Legal Valuation Portfolio in Non-Life Insurance

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Topic of the presentation

Construction of a legal valuation portfolio to value the runoff situation of insurance risks.

Motivation

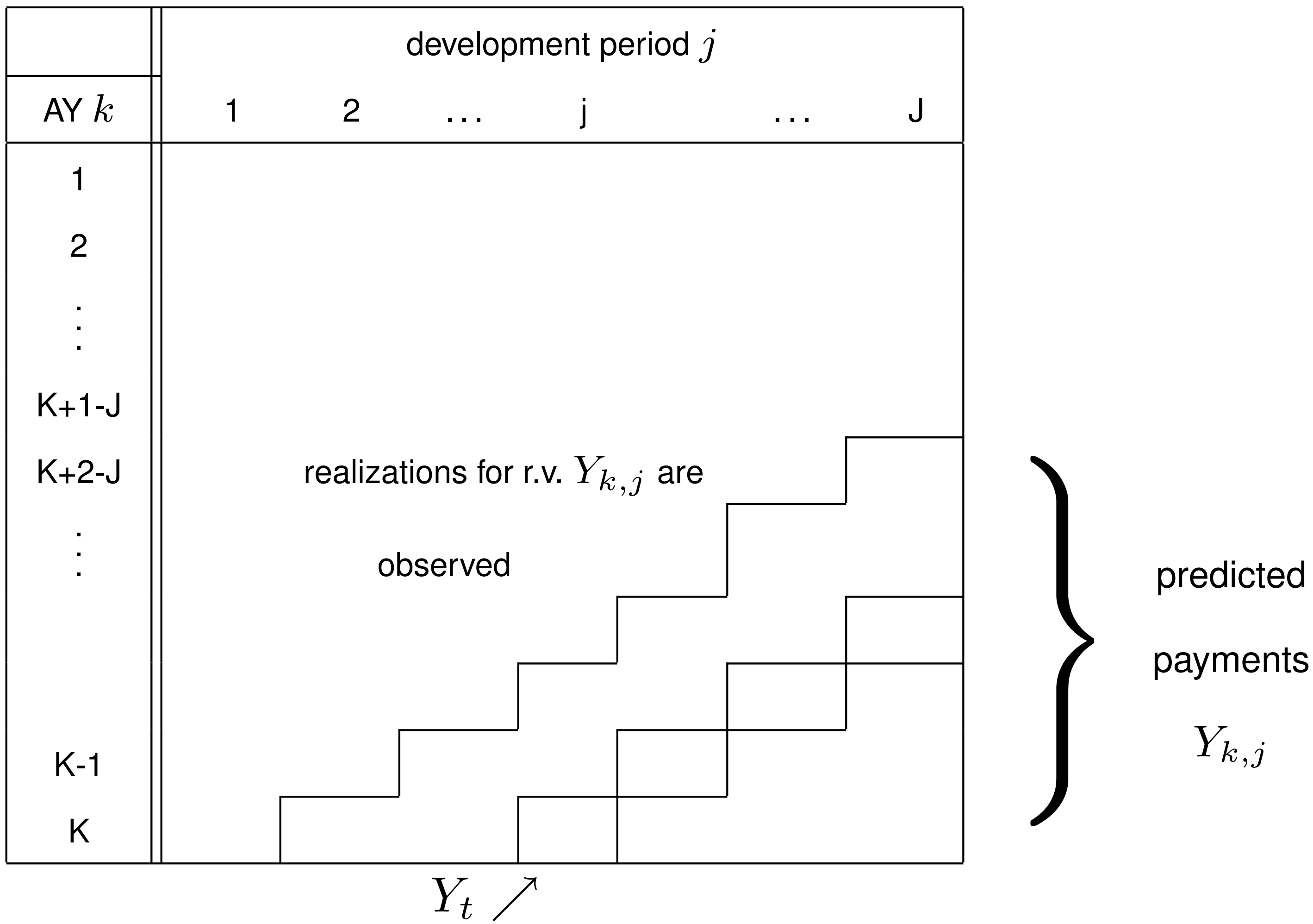
- New risk based solvency requirements

- Protect policyholder respectively the injured from the consequences of insolvency of an insurance company

Notation

$Y_{k,j}$	payments for accident year k in development period j where $1 \leq k \leq K$ and $1 \leq j \leq J \leq K$
$X_{k,j} = \sum_{l=1}^{j} Y_{k,l}$	cumulative payments for accident year k
$Y_t := \sum_{k+j=t+1} Y_{k,j}$	payments of the accounting year t
$\{\mathcal{F}_t\}_t$	Filtration of information for accounting years $t = 1, \ldots, K + J - 1$
$\mathcal{D}_K$	realizations $x_{k,j}$ of the cumulative payments $X_{k,j}$ at time $t = K$, i.e. $\{x_{k,j} \mid 1 \leq k \leq K$ and $1 \leq j \leq \min\{J, K - k + 1\}\}$

Upper trapezoid and lower triangle



Legal valuation portfolio (VaPo)

$E_t^{(K)} := E\left[Y_t \mid \mathcal{F}_K\right]$	expected payments for accounting year $t > K$
ρ_t^K	risk measure, depending on the information up to time K
$Z^{(t)}$	Zero Coupon Bond maturing at time t
i	risk-adjusted cost of capital rate

instrument/basis	Number of units
$Z^{(K+1)}$	$E_{K+1}^{(K)} + \rho_{K+1}^{(K)}$
$Z^{(K+2)}$	$E_{K+2}^{(K)} + i \cdot \rho_{K+2}^{(K)}$
$\vdots$	$\vdots$
$Z^{(K+\tau)}$	$E_{K+\tau}^{(K)} + i \cdot \rho_{K+\tau}^{(K)}$
$\vdots$	$\vdots$
$Z^{(K+J-1)}$	$E_{K+J-1}^{(K)} + i \cdot \rho_{K+J-1}^{(K)}$

Remarks

1) $\mathsf{E}_t^{(K)}$ should contain LAE to guarantee a smooth runoff of the liabilities

$\longrightarrow$ see Appendix A of the submitted paper; here ULAE are not taken into consideration

2) For the numeric example we use a risk measure of the form $\beta \cdot \sigma$.

Motivation: if $Y_t \sim \mathsf{N}(\mu_t, \sigma_t^2)$ then the VaR and the TailVaR are both of the form $\mu_t + \beta \cdot \sigma_t$.

$\longrightarrow$ Stochastic Model

Stochastic Model

Assumption 1 *For* $1 \leq j \leq J$ *and* $1 \leq k \leq K$ *let*

$$X_{k,j} = f_{j-1} \cdot X_{k,j-1} + \sigma_{j-1} \cdot \sqrt{X_{k,j-1}} \cdot \varepsilon_{k,j},$$

with $\sigma_{j-1} > 0$ *and* $\varepsilon_{k,j}$ *independent random variables with* $E[\varepsilon_{k,j}] = 0$ *and* *Var*$[\varepsilon_{k,j}] = 1$.

MACK assumptions for $1 \leq k \leq K, 1 \leq j \leq J$:

1) $E\left(X_{k,j} \middle| X_{k,1}, \ldots, X_{k,j-1}\right) = f_{j-1} \cdot X_{k,j-1}$

2) $\left(X_{k,1}, \ldots, X_{k,J}\right)$ and $\left(X_{l,1}, \ldots, X_{l,J}\right)$ are independent for $k \neq l$

3) Var $\left(X_{k,j} \middle| X_{k,1}, \ldots, X_{k,j-1}\right) = \sigma_{j-1}^2 \cdot X_{k,j-1}$.

Estimators for f_{j-1} and σ^2_{j-1}

$$\widehat{f}_{j-1} := \frac{\sum_{k=1}^{K-j+1} X_{k,j}}{S_{j-1}} \qquad \text{with} \quad S_{j-1} := \sum_{k=1}^{K-j+1} X_{k,j-1} \quad \text{(Chain-ladder estimator)}$$

$$\widehat{\sigma}^2_{j-1} := \frac{1}{K-j} \cdot \sum_{k=1}^{K-j+1} X_{k,j-1} \cdot \left(\frac{X_{k,j}}{X_{k,j-1}} - \widehat{f}_{j-1} \right)^2 \quad \text{for } 1 \le j \le J-1$$

Estimators for cumulative and non-cumulative payments

For

$$E\left[X_{k,j}\middle|\,\mathcal{D}_K\right] = x_{k,k^*} \cdot f_{k^*} \cdot \ldots \cdot f_{j-1}$$

where $k^* := K - k + 1$, we use the estimator

$$\widehat{X}_{k,j} := x_{k,k^*} \cdot \widehat{f}_{k^*} \cdot \ldots \cdot \widehat{f}_{j-1}.$$

Using $Y_{k,j} = X_{k,j} - X_{k,j-1}$ and $\widehat{Y}_{k,j} := \widehat{X}_{k,j} - \widehat{X}_{k,j-1}$ we obtain

- $E\left[Y_{k,j}\middle|\,\mathcal{D}_K\right] = x_{k,k^*} \cdot f_{k^*} \cdot \ldots \cdot f_{j-2} \cdot \big(f_{j-1} - 1\big)$
- $\widehat{Y}_{k,j} = x_{k,k^*} \cdot \widehat{f}_{k^*} \cdot \ldots \cdot \widehat{f}_{j-2} \cdot \big(\widehat{f}_{j-1} - 1\big).$

Prediction error for single accident years

$$E\left[\left(\widehat{X}_{k,j}-X_{k,j}\right)^2\middle|\mathcal{D}_K\right]=\underbrace{E\left[\left(X_{k,j}-E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]}_{\text{process variance}}$$

$$+\underbrace{\left(\widehat{X}_{k,j}-E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2}_{\text{estimation error}}$$

where the observations in the upper trapezoid

$$\mathcal{D}_K=\{x_{k,j}|\,1\le k\le K \text{ and } 1\le j\le \min\{J,K-k+1\}\}$$

are given.

Estimator of the process variance for $X_{k,j}$

$$\widehat{E}\left[\left(X_{k,j} - E\left[X_{k,j}\middle|\,\mathcal{D}_K\right]\right)^2\middle|\,\mathcal{D}_K\right] := \underbrace{\left(\widehat{X}_{k,j}\right)^2 \cdot \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}_l^2}{\widehat{X}_{k,l} \cdot \widehat{f}_l^2}}_{=:\Gamma_{k,j}^2}.$$

Estimation error for $X_{k,j}$

$$\left(\widehat{X}_{k,j} - E\left[X_{k,j} \middle| \mathcal{D}_K\right]\right)^2 = x_{k,k^*}^2 \cdot \left(\widehat{f}_{k^*} \cdot \ldots \cdot \widehat{f}_{j-1} - f_{k^*} \cdot \ldots \cdot f_{j-1}\right)^2$$

In order to determine the estimation error, we average over the possible values $\widehat{f}_{k^*}, \widehat{f}_{k^*+1}, \ldots, \widehat{f}_{j-1}$, given the upper trapezoid $\mathcal{D}_K$.

We propose the following estimator for the estimation error:

$$\left(\widehat{X}_{k,j} - \widehat{E\left[X_{k,j} \middle| \mathcal{D}_K\right]}\right)^2 := \widehat{X}_{k,j}^2 \cdot \underbrace{\left(\prod_{l=k^*}^{j-1}\left(1 + \frac{\widehat{\sigma}_l^2}{\widehat{f}_l^2 \cdot s_l}\right) - 1\right)}_{=:\triangle_{k,j}^2}.$$

MACK's estimator of the estimation error

$$\left(\widehat{X}_{k,j} - \widehat{E\left[X_{k,j}\middle|\mathcal{D}_K\right]}\right)^2 := \widehat{X}^2_{k,j} \cdot \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}^2_l}{\widehat{f}^2_l \cdot s_l}.$$

Remark: MACK's estimator can be seen as a linear approximation to the above estimator:

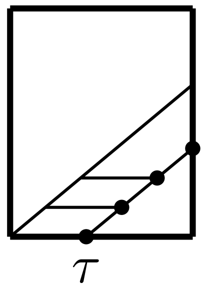
$$\Delta^2_{k,j} = \prod_{l=k^*}^{j-1}\left(1 + \frac{\widehat{\sigma}^2_l}{\widehat{f}^2_l \cdot s_l}\right) - 1 = 1 + \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}^2_l}{\widehat{f}^2_l \cdot s_l} + \cdots - 1$$

$$\approx \sum_{l=k^*}^{j-1} \frac{\widehat{\sigma}^2_l}{\widehat{f}^2_l \cdot s_l}.$$

Prediction error for accounting years (non-cumulative payments)

For our VaPo we consider pooled, non-cumulative payments $Y_t = \sum_{k+j=t+1} Y_{k,j}$.

Analogously as before we obtain



a) best-estimates $\widehat{Y}_t := \sum_{k+j=t+1} \widehat{Y}_{k,j}$ for these non-cumulative payments

b) we estimate the prediction error (special attention has to be paid to the dependence when pooling accident years)

$$E\left[\left(\widehat{Y}_{K+\tau} - Y_{K+\tau}\right)^2 \middle| \mathcal{D}_K\right]$$
$$= \underbrace{E\left[\left(Y_{K+\tau} - E\left[Y_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2 \middle| \mathcal{D}_K\right]}_{\text{process variance}} + \underbrace{\left(\widehat{Y}_{K+\tau} - E\left[Y_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2}_{\text{estimation error}}.$$

Estimator for the Prediction error for accounting years ($Y_{K+\tau}$)

We obtain the following estimator

$$\widehat{E}\left[\left(\widehat{Y}_{K+\tau} - Y_{K+\tau}\right)^2 \middle| \mathcal{D}_K\right] := \sum_{\substack{k+j=K+1+\tau \\ \tau+1 \le j \le J}} \left(\Gamma_{k,j}^2 + \Gamma_{k,j-1}^2 \cdot \left(1 - 2 \cdot \widehat{f}_{j-1}\right)\right) \longleftarrow$$

process variance

estimation error

$$+ \sum_{k^*=1}^{J-\tau} \Delta_{k,k^*+\tau-1}^2 \cdot \widehat{Y}_{k,k^*+\tau}^2 + \left(\Delta_{k,k^*+\tau}^2 - \Delta_{k,k^*+\tau-1}^2\right) \cdot \widehat{X}_{k,k^*+\tau}$$

$$+ 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1, J-\tau\}} \Big(\Delta_{l,k^*+\tau-1}^2 \cdot \widehat{Y}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau}$$

$$+ \left(\Delta_{l,k^*+\tau}^2 - \Delta_{l,k^*+\tau-1}^2\right) \cdot \widehat{X}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau}\Big).$$

Example: Run-off-trapezoid (non cumulative payments)

AY k	$j=$1	2	3	4	5	6	7	8	9	10
1	12666	5450	868	199	257	22	18	10	4	1
2	11890	6766	824	257	46	-25	0	0	2	0
3	11618	7784	1113	257	53	9	6	74	2	0
4	13442	11807	1558	327	120	33	-3	1	0	30
5	11643	6125	744	764	99	7	5	0	-10	8
6	11419	7298	792	327	-5	22	8	-1	12	
7	11168	7489	1089	357	52	27	24	34		
8	11020	5942	551	161	35	24	9			
9	10285	5300	519	133	203	12				
10	13933	11110	1297	430	113					
11	11812	9018	1606	141						
12	10013	5832	486							
13	11684	6384								
14	10646									

Example: Estimated non cumulative payments

	development period j								
AY k	2	3	4	5	6	7	8	9	10
6									7
7								2	7
8							14	1	**6**
9						7	13	**1**	6
10					19	11	**22**	2	10
11				106	16	**9**	18	2	8
12			242	78	**12**	7	13	1	6
13		896	281	**91**	14	8	16	2	7
14	6719	861	**270**	87	13	8	15	1	7

τ	1	2	3	4	5	6	7	8	9
$E_{K+\tau}^{(K)}$	8013	1270	**412**	134	46	33	23	9	7

Prediction error of $\widehat{Y}_{K+\tau}$ and the estimator of $\sigma^{(K)}_{K+\tau}$

Process variance for accounting year $K+\tau$								
15	16	17	18	19	20	21	22	23
2296569	107372	40002	8340	1184	989	858	155	103

Estimation error for accounting year $K+\tau$								
15	16	17	18	19	20	21	22	23
161265	8592	3466	825	157	139	113	26	18

Prediction error of $\widehat{Y}_{K+\tau}$								
15	16	17	18	19	20	21	22	23
2457834	115964	43468	9165	1341	1129	971	181	121

$\widehat{\widehat{\sigma}}^{(K)}_{K+\tau} := \sqrt{\widehat{E}\left[\left(\widehat{Y}_{K+\tau} - Y_{K+\tau}\right)^2 \middle\vert \mathcal{D}_K\right]}$								
15	16	17	18	19	20	21	22	23
1568	341	208	96	37	34	31	13	11

Example of the valuation portfolio: with $\rho_{K+\tau}^{(K)} = 2.326 \cdot \widehat{\widehat{\sigma}}_{K+\tau}^{(K)}$

$$i = 6\%.$$

$Z^{(K+\tau)}$	$E_{K+\tau}^{(K)}$	$\widehat{\widehat{\sigma}}_{K+\tau}^{(K)}$	$\rho_{K+\tau}^{(K)}$	$\rho_{K+\tau}^{K}$ or $i \cdot \rho_{K+\tau}^{K}$	number of units
$Z^{(K+1)}$	8013	1568	3647	3647	11660
$Z^{(K+2)}$	1270	341	792	48	1318
$Z^{(K+3)}$	412	208	485	29	441
$Z^{(K+4)}$	134	96	223	13	148
$Z^{(K+5)}$	46	37	85	5	51
$Z^{(K+6)}$	33	34	78	5	37
$Z^{(K+7)}$	23	31	72	4	27
$Z^{(K+8)}$	9	13	31	2	10
$Z^{(K+9)}$	7	11	26	2	8
sum	9946			3755	13701

Different Term structures

	1	2	3	4	5	6	7	8	9
BPV	0.88%	1.14%	1.36%	1.57%	1.75%	1.91%	2.05%	2.18%	2.29%
3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%
0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%

Prices of the legal VaPo

	BPV	3.5%	0.0%
price of the VaPo	13528	13131	13701
difference in %	1.26%	4.16%	0.00%