Legal Valuation Portfolio in Non-Life Insurance

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Abstract

We study the Legal Valuation Portfolio (VaPo) for the runoff of a non-life insurance company. Therefore we introduce financial instruments as basis elements to value the single runoff payments (cash flows). Since from a regulatory point of view it is not sufficient to just have best-estimates reserves we consider in addition a risk-adjusted margin to protect the policyholder from adverse runoff developments. This leads to an investment portfolio which is needed to guarantee a smooth runoff of all insurance liabilities.

1 Legal Valuation Portfolio (VaPo)

1.1 Motivation

Worldwide, regulators look for new methods to calculate solvency requirements. It is generally understood that the new methods should lead to risk-adjusted solvency margins. The general idea behind all these initiatives is to protect the policyholder, respectively the injured, from the consequences of an insolvency or a bankruptcy of an insurance company. In the present paper we give general legal valuation principles to value the runoff situation of insurance risks. Our main scope is to develop a legal valuation portfolio, i.e. with the help of financial instruments we determine the necessary reserves to guarantee (from a regulatory point of view) the smooth runoff of all liablities. These reserves contain the best-estimate provisions for the pure claims payments, the claims handling costs and the cost-of-capital charge generated by the risk margin, which covers certain adverse developments in the reserves.

1.2 Legal Valuation Portfolio construction

We denote by $Y_{k,j}$ the non-cumulative payments for accident year k ($1 \leq k \leq K$) in development period j ($1 \leq j \leq J \leq K$). The cumulative payments of accident year k up to development period j are denoted by

$$X_{k,j} := \sum_{l=1}^{j} Y_{k,l}. \tag{1.1}$$

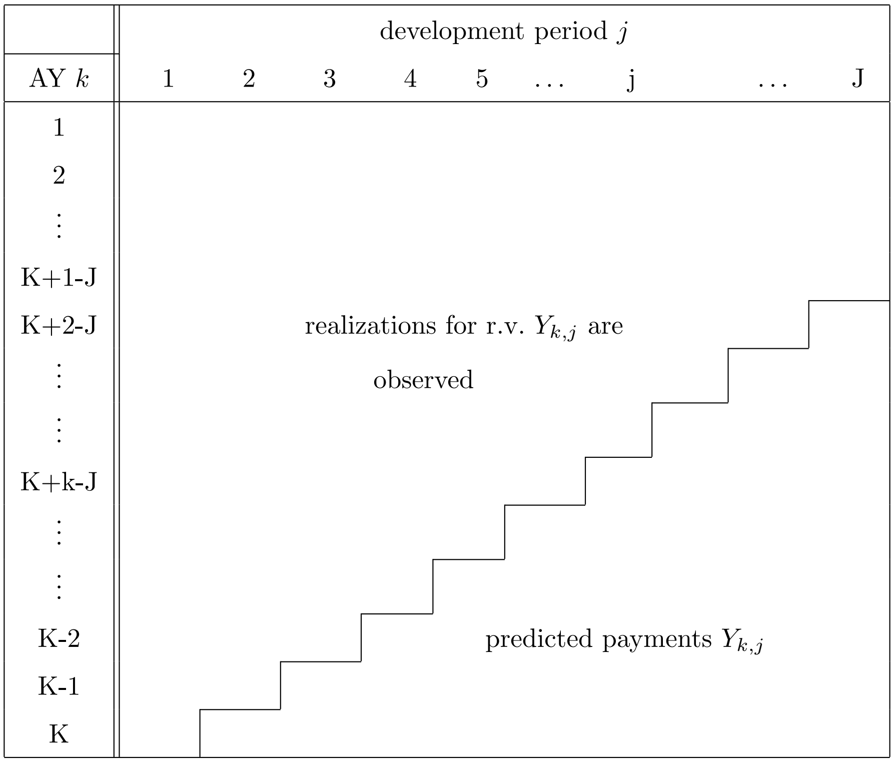
Denote by Y_t the payments in the accounting year t ($1 \leq t \leq K+J-1$), i.e. this is the sum of payments on the t-diagonal of the claims development trapezoid:

$$Y_t := \sum_{\substack{k+j=t+1 \\ 1\leq j\leq J}} Y_{k,j}. \tag{1.2}$$

Let $\{\mathcal{F}_t\}_{1\leq t\leq K+J-1}$ denote the filtration of information known at time t ($1 \leq t \leq K+J-1$). This is – unless we have additional information besides the observed claim payments – the information in an upper trapezoid including the t-diagonal.

We assume that we have realizations for the upper trapezoid $\{Y_{k,j} |\ 1 \leq k \leq K$ and $1 \leq j \leq \min\{J, K-k+1\}\}$ and we want to predict the lower triangle $\{Y_{k,j} |\ 1 \leq k \leq$

K and $K-k+1 < j \leq J\}$. This also means that K denotes the last observed accounting year, or more intuitively that we are at time K.



For future accounting years $t > K$ we want to estimate the (conditional) expected payments

$$E_t^{(K)} \quad := \quad E\left[Y_t \middle| \mathcal{F}_K\right] \tag{1.3}$$

with an appropriate corresponding risk measure ρ_t^K depending on the information up to time K. This allows us to construct the *Legal Valuation Portfolio (VaPo)* at time K:

instrument/basis	Number of units
$Z^{(K+1)}$	$E_{K+1}^{(K)} + 1 \cdot \rho_{K+1}^{(K)}$
$Z^{(K+2)}$	$E_{K+2}^{(K)} + i \cdot \rho_{K+2}^{(K)}$
$\vdots$	$\vdots$
$Z^{(K+\tau)}$	$E_{K+\tau}^{(K)} + i \cdot \rho_{K+\tau}^{(K)}$
$\vdots$	$\vdots$
$Z^{(K+J-1)}$	$E_{K+J-1}^{(K)} + i \cdot \rho_{K+J-1}^{(K)}$

The construction of the Valuation Portfolio is the first and most important step in the process of assessing the liabilities of a risk carrier. The Valuation Portfolio contains a multitude of financial instruments (basic units) which generate the liability cashflow. The choice of these basic units is crucial. In this paper we have chosen the instruments

$$Z^{(t)} \sim \text{Zero Coupon Bond maturing at time t}$$

for $t = K+1, K+2, \ldots, K+J-1$. One might as well have tried to use inflation adjusted Zero Coupon Bonds, but this idea is not further pursued here.

It is important to realize that in our approach one discusses the question, how to assign money values to $Z^{(t)}$ (e.g. marked prizes, conventional actuarial values etc.) only at the last step of the analysis. Hence the results obtained are applicable for any arbitrary accounting rule.

Remark 1.1

- Depending on the situation there are different methods to estimate the (conditional) expected payments $E^{(K)}_{K+\tau}$, e.g. the choice of an appropriate method will depend on the line of business. In practice, the resulting estimate of $E^{(K)}_{K+\tau}$ by an appropriate method is often called "best-estimate".

- i denotes the risk-adjusted cost of capital rate. This means that i is the risk spread to the risk free interest rate, which reflects the price for the risk exposure of the solvency margin $\rho^{(K)}_{K+\tau}$.

- It is convenient to first derive the VaPo for each accident year. When pooling accident years for an accounting year one has to take into account correlation effects between the accident years.

Interpretation: Every expected future cashflow $E^{(K)}_{K+\tau}$ is evaluated with its instrument $Z^{(K+\tau)}$. From a regulatory point of view, it is not sufficient to just hold the best-estimate reserves for $E^{(K)}_{K+\tau}$. In fact, we need to hold a risk margin $\rho^{(K)}_{K+\tau}$ for every of these (stochastic) cashflows, which guarantees that also certain adverse developments can be covered. In the first period $\tau = 1$ we hold the whole capital $\rho^{(K)}_{K+\tau}$, whereas for later periods we only hold the cost-of-capital $i \cdot \rho^{(K)}_{K+\tau}$.

Organisation of the paper: Our goal is to calculate the sensitivities of the legal valuation portfolio VaPo. We therefore use different interest rate curves for $Z^{(K+\tau)}$ to analyze the interest rate sensitivity and we allow different risk measures $\rho_{K+\tau}^{(K)}$. We proceed as follows: 1) We analyze the mean square error of prediction, which leads to a result that we compare to the famous Mack formula. 2) In the example we chose – for illustrative purposes – a specific risk measure. Other choices could have been taken. On the other hand we use different interest rate curves to demonstrate the sensitivity of the value of the VaPo. 3) In order to obtain a legal VaPo also run-off costs have to be taken into account (they guarantee a smooth runoff of the liabilities). We briefly describe the New York method to calculate (unallocated) loss adjustment expenses ULAE. This is done in the appendix.

2 Mean square error of prediction

At time $t = K$ the realizations $x_{k,j}$ of the cumulative payments $X_{k,j}$ are given by the upper trapezoid $\mathcal{D}_K := \{x_{k,j} |\ 1 \leq k \leq K \text{ and } 1 \leq j \leq \min\{J, K-k+1\}\}$, moreover $\widehat{X}_{k,j}$ denote the estimators of the future cumulative payments $X_{k,j}$ ($1 \leq k \leq K$ and $K-k+1 < j \leq J$).

2.1 Prediction error for single accident years

As usual the prediction error of $\widehat{X}_{k,j}$ can be decomposed into two parts: a) stochastic error (process variance) and b) estimation error (c.f. Mack [4], Section 3):

$$E\left[\left(\widehat{X}_{k,j} - X_{k,j}\right)^2 \middle| \mathcal{D}_K\right] = \underbrace{E\left[\left(X_{k,j} - E\left[X_{k,j}|\,\mathcal{D}_K\right]\right)^2 \middle| \mathcal{D}_K\right]}_{\text{process variance}} + \underbrace{\left(\widehat{X}_{k,j} - E\left[X_{k,j}|\,\mathcal{D}_K\right]\right)^2}_{\text{estimation error}}. \tag{2.1}$$

The process variance $\mathrm{Var}\left(X_{k,j}|\,\mathcal{D}_K\right) = E\left[\left(X_{k,j} - E\left[X_{k,j}|\,\mathcal{D}_K\right]\right)^2 \middle| \mathcal{D}_K\right]$ originate from the stochastic movement of the process, whereas the estimation error reflects the uncertainty in the estimation of the expectation (mean value). For the cumulative payments $X_{k,j}$ we make the following assumptions:

Assumption 2.1 *For $1 \leq j \leq J$ and $1 \leq k \leq K$ let*

$$X_{k,j} = f_{j-1} \cdot X_{k,j-1} + \sigma_{j-1} \cdot \sqrt{X_{k,j-1}} \cdot \varepsilon_{k,j}, \tag{2.2}$$

with $\sigma_{j-1} > 0$ and $\varepsilon_{k,j}$ independent random variables with $E\left[\varepsilon_{k,j}\right] = 0$ and $Var\left[\varepsilon_{k,j}\right] = 1$.

The parameters $f_0, \ldots, f_{J-1}$ are called development factors, chain-ladder factors or "age-to-age" factors.

Remark 2.2

- $X_{k,0}$ $(1 \leq k \leq K)$ can be interpreted as the (deterministic) premium P_k earned for accident year k. In an enlarged trapezoid this allows the modelling of next year business.

- Theoretically, in assumption 2.1 we could have negative values for the cumulative payments $X_{k,j}$. We regard this problem of no practical relevance.

- The assumptions for the derivation of the well-known Mack formula to estimate the mean square error follow from assumption 2.1. In fact, the assumptions of Mack are more general.

- The model assumptions are of importance especially for the derivation of the estimation error. They reflect the mechanism of generating sets of "other possible" observations: Having an observation $x_{k,j-1}$ formula (2.2) tells us how to model/generate the succeeding observations $X_{k,j}$.

Assumption (2.2) implies for $X_{k,j}$ with $K - k + 1 < j \leq J$ that

$$E\left[X_{k,j} \middle| \mathcal{D}_K\right] = x_{k,K-k+1} \cdot f_{K-k+1} \cdot \ldots \cdot f_{j-1}. \tag{2.3}$$

In the sequel $\widehat{f}_0, \ldots, \widehat{f}_{J-1}$ denote the estimators of the development factors $f_0, \ldots, f_{J-1}$ and for the estimation of the future payments $X_{k,j}$ $(K - k + 1 < j \leq J)$ we use the classical *chain-ladder model*. I.e. we have

$$\widehat{X}_{k,j} := x_{k,K-k+1} \cdot \widehat{f}_{K-k+1} \cdot \ldots \cdot \widehat{f}_{j-1} \tag{2.4}$$

as the (chain-ladder) estimator of $E\left[X_{k,j}\middle|\mathcal{D}_K\right]$ and we use

$$\widehat{f}_{l-1} := \frac{\sum_{k=1}^{K-l+1} X_{k,l}}{S_{l-1}} \tag{2.5}$$

with

$$S_{l-1} := \sum_{k=1}^{K-l+1} X_{k,l-1} \tag{2.6}$$

and realizations $x_{k,l}$ and s_{l-1} to estimate f_{l-1}.

The appropriate estimator for the variance σ_{j-1}^2 is given by

$$\widehat{\sigma}_{j-1}^2 := \frac{1}{K-j} \cdot \sum_{k=1}^{K-j+1} X_{k,j-1} \cdot \left(\frac{X_{k,j}}{X_{k,j-1}} - \widehat{f}_{j-1}\right)^2 \tag{2.7}$$

for $1 \leq j \leq J-1$.

Remark 2.3

- Estimator (2.5) is minimum variance among linear estimators.
- Estimator (2.7) is minimum variance for normal $\varepsilon_{k,j}$.
- If we have enough data (i.e. $K > J$), we are able to estimate σ_{J-1}^2 with (2.7), otherwise there are several possibilities to estimate σ_{J-1}^2 (see e.g. [4]).

2.1.1 Process variance

Using (2.3) we obtain for the process variance in formula (2.1): Choose $1 \leq k \leq K$ and $K-k+1 < j \leq J$

$$E\left[\left(X_{k,j} - E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right] = E\left(X_{k,j}^2\middle|\mathcal{D}_K\right) - x_{k,K-k+1}^2 \cdot f_{K-k+1}^2 \cdot \ldots \cdot f_{j-1}^2.$$

Moreover, using (2.2) we have

$$\begin{aligned} E\left(X_{k,j}^2\middle|\mathcal{D}_K\right) &= f_{j-1}^2 \cdot E\left(X_{k,j-1}^2\middle|\mathcal{D}_K\right) + \sigma_{j-1}^2 \cdot E\left(X_{k,j-1} \cdot \varepsilon_{k,j}^2\middle|\mathcal{D}_K\right) \\ &\vdots \\ &= f_{j-1}^2 \cdot f_{j-2}^2 \cdot \ldots \cdot f_{K-k+1}^2 \cdot x_{k,K-k+1}^2 \\ &\quad + \sigma_{j-1}^2 \cdot f_{j-2} \cdot f_{j-3} \cdot \ldots \cdot f_{K-k+1} \cdot x_{k,K-k+1} \\ &\quad + f_{j-1}^2 \cdot \sigma_{j-2}^2 \cdot f_{j-3} \cdot \ldots \cdot f_{K-k+1} \cdot x_{k,K-k+1} \\ &\quad + \ldots + f_{j-1}^2 \cdot f_{j-2}^2 \cdot \ldots \cdot f_{K-k+2}^2 \cdot \sigma_{K-k+1}^2 \cdot x_{k,K-k+1}. \end{aligned} \tag{2.8}$$

Using (2.8) we obtain for the process variance:

$$E\left[\left(X_{k,j}-E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]=x_{k,K-k+1}\cdot\sum_{l=K-k+1}^{j-1}\sigma_l^2\cdot\prod_{m=l+1}^{j-1}f_m^2\cdot\prod_{m=K-k+1}^{l-1}f_m$$

Finally, using (2.4) the process variance can be rewritten in the form:

$$E\left[\left(X_{k,j}-E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]=\left(E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2\cdot\sum_{l=K-k+1}^{j-1}\frac{\sigma_l^2}{E\left[X_{k,l}\middle|\mathcal{D}_K\right]\cdot f_l^2}. \tag{2.9}$$

Replacing $E\left[X_{k,j}\middle|\mathcal{D}_K\right]$, $E\left[X_{k,l}\middle|\mathcal{D}_K\right]$ and the parameters $\sigma_{K-k+1}^2,\ldots,\sigma_{j-1}^2, f_{K-k+1},\ldots,f_{j-1}$ by their estimators leads to the following estimator of the process variance

$$\widehat{E}\left[\left(X_{k,j}-E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]:=\underbrace{\left(\widehat{X}_{k,j}\right)^2\cdot\sum_{l=K-k+1}^{j-1}\frac{\widehat{\sigma}_l^2}{\widehat{X}_{k,l}\cdot\widehat{f}_l^2}}_{=:\Gamma_{k,j}^2}. \tag{2.10}$$

$\Gamma_{k,j}^2$ can be rewritten in a recursive form:

$$\Gamma_{k,j+1}^2=\Gamma_{k,j}^2\cdot\widehat{f}_j^2+\widehat{\sigma}_j^2\cdot\widehat{X}_{k,j}. \tag{2.11}$$

Using (2.10), we obtain for the non-cumulative payments $Y_{k,j}$ the following estimator of the process variance

$$\begin{aligned}\widehat{E}\left[\left(Y_{k,j}-E\left[Y_{k,j}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]&:=\Gamma_{k,j}^2+\Gamma_{k,j-1}^2\cdot\left(1-2\cdot\widehat{f}_{j-1}\right)\\&=\Gamma_{k,j-1}^2\cdot(\widehat{f}_{j-1}-1)^2+\widehat{\sigma}_{j-1}^2\cdot\widehat{X}_{k,j-1}.\end{aligned} \tag{2.12}$$

2.1.2 Estimation error

Using (2.3) and (2.4) leads to the following formula for the estimation error:

$$\left(\widehat{X}_{k,j}-E\left[X_{k,j}\middle|\mathcal{D}_K\right]\right)^2=x_{k,K-k+1}^2\cdot\left(\widehat{f}_{K-k+1}\cdot\ldots\cdot\widehat{f}_{j-1}-f_{K-k+1}\cdot\ldots\cdot f_{j-1}\right)^2, \tag{2.13}$$

where the realizations of the estimators $\widehat{f}_{K-k+1},\ldots,\widehat{f}_{j-1}$ are known at time $t=K$. In order to determine the estimation error we average (2.13) over the possible values $\widehat{f}_{K-k+1},\ldots,\widehat{f}_{j-1}$ given the upper trapezoid $\mathcal{D}_K$ (see Remark 2.2).

From formula (2.2) follows that we have the following representation for the development factors

$$\widehat{f}_{j-1}=f_{j-1}+\frac{\sigma_{j-1}}{S_{j-1}}\cdot\sum_{k=1}^{K-j+1}\sqrt{X_{k,j-1}}\cdot\varepsilon_{k,j}\qquad(1\le j\le J) \tag{2.14}$$

from which we see

1) the estimators $\widehat{f}_0, \ldots, \widehat{f}_{J-1}$ are conditionally independent w.r.t. $\mathcal{D}_K$,

2) $E\left[\widehat{f}_{j-1}|\mathcal{D}_K\right] = f_{j-1}$ for $1 \le j \le J$ and

3) $E\left[\widehat{f}^2_{j-1}|\mathcal{D}_K\right] = f^2_{j-1} + \frac{\sigma^2_{j-1}}{s_{j-1}}$ for $1 \le j \le J$.

Henceforth, using 1)-3), we obtain

$$
\begin{aligned}
E & \left[x^2_{k,K-k+1} \cdot \left(\widehat{f}_{K-k+1} \cdot \ldots \cdot \widehat{f}_{j-1} - f_{K-k+1} \cdot \ldots \cdot f_{j-1} \right)^2 \middle| \mathcal{D}_K \right] \\
&= x^2_{k,K-k+1} \cdot \left(E\left[\widehat{f}^2_{K-k+1}|\mathcal{D}_K\right] \cdot \ldots \cdot E\left[\widehat{f}^2_{j-1}|\mathcal{D}_K\right] - \prod_{l=K-k+1}^{j-1} f_l^2 \right) \qquad (2.15) \\
&= x^2_{k,K-k+1} \cdot \left(\prod_{l=K-k+1}^{j-1} \left(f_l^2 + \frac{\sigma_l^2}{s_l} \right) - \prod_{l=K-k+1}^{j-1} f_l^2 \right).
\end{aligned}
$$

Next, we replace the parameters $\sigma^2_{K-k+1}, \ldots, \sigma^2_{j-1}$ and $f_{K-k+1}, \ldots, f_{j-1}$ by their estimators, and we obtain the following estimator for the estimation error

$$
\begin{aligned}
\widehat{E} & \left[\left(\widehat{X}_{k,j} - E\left[X_{k,j} \middle| \mathcal{D}_K \right] \right)^2 \right] \\
&:= x^2_{k,K-k+1} \cdot \underbrace{\left(\prod_{l=K-k+1}^{j-1} \left(\widehat{f}_l^2 + \frac{\widehat{\sigma}_l^2}{s_l} \right) - \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2 \right)}_{=: \triangle^2_{k,j}}. \qquad (2.16)
\end{aligned}
$$

$\triangle^2_{k,j}$ can be rewritten in a recursive form:

$$
\triangle^2_{k,j} = \triangle^2_{k,j-1} \cdot \widehat{f}^2_{j-1} + \prod_{l=K-k+1}^{j-2} \left(\widehat{f}_l^2 + \frac{\widehat{\sigma}_l^2}{s_l} \right) \cdot \frac{\widehat{\sigma}^2_{j-1}}{s_{j-1}}. \qquad (2.17)
$$

Analogously we define for non-cumulative payments the following estimator:

$$
\widehat{Y}_{k,j} := \widehat{X}_{k,j} - \widehat{X}_{k,j-1}. \qquad (2.18)
$$

Hence the estimation error is estimated by

$$
\begin{aligned}
\widehat{E} & \left[\left(\widehat{Y}_{k,j} - E\left[Y_{k,j} \middle| \mathcal{D}_K \right] \right)^2 \right] \\
&:= E\left[x^2_{k,K-k+1} \cdot \left(\widehat{f}_{K-k+1} \cdot \ldots \cdot \widehat{f}_{j-2} \cdot \left(\widehat{f}_{j-1} - 1 \right) - f_{K-k+1} \cdot \ldots \cdot f_{j-2} \cdot \left(f_{j-1} - 1 \right) \right)^2 \middle| \mathcal{D}_K \right] \\
&= x^2_{k,K-k+1} \cdot \left(\triangle^2_{k,j-1} \cdot (\widehat{f}_{j-1} - 1)^2 + \prod_{l=K-k+1}^{j-2} \left(\widehat{f}_l^2 + \frac{\widehat{\sigma}_l^2}{s_l} \right) \cdot \frac{\widehat{\sigma}^2_{j-1}}{s_{j-1}} \right).
\end{aligned}
\qquad (2.19)
$$

Using (2.17) and

$$\begin{aligned}\triangle_{k,j}^2 &= \widehat{f}_{j-1}^2 \cdot \ldots \cdot \widehat{f}_{K-k+1}^2 \cdot \prod_{l=K-k+1}^{j-1}\left(1+\frac{\widehat{\sigma}_l^2}{s_l \cdot \widehat{f}_l^2}\right) - \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2 \\ &= \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2 \cdot \underbrace{\left(\prod_{l=K-k+1}^{j-1}\left(1+\frac{\widehat{\sigma}_l^2}{s_l \cdot \widehat{f}_l^2}\right)-1\right)}_{=:\widetilde{\triangle}_{k,j}^2}\end{aligned} \tag{2.20}$$

the estimator of the estimation error (2.19) can be rewritten in the alternative form:

$$\begin{aligned}\widehat{E}\left[\left(\widehat{Y}_{k,j} - E\left[Y_{k,j}\middle|\,\mathcal{D}_K\right]\right)^2\right] &= x_{k,K-k+1}^2 \cdot \left(\triangle_{k,j-1}^2 \cdot (\widehat{f}_{j-1}-1)^2 + \triangle_{k,j}^2 - \triangle_{k,j-1}^2 \cdot \widehat{f}_{j-1}^2\right) \\ &= \widetilde{\triangle}_{k,j-1}^2 \cdot \widehat{Y}_{k,j}^2 + \left(\widetilde{\triangle}_{k,j}^2 - \widetilde{\triangle}_{k,j-1}^2\right) \cdot \widehat{X}_{k,j}^2.\end{aligned} \tag{2.21}$$

2.2 Result and comparison to the Mack formula

From our results we obtain the following estimator for the prediction error of a single accident year (cf. (2.10) and (2.16)):

Result 2.4 *Under Assumption 2.1 we have the following estimator for the mean square prediction error (2.1) of a single accident year*

$$\begin{aligned}\widehat{E}\left[\left(\widehat{X}_{k,j} - X_{k,j}\right)^2\middle|\,\mathcal{D}_K\right] &:= \Gamma_{k,j}^2 + x_{k,K-k+1}^2 \cdot \Delta_{k,j}^2 \\ &= \underbrace{\left(\widehat{X}_{k,j}\right)^2 \cdot \sum_{l=K-k+1}^{j-1} \frac{\widehat{\sigma}_l^2}{\widehat{X}_{k,l} \cdot \widehat{f}_l^2}}_{\textit{process variance}} \\ &\quad + \underbrace{x_{k,K-k+1}^2 \cdot \left(\prod_{l=K-k+1}^{j-1}\left(\widehat{f}_l^2 + \frac{\widehat{\sigma}_l^2}{s_l}\right) - \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2\right)}_{\textit{estimation error}}.\end{aligned} \tag{2.22}$$

The recursive form of the estimator of the prediction error is given by (cf. (2.11) and (2.17))

$$\begin{aligned}\widehat{E}\left[\left(\widehat{X}_{k,j} - X_{k,j}\right)^2\middle|\,\mathcal{D}_K\right] &:= \Gamma_{k,j-1}^2 \cdot \widehat{f}_{j-1}^2 + \sigma_{j-1}^2 \cdot \widehat{X}_{k,j-1} \\ &\quad + x_{k,K-k+1}^2 \cdot \left(\triangle_{k,j-1}^2 \cdot \widehat{f}_{j-1}^2 + \prod_{l=K-k+1}^{j-2}\left(\widehat{f}_l^2 + \frac{\widehat{\sigma}_l^2}{s_l}\right) \cdot \frac{\widehat{\sigma}_{j-1}^2}{s_{j-1}}\right).\end{aligned} \tag{2.23}$$

$\Gamma_{k,j-1}^2$ *and* $\triangle_{k,j-1}^2$ *were defined in (2.10) and (2.16) respectively.*

Formula (2.22) looks very similar to the famous MACK formula [4].

MACK uses the following assumptions for the calculation of the mean square error of prediction (see [4]):

1) $E\left(X_{k,j}\middle| X_{k,1},\ldots,X_{k,j-1}\right) = f_{j-1}\cdot X_{k,j-1}$ for $1\le k\le K$ and $1\le j\le J$;

2) the accident years are independent, i.e. $\left(X_{k,1},\ldots,X_{k,J}\right)$ and $\left(X_{l,1},\ldots,X_{l,J}\right)$ are independent for $k\neq l$;

3) $\operatorname{Var}\left(X_{k,j}\middle| X_{k,1},\ldots,X_{k,j-1}\right) = \sigma^2_{j-1}\cdot X_{k,j-1}$ for $1\le k\le K$ and $1\le j\le J$.

These assumptions are more general compared to our Assumptions 2.1. Thus, our model satisfies the assumptions of the MACK model. Henceforth the estimator given in [4] can also be used in our model.

But, in the first moment surprisingly, the formula for the estimation error given in MACK [4] differs from our formula (2.17). In [4] the recursive formula for $\triangle^2_{k,j}$ only considers the following terms

$$\widehat{\triangle}^2_{k,j} = \widehat{\triangle}^2_{k,j-1}\cdot \widehat{f}^2_{j-1} + \prod_{l=K-k+1}^{j-2}\widehat{f}^2_l\cdot\frac{\widehat{\sigma}^2_{j-1}}{s_{j-1}} \tag{2.24}$$

whereas we consider for the second term the following expression (cf. (2.17))

$$\prod_{l=K-k+1}^{j-2}\left(\widehat{f}^2_l+\frac{\widehat{\sigma}^2_l}{s_l}\right)\cdot\frac{\widehat{\sigma}^2_{j-1}}{s_{j-1}}.$$

The difference comes from the fact that MACK does not consider terms coming from the conditional variances

$$Var\left[\widehat{f}_l\middle|\mathcal{D}_K\right] = \frac{\sigma^2_l}{s_l}$$

in (2.24). In practice the difference between the full formula presented in (2.17) and the MACK formula is rather small; in most cases the differences are even negligible.

In fact, the MACK formula is a linear approximation to our result:

$$\begin{aligned}
&\widehat{E}\left[x_{k,K-k+1}^2 \cdot \left(\widehat{f}_{K-k+1} \cdot \ldots \cdot \widehat{f}_{j-1} - f_{K-k+1} \cdot \ldots \cdot f_{j-1}\right)^2 \middle| \mathcal{D}_K\right] \\
&\quad = x_{k,K-k+1}^2 \cdot \left(\prod_{l=K-k+1}^{j-1} \left(\widehat{f}_l^2 + \frac{\widehat{\sigma}_l^2}{s_l}\right) - \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2\right) \\
&\quad = x_{k,K-k+1}^2 \cdot \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2 \cdot \underbrace{\left(\prod_{l=K-k+1}^{j-1} \left(1 + \frac{\widehat{\sigma}_l^2}{s_l \cdot \widehat{f}_l^2}\right) - 1\right)}_{\approx \sum_{l=K-k+1}^{j-1} \frac{\widehat{\sigma}_l^2 / \widehat{f}_l^2}{s_l}} \qquad (2.25) \\
&\quad \approx x_{i,K-k+1}^2 \cdot \prod_{l=K-k+1}^{j-1} \widehat{f}_l^2 \cdot \sum_{l=K-k+1}^{j-1} \frac{\widehat{\sigma}_l^2 / \widehat{f}_l^2}{s_l}
\end{aligned}$$

which is the MACK term for the estimation error (cf. [4], page 219).

2.3 Prediction Error for accounting years

In the sequel X_t denotes the total of the cumulative payments originating from all accident years up to accounting year t $(1 \leq t \leq K + J - 1)$. Formally this corresponds to the sum of the $X_{k,j}$ over the t-diagonal

$$X_t := \sum_{\substack{k+j=t+1 \\ 1 \leq j \leq J}} X_{k,j}. \qquad (2.26)$$

We are at time K. I.e. the total payments X_t are observed for $t \leq K$ and for accident year k we denote by x_{k,k^*} with $k + k^* = K + 1$ the last observation.

We want to predict $X_{K+\tau}$ for $1 \leq \tau \leq J - 1$ and the corresponding estimator is given by

$$\widehat{X}_{K+\tau} := \sum_{\substack{k+j=K+\tau+1 \\ \tau+1 \leq j \leq J}} \widehat{X}_{k,j}. \qquad (2.27)$$

Sometimes it is easier to write this sum

$$\begin{aligned}
\widehat{X}_{K+\tau} &= \sum_{k=K+\tau+1-J}^{K} \widehat{X}_{k,k^*+\tau} \\
&= \sum_{k^*=1}^{J-\tau} \widehat{X}_{k,k^*+\tau}.
\end{aligned} \qquad (2.28)$$

The prediction error of $\widehat{X}_{K+\tau}$ can be decomposed as follows (see (2.1))

$$
\begin{aligned}
&E\left[\left(\widehat{X}_{K+\tau}-X_{K+\tau}\right)^2\middle|\mathcal{D}_K\right]\\
&\qquad=\underbrace{E\left[\left(X_{K+\tau}-E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]}_{\text{process variance}}+\underbrace{\left(\widehat{X}_{K+\tau}-E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2}_{\text{estimation error}}
\end{aligned}
\tag{2.29}
$$

Using (2.27), (2.28) and the independence of the accident years we obtain for the process variance of $\widehat{X}_{K+\tau}$

$$
\begin{aligned}
E\left[\left(X_{K+\tau}-E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]&=\sum_{\substack{k+j=K+\tau+1\\ \tau+1\le j\le J}}\operatorname{Var}\left[X_{k,j}\middle|\mathcal{D}_K\right]\\
&=\sum_{k=K+\tau+1-J}^{K}\operatorname{Var}\left[X_{k,k^*+\tau}\middle|\mathcal{D}_K\right].
\end{aligned}
\tag{2.30}
$$

Finally, using (2.10) we obtain as estimator for the process variance of $\widehat{X}_{K+\tau}$

$$
\widehat{E}\left[\left(X_{K+\tau}-E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]:=\sum_{\substack{k+j=K+\tau+1\\ \tau+1\le j\le J}}\Gamma_{k,j}^2.
\tag{2.31}
$$

For the estimation error we proceed as in Subsection 2.1.2. We average

$$
\left(\widehat{X}_{K+\tau}-E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2
$$

over the possible values of $\widehat{f}_{k^*},\dots,\widehat{f}_{k^*+\tau-1}$, given the upper trapezoid $\mathcal{D}_K$, to determine the estimation error of $X_{K+\tau}$. Using (2.28) and

$$
\widehat{g}_{k^*,\tau}:=\widehat{f}_{k^*}\cdot\ldots\widehat{f}_{k^*+\tau-1}\qquad\text{and}\qquad g_{k^*,\tau}:=f_{k^*}\cdot\ldots\cdot f_{k^*+\tau-1}
\tag{2.32}
$$

we obtain

$$
\begin{aligned}
&E\left[\left(\widehat{X}_{K+\tau}-E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]\\
&\quad=E\left[\left(\sum_{k^*=1}^{J-\tau}\left(\widehat{X}_{k,k^*+\tau}-E\left[X_{k,k^*+\tau}\middle|\mathcal{D}_K\right]\right)\right)^2\middle|\mathcal{D}_K\right]\\
&\quad=E\left[\left(\sum_{k^*=1}^{J-\tau}x_{k,k^*}\cdot\left(\widehat{f}_{k^*}\cdot\ldots\widehat{f}_{k^*+\tau-1}-f_{k^*}\cdot\ldots\cdot f_{k^*+\tau-1}\right)\right)^2\middle|\mathcal{D}_K\right]\\
&\quad=E\left[\sum_{k^*,l^*=1}^{J-\tau}x_{k,k^*}\cdot x_{l,l^*}\cdot\left(\widehat{g}_{k^*,\tau}-g_{k^*,\tau}\right)\cdot\left(\widehat{g}_{l^*,\tau}-g_{l^*,\tau}\right)\middle|\mathcal{D}_K\right].
\end{aligned}
\tag{2.33}
$$

Next, we use $\Delta^2_{k,k^*+\tau}$ (cf. (2.16)) and replace $f_{k^*}, \ldots, f_{l^*-1}, f_{k^*+\tau}, \ldots, f_{l^*+\tau-1}$ by their estimators. Thus we obtain the following estimator for the estimation error:

$$\widehat{E}\left[\left(\widehat{X}_{K+\tau} - E\left[X_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]$$
$$:= \sum_{k^*=1}^{J-\tau} x^2_{k,k^*} \cdot \Delta^2_{k,k^*+\tau} \tag{2.34}$$
$$+ 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1,J-\tau\}} x_{k,k^*} \cdot x_{l,l^*} \cdot \prod_{m=k^*}^{l^*-1} \widehat{f}_m \cdot \Delta^2_{l,k^*+\tau} \cdot \prod_{m=k^*+\tau}^{l^*+\tau-1} \widehat{f}_m.$$

Remark 2.5

- We have used the independence of the accident years for the derivation of (2.31). For (2.34) we can not use this independence because the estimators $\widehat{f}_l$ are positively correlated (they depend on the same data).

From our results (2.31) and (2.34) we obtain for the prediction error of a accounting year:

Result 2.6 *For the estimator of the prediction error (2.29) we have*

$$\widehat{E}\left[\left(\widehat{X}_{K+\tau} - X_{K+\tau}\right)^2\middle|\mathcal{D}_K\right] := \sum_{\substack{k+j=K+\tau+1\\ \tau+1\le j\le J}} \Gamma^2_{k,j} + \sum_{k^*=1}^{J-\tau} x^2_{k,k^*} \cdot \Delta^2_{k,k^*+\tau}$$
$$+ 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1,J-\tau\}} x_{k,k^*} \cdot x_{l,l^*} \cdot \prod_{m=k^*}^{l^*-1} \widehat{f}_m \cdot \Delta^2_{l,k^*+\tau} \cdot \prod_{m=k^*+\tau}^{l^*+\tau-1} \widehat{f}_m, \tag{2.35}$$

The recursive form is given by (see (2.10), (2.11), (2.16) and (2.17))

$$\widehat{E}\left[\left(\widehat{X}_{K+\tau} - X_{K+\tau}\right)^2\middle|\mathcal{D}_K\right] = \sum_{\substack{k+j=K+\tau+1\\ \tau+1\le j\le J}} \left(\Gamma^2_{k,j-1} \cdot \widehat{f}^2_{j-1} + \widehat{\sigma}^2_{j-1} \cdot \widehat{X}_{k,j-1}\right)$$
$$+ \sum_{k^*=1}^{J-\tau} x^2_{k,k^*} \cdot \left(\Delta^2_{k,k^*+\tau-1} \cdot \widehat{f}^2_{k^*+\tau-1} + \prod_{m=k^*}^{k^*+\tau-2} \left(\widehat{f}^2_m + \frac{\widehat{\sigma}^2_m}{s_m}\right) \cdot \frac{\widehat{\sigma}^2_{k^*+\tau-1}}{s_{k^*+\tau-1}}\right) \tag{2.36}$$
$$+ 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1,J-\tau\}} x_{k,k^*} \cdot x_{l,l^*} \cdot \prod_{m=k^*}^{l^*-1} f_m$$
$$\cdot \left(\Delta^2_{l,k^*+\tau-1} \cdot \widehat{f}^2_{k^*+\tau-1} + \prod_{m=l^*}^{k^*+\tau-2} \left(\widehat{f}^2_m + \frac{\widehat{\sigma}^2_m}{s_m}\right) \cdot \frac{\widehat{\sigma}^2_{k^*+\tau-1}}{s_{k^*+\tau-1}}\right) \cdot \prod_{m=k^*+\tau}^{l^*+\tau-1} f_m.$$

Analogously for the estimated non-cumulative payments in accounting year $K+\tau$ we have

$$\begin{aligned}\widehat{Y}_{K+\tau} &= \sum_{\substack{k+j=K+1+\tau\\ \tau+1\le j\le J}} \widehat{Y}_{k,j}\\ &= \sum_{k^*=1}^{J-\tau}\left(\widehat{X}_{k,k^*+\tau}-\widehat{X}_{k,k^*+\tau-1}\right).\end{aligned} \tag{2.37}$$

The corresponding prediction error can again be decomposed:

$$E\left[\left(\widehat{Y}_{K+\tau}-Y_{K+\tau}\right)^2\middle|\mathcal{D}_K\right] = \underbrace{E\left[\left(Y_{K+\tau}-E\left[Y_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]}_{\text{process variance}} + \underbrace{\left(\widehat{Y}_{K+\tau}-E\left[Y_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2}_{\text{estimation error}}. \tag{2.38}$$

Using (2.37), (2.12) and the independence of the accident years we obtain for the process variance of $\widehat{Y}_{K+\tau}$ the estimator

$$\widehat{E}\left[\left(Y_{K+\tau}-E\left[Y_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right] := \sum_{\substack{k+j=K+1+\tau\\ \tau+1\le j\le J}}\left(\Gamma_{k,j}^2+\Gamma_{k,j-1}^2\cdot\left(1-2\cdot\widehat{f}_{j-1}\right)\right). \tag{2.39}$$

Using (2.37) and (2.32) we obtain for the average estimation error

$$\begin{aligned}
&E\left[\left(\widehat{Y}_{K+\tau}-E\left[Y_{K+\tau}\middle|\mathcal{D}_K\right]\right)^2\middle|\mathcal{D}_K\right]\\
&\quad= E\Bigg[\Bigg(\sum_{k^*=1}^{J-\tau} x_{k,k^*}\cdot\Big(\widehat{f}_{k^*}\cdot\ldots\cdot\widehat{f}_{k^*+\tau-2}\cdot\left(\widehat{f}_{k^*+\tau-1}-1\right)\\
&\qquad\qquad - f_{k^*}\cdot\ldots\cdot f_{k^*+\tau-2}\cdot\left(f_{k^*+\tau-1}-1\right)\Big)\Bigg)^2\Bigg|\mathcal{D}_K\Bigg]\\
&\quad= \sum_{k^*=1}^{J-\tau} x_{k,k^*}^2\cdot E\left[\left(\widehat{g}_{k^*,\tau-1}\cdot\left(\widehat{f}_{k^*+\tau-1}-1\right)-g_{k^*,\tau-1}\cdot\left(f_{k^*+\tau-1}-1\right)\right)^2\middle|\mathcal{D}_K\right]\\
&\qquad + 2\cdot\sum_{k^*=1}^{J-\tau}\sum_{l^*>k^*}^{\min\{k^*+\tau-1,J-\tau\}} E\Big[x_{k,k^*}\cdot x_{l,l^*}\cdot\Big(\widehat{g}_{k^*,\tau-1}\cdot\left(\widehat{f}_{k^*+\tau-1}-1\right)-g_{k^*,\tau-1}\\
&\qquad\qquad\cdot\left(f_{k^*+\tau-1}-1\right)\Big)\cdot\Big(\widehat{g}_{l^*,\tau-1}\cdot\left(\widehat{f}_{l^*+\tau-1}-1\right)-g_{l^*,\tau-1}\cdot\left(f_{l^*+\tau-1}-1\right)\Big)\Big|\mathcal{D}_K\Big].
\end{aligned} \tag{2.40}$$

Finally, using (2.19), (2.20), (2.21) and replacing the parameters by their estimators we obtain the following estimator for the estimation error:

$$\begin{aligned}
&\widehat{E}\left[\left(\widehat{Y}_{K+\tau} - E\left[Y_{K+\tau}\middle|\,\mathcal{D}_K\right]\right)^2\middle|\,\mathcal{D}_K\right] \\
&\quad := \sum_{k^*=1}^{J-\tau} \widetilde{\Delta}^2_{k,k^*+\tau-1} \cdot \widehat{Y}^2_{k,k^*+\tau} + \left(\widetilde{\Delta}^2_{k,k^*+\tau} - \widetilde{\Delta}^2_{k,k^*+\tau-1}\right) \cdot \widehat{X}_{k,k^*+\tau} \\
&\qquad + 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1,J-\tau\}} \Big(\widetilde{\Delta}^2_{l,k^*+\tau-1} \cdot \widehat{Y}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau} \\
&\qquad\qquad + \left(\widetilde{\Delta}^2_{l,k^*+\tau} - \widetilde{\Delta}^2_{l,k^*+\tau-1}\right) \cdot \widehat{X}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau}\Big).
\end{aligned} \tag{2.41}$$

From our results (2.39) and (2.41) we have:

Result 2.7 *For the prediction error (2.38) we have the estimator*

$$\begin{aligned}
\widehat{E}\left[\left(\widehat{Y}_{K+\tau} - Y_{K+\tau}\right)^2\middle|\,\mathcal{D}_K\right] &:= \sum_{\substack{k+j=K+1+\tau \\ \tau+1\le j\le J}} \left(\Gamma^2_{k,j} + \Gamma^2_{k,j-1} \cdot \left(1 - 2 \cdot \widehat{f}_{j-1}\right)\right) \\
&\quad + \sum_{k^*=1}^{J-\tau} \widetilde{\Delta}^2_{k,k^*+\tau-1} \cdot \widehat{Y}^2_{k,k^*+\tau} + \left(\widetilde{\Delta}^2_{k,k^*+\tau} - \widetilde{\Delta}^2_{k,k^*+\tau-1}\right) \cdot \widehat{X}_{k,k^*+\tau} \\
&\qquad + 2 \cdot \sum_{k^*=1}^{J-\tau} \sum_{l^*>k^*}^{\min\{k^*+\tau-1,J-\tau\}} \Big(\widetilde{\Delta}^2_{l,k^*+\tau-1} \cdot \widehat{Y}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau} \\
&\qquad\qquad + \left(\widetilde{\Delta}^2_{l,k^*+\tau} - \widetilde{\Delta}^2_{l,k^*+\tau-1}\right) \cdot \widehat{X}_{k,k^*+\tau} \cdot \widehat{Y}_{l,l^*+\tau}\Big).
\end{aligned} \tag{2.42}$$

3 Example

In section 1.2 a Legal Valuation Portfolio is defined. The ideas derived in the preceding sections are now applied to calculate an example of such a VaPo.

We consider the case of the run off of an insurance portfolio with given experience of 14 accident years, and assume, that all claims are settled within 10 years (i.e. $K = 14$ and $J = 9$). The trapezoid of cumulative and non-cumulative payments are given in the Appendix, tables 10 and 12 respectively. In order to construct the VaPo, we evaluate in a first step the expected future payments by calender year, i.e. we evaluate the values $E_{K+\tau}^{(K)} = E\left[Y_{K+\tau}\middle|\mathcal{F}_K\right]$ for $\tau = 1, \ldots, J-1$.

Within our stochastic model (see Assumption 2.1), the expected future non-cumulative payments $E[Y_{k,j}|\mathcal{F}_K]$ are calculated by applying the Chain-Ladder technique to the trapezoid of cumulative payments; see table 1 and 2 for the estimated cumulative and non-cumulative future payments respectively. The last column in table 1 indicates the estimated ultimates of the corresponding accident years. By subtracting the cumulative payments of the actual calender year K we derive the outstanding claims reserves. The same result is obtained of course through row-wise summation of the numbers in table 2. From the estimators of the development-factors $\widehat{f}_j$ (see appendix B, table 11) we see, that

	development period j								
AY k	2	3	4	5	6	7	8	9	10
6									19879
7								20242	20249
8							17756	17758	17764
9						16459	16472	16473	16479
10					26902	26913	26935	26937	26947
11				22683	22700	22709	22727	22729	22737
12			16573	16651	16663	16670	16684	16685	16691
13		18964	19245	19336	19350	19358	19373	19375	19382
14	17365	18226	18497	18584	18597	18605	18620	18621	18628

Table 1: Estimated cumulative payments $\widehat{X}_{k,j}$.

we are considering a short-tail insurance branch. We did this in order to keep the tables in an workable size. To study the effect of different interest rate curves, it would be more interesting to consider a long-tail business. By summation over the diagonals in table 2

	development period j								
AY k	2	3	4	5	6	7	8	9	10
6									7
7								2	7
8							14	1	6
9						7	13	1	6
10					19	11	22	2	10
11				106	16	9	18	2	8
12			242	78	12	7	13	1	6
13		896	281	91	14	8	16	2	7
14	6719	861	270	87	13	8	15	1	7

Table 2: Estimated non-cumulative payments $\widehat{Y}_{k,j}$.

we get the expected cash flow pattern of the outstanding claims reserves (see table 3). Considering different interest rate curves (see appendix B, table 16), the prizes of the Portfolio of Zero Coupon Bonds $(Z^{(K+\tau)})_{\tau=1\dots,J-1}$ needed to model exactly this cash flow pattern are: 9810 for the BPV interest curve, 9511 for the constant interest curve (3.5%) and 9946 for the constant interest curve (0.0%, i.e. nominal value).
In the next step we have to determine the loadings $\rho_{K+\tau}^{(K)}$ for $\tau = 1 + \dots, J-1$. Different risk measures could be considered. In this example we concentrate on a risk measure of the form $\beta \cdot \sigma$ (where σ stands for the standard deviation). A risk loading of this form is known from premium calculation as standard deviation principle. To motivate this risk measure in our context let us consider, that the random variables $Y_{k,j}$ are normally distributed (i.e. $Y_{k,j} \sim \mathrm{N}(\mu, \sigma)$). Then the Value at Risk as well as the Expected Shortfall (or TailVaR) of $Y_{k,j}$ both are of the form $\beta \cdot \sigma$, with $\beta = \Phi(\alpha)^{-1}$ for the Value at Risk

τ	1	2	3	4	5	6	7	8	9
$E_{K+\tau}^{(K)}$	8013	1270	412	134	46	33	23	9	7

Table 3: Cash flow pattern of the outstanding claims reserves $E_{K+\tau}^{(K)}$.

and $\beta = \frac{\varphi\left(\Phi(\alpha)^{-1}\right)}{1-\alpha}$ for the Expected Shortfall. Here φ and Φ denote the density and the distribution function of the standard normal distribution respectively, and α denotes a prescribed security level, for instance 99%. Because $Y_{k,j} \sim \mathrm{N}(\mu, \sigma)$ the total payments $Y_{K+\tau}$ of an accounting year is normally distributed as well. To derive the estimator

$$\widehat{\widehat{\sigma}}^{(K)}_{K+\tau} := \sqrt{\widehat{E}\left[\left(\widehat{Y}_{K+\tau} - Y_{K+\tau}\right)^2 \middle| \mathcal{D}_K\right]}$$

for the standard deviation σ of $\widehat{Y}_{K+\tau}$ for $\tau = 1, \ldots, J-1$ we use the techniques introduced in the previous sections. It is important to notice, that by assigning only the square root of the estimated process variance to $\widehat{\widehat{\sigma}}^{(K)}_{K+\tau}$, we underestimate the standard deviation of $\widehat{Y}_{K+\tau}$. We have to take the estimation error into account as well. We omit the description of the intermediate results and refer to appendix B: table 13 for the process variances and table 14 for the estimation error of the $\widehat{Y}_{k,j}$. In table 15 the process variances and estimation errors for the total payments $\widehat{Y}_{K+\tau}$ of accounting years $K+\tau$ $(\tau = 1, \ldots, J-1)$ are given. The process variance of $\widehat{Y}_{K+\tau}$ is simply the sum of the process variances of its summands $\widehat{Y}_{k,j}$, whereas in the calculation of the estimation error we have to take into account certain dependencies as described in section 2.3 (see Remark 2.5). From this table we derive the prediction error for the $\widehat{Y}_{K+\tau}$ $(\tau = 1, \ldots, J-1)$ simply as the sum of process variance and estimation error:

Prediction error of $\widehat{Y}_{K+\tau}$								
15	16	17	18	19	20	21	22	23
2457834	115964	43468	9165	1341	1129	971	181	121

$\widehat{\widehat{\sigma}}^{(K)}_{K+\tau}$								
15	16	17	18	19	20	21	22	23
1568	341	208	96	37	34	31	13	11

Table 4: Prediction error of $\widehat{Y}_{K+\tau}$ and $\widehat{\widehat{\sigma}}^{(K)}_{K+\tau}$.

We now have the values needed to evaluate the loadings

$$\rho^{(K)}_{K+\tau} := \beta \cdot \widehat{\widehat{\sigma}}^{(K)}_{K+\tau}. \tag{3.1}$$

In table 5 we construct our legal valuation portfolio with a given risk-adjusted cost of capital rate $i = 6\%$ and $\beta = 2.3263\ldots$, which corresponds to a Value at Risk with a security level of 99%.

$Z^{(K+\tau)}$	$E_{K+\tau}^{(K)}$	$\widehat{\widehat{\sigma}}_{K+\tau}^{(K)}$	$\rho_{K+\tau}^{(K)}$	$\rho_{K+\tau}^{K}$ or $i \cdot \rho_{K+\tau}^{K}$	number of units
$Z^{(K+1)}$	8013	1568	3647	3647	11660
$Z^{(K+2)}$	1270	341	792	48	1318
$Z^{(K+3)}$	412	208	485	29	441
$Z^{(K+4)}$	134	96	223	13	148
$Z^{(K+5)}$	46	37	85	5	51
$Z^{(K+6)}$	33	34	78	5	37
$Z^{(K+7)}$	23	31	72	4	27
$Z^{(K+8)}$	9	13	31	2	10
$Z^{(K+9)}$	7	11	26	2	8
sum	9946			3755	13701

Table 5: Number of units for different accounting years.

In table 17 (see appendix B) we prize the zero-coupon bonds with different interest rate curves. Adding these number we obtain the prices of the legal VaPo, depending on the underlying interest rate curves (see table 6).

	BPV	3.5%	0.0%
price of the VaPo	13528	13131	13701
difference in %	1.26%	4.16%	0.00%

Table 6: Prices of the legal VaPo for different interest rate curves.

Remark 3.1

- The small differences in table 6 are the result of considering a short-tail business.

A Expected Payments

In this appendix we reinterpret the meaning of $E_{K+\tau}^{(K)}$, the expected payments in the periods $\tau \geq 1$. We also change philosophy how to estimate the quantities. In the first part we have always assumed that we are able to directly estimate the cashflow $\left(E_{K+1}^{(K)}, E_{K+2}^{(K)}, \ldots\right)$ using the chainladder method on cumulative payments. However in practice, the chainladder method on cumulative payments is very often not an appropriate method to estimate outstanding liabilities, especially for long-tailed lines of business. Henceforth, often one uses different methods to estimate the outstanding reserves, which in most of the cases do not directly lead to a cashflow pattern but rather to an estimate of the total reserves for open claims (including IBNyR and reopened claims).

Let us introduce the following notation

$$R_{K+\tau}^{(K)} := \sum_{s \geq \tau} E_{K+s}^{(K)}. \tag{A.1}$$

$R_{K+\tau}^{(K)}$ denotes the necessary expected total reserves to meet all payments after $K + \tau - 1$ (based on the information up to time K, $\mathcal{F}_K$). Usually $R_{K+\tau}^{(K)}$ are called nominal (undiscounted) total claims reserves at time $K+\tau-1$. There are many different methods to obtain "best estimates" for these total reserves (see e.g. [5], [1]). Here we want to stress two special difficulties setting these total claims reserves:

1. From a regulatory point of view these reserves should contain claims handling costs.

2. As already mentioned, one usually estimates the total reserves at time K(i.e. $R_{K+1}^{(K)}$), not the single payments $E_{K+\tau}^{(K)}$. Hence we need to estimate a payout/cashflow pattern for $R_{K+1}^{(K)}$.

A.1 ULAE reserves

Usually there are two different types of claims handling costs, external ones and internal ones. External costs like costs for external layers etc. can usually be allocated to single claims and are therefore contained in the claims payments (allocated loss adjustment expenses ALAE). Typically, internal loss adjustment expenses (income of claims handling department, maintenance of claims handling system, etc.) are not contained in claims figures and therefore have to be estimated separately. We call these costs unallocated loss adjustment expenses ULAE. From a regulatory point of view, we need to build reserves also for these costs because they are as well part of the claims handling process which guarantees that we are able to meet our obligations.

We denote by $R_{K+\tau}^{pure,K}$ the "pure" claims reserves excluding the ULAE reserves and with $R_{K+\tau}^{ULAE,K}$ the ULAE reserves. Henceforth the total claims reserves are given by

$$\widetilde{R}_{K+\tau}^{(K)} := R_{K+\tau}^{pure,K} + R_{K+\tau}^{ULAE,K}. \tag{A.2}$$

As already mentioned, there are many different methods to estimate $R_{K+\tau}^{pure,K}$ such as the chain-ladder method on cumulative payments, the chain-ladder method on claims incurred, Cape-Cod method, Bornhuetter-Ferguson method, etc. All these methods are usually based on figures excluding ULAE costs. Our proposal is that one uses these estimates and that one adds an extra estimate $R_{K+\tau}^{ULAE,K}$ for the ULAE costs.
One method to estimate the ULAE reserves is the so-called "New York" method (see e.g. [2], p. 219-221 and [3], p. 387-388). This method assumes that one part of the ULAE charge is proportional to the claims registration (denote the proportion by r) and the other part is proportional to the settlement (payments) of the claims (proportion $1-r$). Therefore it is necessary that the actuary is able to split the "pure" claims reserves $R_{K+\tau}^{pure,K}$ into reserves for reported claims $R_{K+\tau}^{rep,K}$ and reserves for IBNyR claims $R_{K+\tau}^{IBNR,K}$ (incurred but not yet reported), i.e. $R_{K+\tau}^{pure,K} = R_{K+\tau}^{rep,K} + R_{K+\tau}^{IBNR,K}$.
The estimation procedure works as follows:

- Determine the "paid-to-paid" ratio π. This is the ratio between ULAE payments per accounting year and pure claims payments. This can be done e.g. via an activity-based cost allocation split.

- The ULAE reserve is then given by

$$\begin{aligned} R_{K+\tau}^{ULAE,K} &= \pi \cdot \Big((1-r) \cdot R_{K+\tau}^{rep,K} + R_{K+\tau}^{IBNR,K} \Big) \\ &= \pi \cdot (1-r) \cdot \Big(R_{K+\tau}^{rep,K} + R_{K+\tau}^{IBNR,K} \Big) + \pi \cdot r \cdot R_{K+\tau}^{IBNR,K} \\ &= \pi \cdot (1-r) \cdot R_{K+\tau}^{pure,K} + \pi \cdot r \cdot R_{K+\tau}^{IBNR,K}. \end{aligned} \tag{A.3}$$

  Hence

$$\begin{aligned} \widetilde{R}_{K+\tau}^{(K)} &= R_{K+\tau}^{pure,K} + R_{K+\tau}^{ULAE,K} = R_{K+\tau}^{rep,K} + R_{K+\tau}^{IBNR,K} + R_{K+\tau}^{ULAE,K} \\ &= (1 + \pi \cdot (1-r)) \, R_{K+\tau}^{rep,K} + (1+\pi) \, R_{K+\tau}^{IBNR,K} \\ &= (1 + \pi \cdot (1-r)) \, R_{K+\tau}^{pure,K} + \pi \cdot r \cdot R_{K+\tau}^{IBNR,K}. \end{aligned} \tag{A.4}$$

A.2 Cashflow pattern

In (A.4) we have determined the outstanding total payments (total claims reserves). In order to determine the legal VaPo we have to estimate the cashflow pattern for these total outstanding liabilities. Unfortunately, not every claims reserving method leads in a natural way to a payout pattern. In this appendix we estimate the total claims reserves in a first step, using any appropriate method. In a second step we determine the payout pattern for these obligations using payment based methods, i.e. the payment based methods are only used to estimate the payout pattern of the reserves, not for estimating the total reserves itselves.

A.2.1 Pure claims payment

Using the notation of this paper the payout pattern for $R_{K+1}^{pure,K}$ is defined by

$$\alpha^{(K)} = \left(\alpha_2^{(K)}, \alpha_3^{(K)}, \ldots, \alpha_J^{(K)}\right) \qquad \text{with} \qquad \sum_{i\geq 2}^{J} \alpha_i^{(K)} = 1$$

and components

$$\alpha_{\tau+1}^{(K)} := \frac{E_{K+\tau}^{(K)}}{R_{K+1}^{(K)}} \tag{A.5}$$

where both numerator and denominator derive from a paid claims trapezoid and are aggregated over all accident years.

In the philosophy of this appendix we may apply this payout pattern to the reserve $R_{K+1}^{pure,K}$ also if this reserve is obtained by a different estimation method and is based on different data (e.g. incurred figures).

Example A.1 (Payout pattern for pure claims reserves)

For the example treated in this paper the total claims reserves is $R_{K+1}^{pure,K} = 9946$ (see table 3) and the payout pattern is as follows (table 7).

A.2.2 Cashflow pattern ULAE

As described above we assume that one part of the ULAE reserves is proportional to the pure claims payments $R_{K+1}^{pure,K}$ (proportion $1-r$) and the other part is proportional to the

α_2	0.8056	α_5	0.0135	α_8	0.0023
α_3	0.1277	α_6	0.0046	α_9	0.0009
α_4	0.0414	α_7	0.0033	α_{10}	0.0007

Table 7: Payout pattern.

reporting of the claims $R_{K+1}^{IBNR,K}$ (proportion r). Therefore in addition to the $\alpha^{(K)}$ payout pattern, we need to estimate the reporting pattern of the claims load. I.e. we need to estimate how the reserves $R_{K+1}^{IBNR,K}$ are reported. We denote this pattern by

$$\beta^{(K)} = \left(\beta_2^{(K)}, \ldots, \beta_J^{(K)}\right) \qquad \text{with} \qquad \sum_{i=2}^{J} \beta_i^{(K)} = 1, \tag{A.6}$$

i.e. we assume that the present IBNyR claims are reported as follows

$$\left(\text{year } K+1 : \beta_2^{(K)} \cdot R_{K+1}^{IBNR,K},\ \text{year } K+2 : \beta_3^{(K)} \cdot R_{K+1}^{IBNR,K}, \ldots\right). \tag{A.7}$$

There are several different ways to estimate the reporting pattern $\beta^{(K)}$, we do not want to go into further details about this estimates. Hence the ULAE reserves $R_{K+1}^{ULAE,K}$ are paid according to (see also (A.3))

$$\Big(\alpha_2^{(K)} \cdot \pi \cdot (1-r) \cdot R_{K+1}^{pure,K} + \beta_2^{(K)} \cdot \pi \cdot r \cdot R_{K+1}^{IBNR,K}, \tag{A.8}$$
$$\alpha_3^{(K)} \cdot \pi \cdot (1-r) \cdot R_{K+1}^{pure,K} + \beta_3^{(K)} \cdot \pi \cdot r \cdot R_{K+1}^{IBNR,K}, \ldots\Big)$$

Example A.2

We chose the payout pattern and the pure claims reserves as in given tables 8 and 9. The remaining reporting pattern is given by ($\beta_2^{(K)} = 95\%, \beta_3^{(K)} = 5\%$) which is a typical reporting pattern for property insurance. Usually liability insurance is more long-tailed. One important aspect in the reporting pattern is the definition of the accident date. There are different principles for liability insurance in force such as the claims cause principle, the claims occurrence principle or the claims made principle.

Using (A.3) and (A.8) we obtain the total reserves (including ULAE) in table 8 and the corresponding pattern in table 9 respectively.

pure claims reserves for reported claims $R_{K+1}^{rep,K}$	900
pure claims reserves for IBNR claims $R_{K+1}^{IBNR,K}$	100
pure claims reserves $R_{K+1}^{pure,K}$	1'000
paid-to-paid ratio π	5%
proportion r	50%
ULAE reserves $R_{K+1}^{ULAE,K}$	27.5
total claims reserves incl. ULAE $\widetilde{R}_{K+1}^{(K)}$	1'027.5

Table 8: Claims reserves obtained with (A.3) and (A.4).

	total	2	3	4	5
pattern $\alpha^{(K)}$		56.5%	27.6%	12.3%	3.6%
pattern $\beta^{(K)}$		95.0%	5.0%		
cashflow ULAE reserves $R_{K+1}^{ULAE,K}$	27.5	16.5	7.0	3.1	0.9
runoff of total reserves + ULAE $\widetilde{R}_{K+1}^{(K)} \longrightarrow \left(\widetilde{E}_{K+\tau}^{(K)}\right)_{\tau\geq 1}$	1'027.5	581.7	282.8	125.8	37.3

Table 9: Pattern for runoff of total reserves (including ULAE).

A.2.3 Estimation of expected cashflows

Merging (A.5) and (A.8) we obtain the following estimates for $\widetilde{E}_{K+\tau}^{(K)}$ (see also Table 9):

$$\begin{aligned}\widetilde{E}_{K+\tau}^{(K)} &= \alpha_{\tau+1}^{(K)} \cdot R_{K+1}^{pure,K} + \alpha_{\tau+1}^{(K)} \cdot \pi \cdot (1-r) \cdot R_{K+1}^{pure,K} + \beta_{\tau+1}^{(K)} \cdot \pi \cdot r \cdot R_{K+1}^{IBNR,K} \\ &= \alpha_{\tau+1}^{(K)} \cdot (1+\pi\cdot(1-r)) \cdot R_{K+1}^{pure,K} + \beta_{\tau+1}^{(K)} \cdot \pi \cdot r \cdot R_{K+1}^{IBNR,K}. \end{aligned} \tag{A.9}$$

$\widetilde{E}_{K+\tau}^{(K)}$ is the (at time K) estimated payment in period $K+\tau$ containing also the unallocated loss adjustment expenses ULAE, which is a common regulatory requirement. Of course we have (see also (A.4))

$$\sum_{s\geq 1} \widetilde{E}_{K+s}^{(K)} = (1+\pi\cdot(1-r)) \cdot R_{K+1}^{pure,K} + \pi \cdot r \cdot R_{K+1}^{IBNR,K} = \widetilde{R}_{K+1}^{(K)}, \tag{A.10}$$

which are the best estimate total reserves at time K for the outstanding liabilities containing ULAE reserves.

Remark A.3

In this appendix we have presented a method for estimating ULAE reserves and a payout pattern for the total claims reserves resulting in an estimate for $\widetilde{E}_{K+\tau}^{(K)}$. If we now apply our legal VaPo construction presented in the first part of our work (replace $E_{K+\tau}^{(K)}$ by $\widetilde{E}_{K+\tau}^{(K)}$), we obtain a valuation portfolio which is not consistent in itself, since the expected payments $\widetilde{E}_{K+\tau}^{(K)}$ and the risk margin $\rho_{K+\tau}^{(K)}$ (see (3.1)) may not have been estimated by the same method. Nevertheless, we believe that this approach is very useful in practice, since the risk margin $\rho_{K+\tau}^{(K)}$ gives a good idea about the incertainty of the payments, no matter which method we have used to estimate them. If we have better methods than the chain-ladder method to estimate expected payments this will (hopefully) only reduce the estimation error.

B Results for the example

	development period j									
AY k	1	2	3	4	5	6	7	8	9	10
1	12666	18116	18984	19183	19440	19462	19480	19490	19494	19495
2	11890	18656	19480	19737	19783	19758	19758	19758	19760	19760
3	11618	19402	20515	20772	20825	20834	20840	20914	20916	20916
4	13442	25249	26807	27134	27254	27287	27284	27285	27285	27315
5	11643	17768	18512	19276	19375	19382	19387	19387	19377	19385
6	11419	18717	19509	19836	19831	19853	19861	19860	19872	
7	11168	18657	19746	20103	20155	20182	20206	20240		
8	11020	16962	17513	17674	17709	17733	17742			
9	10285	15585	16104	16237	16440	16452				
10	13933	25043	26340	26770	26883					
11	11812	20830	22436	22577						
12	10013	15845	16331							
13	11684	18068								
14	10646									

Table 10: Run-off-trapezoid (cumulative payments).

$\widehat{f}_j$								
1	2	3	4	5	6	7	8	9
1.631123	1.049591	1.014840	1.004707	1.000725	1.000407	1.000804	1.000079	1.000365

$\widehat{\sigma}_j^2$								
1	2	3	4	5	6	7	8	9
206.10627	3.60183	1.68776	0.37389	0.01373	0.00415	0.03839	0.00257	0.00553

Table 11: Parameter estimates $\widehat{f}_j$ and $\widehat{\sigma}_j^2$.

	development period j									
AY k	1	2	3	4	5	6	7	8	9	10
1	12666	5450	868	199	257	22	18	10	4	1
2	11890	6766	824	257	46	-25	0	0	2	0
3	11618	7784	1113	257	53	9	6	74	2	0
4	13442	11807	1558	327	120	33	-3	1	0	30
5	11643	6125	744	764	99	7	5	0	-10	8
6	11419	7298	792	327	-5	22	8	-1	12	
7	11168	7489	1089	357	52	27	24	34		
8	11020	5942	551	161	35	24	9			
9	10285	5300	519	133	203	12				
10	13933	11110	1297	430	113					
11	11812	9018	1606	141						
12	10013	5832	486							
13	11684	6384								
14	10646									

Table 12: Run-off-trapezoid (non-cumulative payments).

	development period j								
AY k	2	3	4	5	6	7	8	9	10
6									110
7								52	112
8							681	46	98
9						68	632	42	91
10					369	112	1033	69	149
11				8441	311	94	872	58	126
12			27563	6197	229	69	640	43	92
13		65078	32021	7198	265	80	743	50	107
14	2194207	67942	31307	6973	256	78	716	48	103

Table 13: Process variance for non-cumulative payments (single accident years).

	development period j								
AY k	2	3	4	5	6	7	8	9	10
6									20
7								8	21
8							82	6	16
9						7	71	5	14
10					55	18	189	15	38
11				922	39	13	135	10	27
12			1992	497	21	7	73	6	14
13		5094	2688	670	28	9	98	8	19
14	153084	5084	2521	623	26	9	91	7	18

Table 14: Estimation error for non-cumulative payments (single accident years).

Process variance for accounting year $K+\tau$								
15	16	17	18	19	20	21	22	23
2296569	107372	40002	8340	1184	989	858	155	103
Estimation error for accounting year $K+\tau$								
15	16	17	18	19	20	21	22	23
161265	8592	3466	825	157	139	113	26	18

Table 15: Process variance and estimation error for $Y_{K+\tau}$.

BPV (Swiss Federal Office of Private Insurance) interest rate curve used for the Swiss Solvency Test 2005								
1	2	3	4	5	6	7	8	9
interest rate curve								
0.88%	1.14%	1.36%	1.57%	1.75%	1.91%	2.05%	2.18%	2.29%
discount factors								
99.13%	97.76%	96.03%	93.96%	91.69%	89.27%	86.76%	84.15%	81.56%
3.5%								
1	2	3	4	5	6	7	8	9
interest rate curve								
3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%
discount factors								
96.62%	93.35%	90.19%	87.14%	84.20%	81.35%	78.60%	75.94%	73.37%
0.0%								
1	2	3	4	5	6	7	8	9
interest rate curve								
0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
discount factors								
100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Table 16: Interest rate curves BPV, 3.5% and 0.0%.

$Z^{(K+\tau)}$	Price for the units (BPV)	Price for the units (3.5%)	Price for the units (0.0%)
$Z^{(K+1)}$	11558	11266	11660
$Z^{(K+2)}$	1288	1230	1318
$Z^{(K+3)}$	423	398	441
$Z^{(K+4)}$	139	129	148
$Z^{(K+5)}$	47	43	51
$Z^{(K+6)}$	33	30	37
$Z^{(K+7)}$	23	21	27
$Z^{(K+8)}$	9	8	10
$Z^{(K+9)}$	7	6	8

Table 17: Price of the units for different interest rate curves.

References

[1] England, P.D., and Verrall, R.J., 2002. Stochastic claims reserving in general insurance, British Actuarial Journal 8, 3, 443–518.

[2] Feldblum, S., 2003. Completing and using schedule P, Casualty Actuarial Society (CAS).

[3] Casualty Actuarial Society (CAS), 1990. Foundations of Casualty Actuarial Science, fourth edition.

[4] Mack, T., 1993. Distribution-free calculation of the standard error of Chain Ladder reserve estimates. Astin Bulletin 23, 2, 213–225.

[5] Taylor, G., 2000. Loss reserving: an actuarial perspective, Kluwer Academic Publishers.