficult to explain according to experts

Dependencies between lines of business : 7 pairs

Dependencies within lines of business : 2 pairs

Detection of dependencies

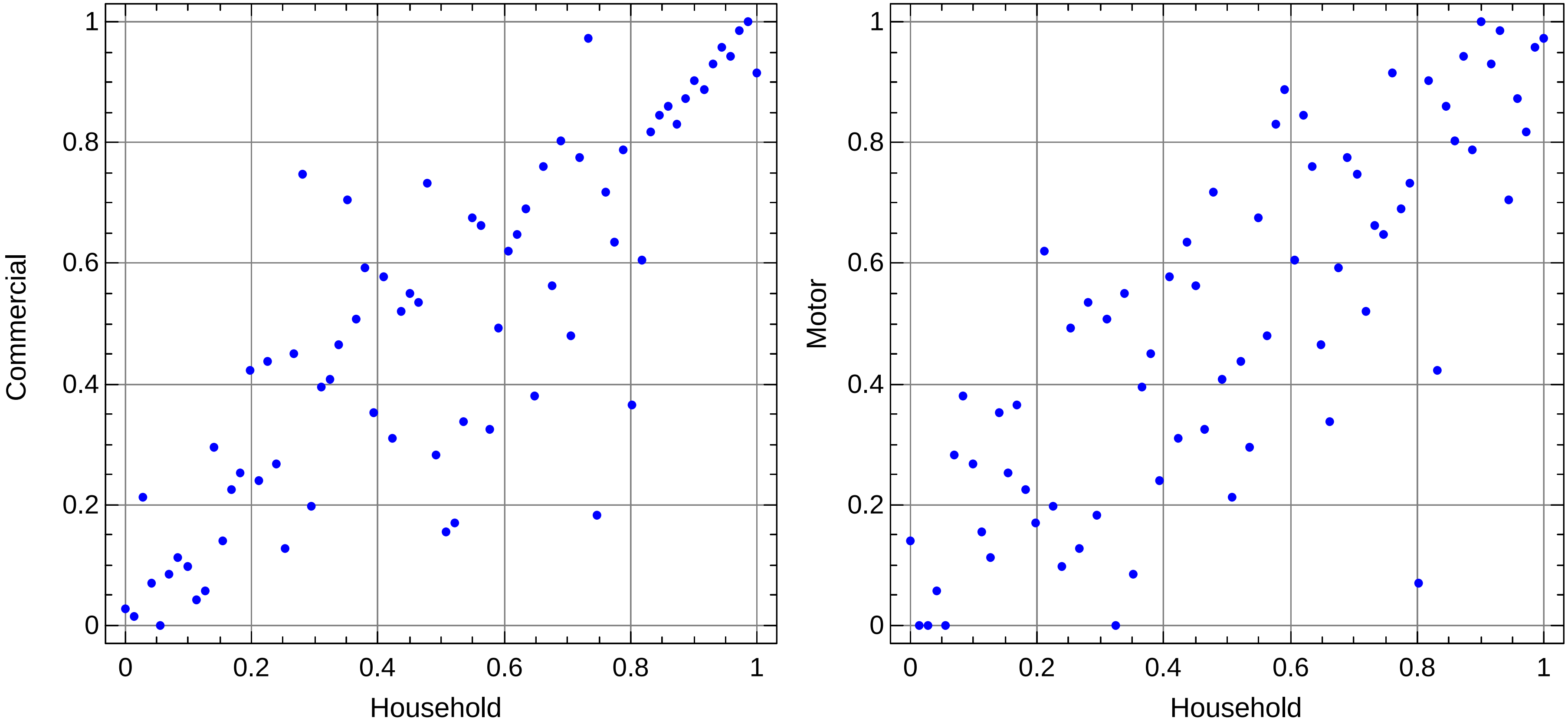
- **Dependency measures on 9 pairs selected**

Pairs sorted by Kendall's tau decreasing values		Kendall's tau	Spearman's rho	Chi-square Prob.
HOUSEHOLD - Storm	COMMERCIAL - Storm	0.68	0.83	0.00%
MOTOR - Storm	HOUSEHOLD - Storm	0.61	0.79	0.00%
MOTOR - Storm	COMMERCIAL - Storm	0.55	0.74	0.00%
MOTOR - Nat. Cat.	COMMERCIAL - Nat. Cat.	0.53	0.70	0.00%
HOUSEHOLD - Nat. Cat.	COMMERCIAL - Nat. Cat.	0.48	0.66	0.00%
COMMERCIAL - Business Interruption	COMMERCIAL - Fire	0.48	0.66	0.00%
MOTOR - Nat. Cat.	HOUSEHOLD - Nat. Cat.	0.42	0.59	0.02%
MOTOR - Theft	COMMERCIAL - Theft	0.34	0.48	0.13%
MOTOR - TPL Property Damage	MOTOR - Own Damage	0.31	0.44	0.00%

Detection of dependencies

- **Graphical example for Storm cover**

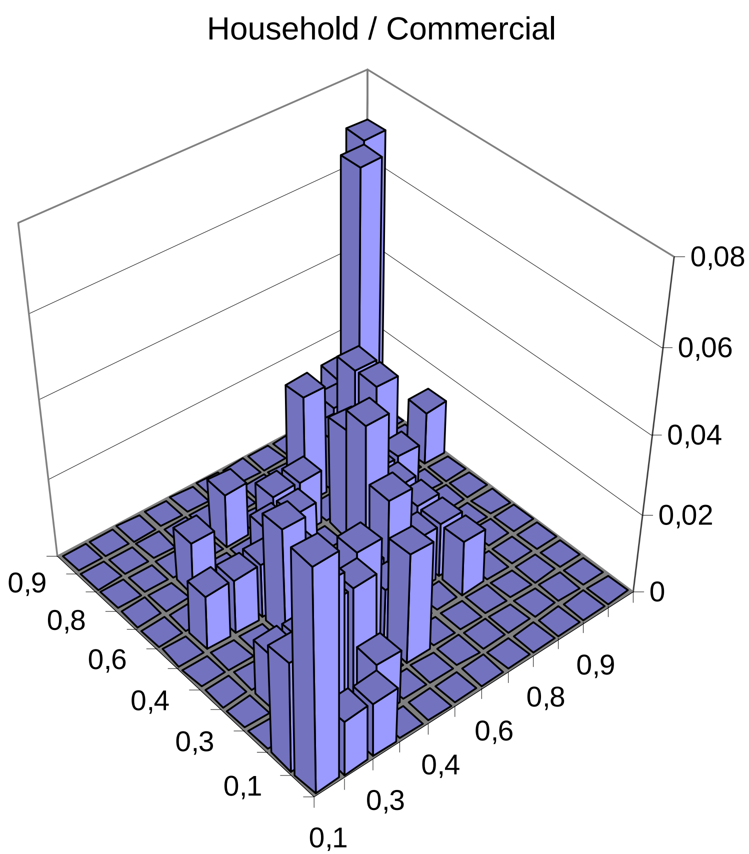
Uniform margins

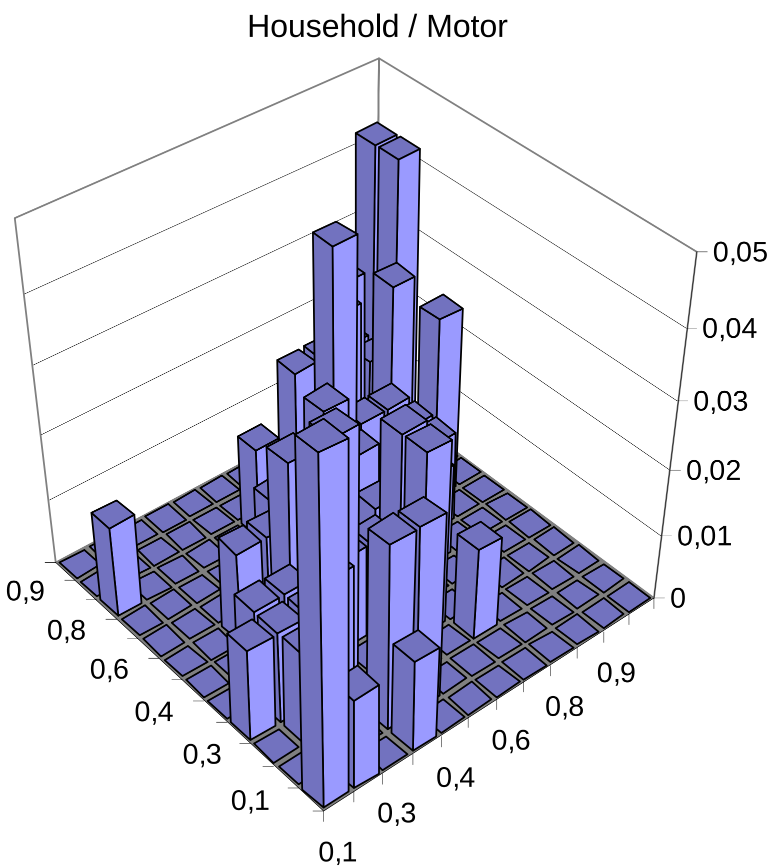


Detection of dependencies

- **Graphical example for Storm cover**

Empirical PDF of copulas





Assessment of copula's parameter

- ✓ Moments
- ✓ Maximum Likelihood

$$f(x_1,\dots,x_n) = c(F_1(x_1),\dots,F_n(x_n))\prod_{i=1}^{n} f_i(x_i)$$

$$l(\theta) = \sum_{t=1}^{T} \ln c(F_1(x_1^t),\dots,F_n(x_n^t)) + \sum_{t=1}^{T}\sum_{k=1}^{n} \ln f_k(x_k^t)$$

- ✓ Inference Functions for Margins

$$\hat{\theta}_k = \arg\max \sum_{t=1}^{T} \ln f_k(x_k^t;\theta_k)$$

$$\hat{a} = \arg\max \sum_{t=1}^{T} \ln c(F_1(x_1^t,\hat{\theta}_1),\dots,F_n(x_n^t,\hat{\theta}_n);a)$$

- ✓ **Canonical Maximum Likelihood**

$$\hat{a} = \arg\max \sum_{t=1}^{T} \ln c(\hat{u}_1^t,\dots,\hat{u}_n^t;a)$$

➤ copula's parameter independent from margins

Selection of best copulas

- **Example of chi-square goodness-of-fit test**

Empirical values

	0 - 0.25	0.25 - 0.50	0.50 - 0.75	0.75 - 1
0 - 0.25	13.00	5.00	0.00	0.00
0.25 - 0.50	2.00	7.00	9.00	0.00
0.50 - 0.75	3.00	5.00	6.00	4.00
0.75 - 1	0.00	1.00	3.00	14.00

Theorical values for Gumbel copula

	0 - 0.25	0.25 - 0.50	0.50 - 0.75	0.75 - 1
0 - 0.25	12.41	4.54	0.97	0.09
0.25 - 0.50	4.54	8.41	4.50	0.55
0.50 - 0.75	0.97	4.50	9.15	3.37
0.75 - 1	0.09	0.55	3.37	13.99

Weighted squared deviations

	Franck	Clayton	Gumbel	HRT	Indep.
C1	0.00	0.02	0.03	0.75	16.06
C2	0.50	0.06	0.24	0.00	1.39
C3	1.08	0.07	1.09	0.75	0.50
C4	0.06	2.08	0.00	0.00	20.06
C5	0.01	0.15	0.05	0.91	1.78
C6	0.16	2.35	0.00	0.01	6.69
C7	6.16	4.15	4.49	4.48	4.50
C8	0.24	0.03	0.05	0.05	0.06
C9	0.16	2.35	0.00	0.01	6.69
C10	0.01	0.15	0.05	0.91	1.78

Sum = chi-square values

8.38	**11.39**	**5.99**	**7.87**	**59.48**

p-value - 8 DF

0.40	**0.18**	**0.65**	**0.45**	**NS**

p-value - 9 DF

0.50	**0.25**	**0.74**	**0.55**	**NS**

Selection of best copulas

	copula	$\hat{a}_{CML}$
HOUSEHOLD - Storm / COMMERCIAL - Storm	Gumbel	2.917
HOUSEHOLD - Storm / MOTOR - Storm	Gumbel	2.194
MOTOR - Nat. Cat. / COMMERCIAL - Nat. Cat.	Franck	5.126
HOUSEHOLD - Nat. Cat. / COMMERCIAL - Nat. Cat.	Gumbel	1.801
COMMERCIAL - Business Interruption / COMMERCIAL - Fire	Franck	5.235
MOTOR - Theft / COMMERCIAL - Theft	Franck	3.163
MOTOR - TPL Property Damage / MOTOR - Own Damage	HRT	0.779

Modelling of capital needs

- **Simple model**

→ For cover i, $X_i^1, \dots, X_i^{72}$, independent and identically distributed : $X_i^j \sim F_{X_i}(x), \forall j$

→ Time horizon = one business year

→ Risk function X = Underwriting Result, function of $\sum_{i=1}^{23} X_i$

- **Risk measures used**

Ruin probability, VaR, TVaR et XTVaR

- **Monte-Carlo simulations**

Main results

Ruin probability	Indep.	Depend.	Dep. / Ind.
	0.005%	0.012%	2.40

α	VaR million of euros		Dep. / Ind.	TVaR million of euros		Dep. / Ind.
	Indep.	Depend.		Indep.	Depend.	
0.5%	22	30	1.34	38	52	1.36
0.1%	44	60	1.38	74	103	1.39
0.05%	59	82	1.39	98	137	1.40
0.01%	117	166	1.41	183	256	1.40

Conclusion

- Assuming independent risks would lead to a capital shortfall of about 40%.
- Upper tail dependency between the 3 lines of business is of utmost importance in the assessment of capital adequacy.
- This is just a first step … Modelling could be improved by making allowance for negative dependencies, through Student copula for instance, and adjusted for other risks…
- Research for multi-dimensional tests more powerful than chi-square goodness-of-fit test could improve the selection of the best copula.

Q&A