Multivariate Extremes
and
Market Risk Scenarios

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The paper can be downloaded via:
`http://www.math.ethz.ch/~embrechts`

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A. Some statements on extremes and correlation

- "A natural consequence of the existence of a lender of last resort is that there will be some sort of allocation of burden of risk of extreme outcomes. Thus, central banks are led to provide what essentially amounts to **catastrophic insurance coverage** ... From the point of view of the risk manager, **inappropriate use of the normal distribution can lead to an understatement of risk**, which must be balanced against the significant advantage of simplification. From the central bank's corner, the consequences are even more serious because we often need to concentrate on the left tail of the distribution in formulating lender-of-last-resort policies. Improving the characterization of the distribution of extreme values is of paramount importance"

(Alan Greenspan, Joint Central Bank Research Conference, 1995)

Some statements on extremes and correlation

- "Extreme, synchronized rises and falls in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - the "perfect storm" scenario" (Business Week, September 1998)

- "Regulators have criticised LTCM and banks for not "stress-testing" risk models against extreme market movements... The markets have been through the financial equivalent of several Hurricane Andrews hitting Florida all at once. Is the appropriate response to accept that it was mere bad luck to run into such a **rare event** - or to get new forecasting models that assume more storms in the future?" (The Economist, October 1998, after the LTCM rescue)

Some statements on extremes and correlation

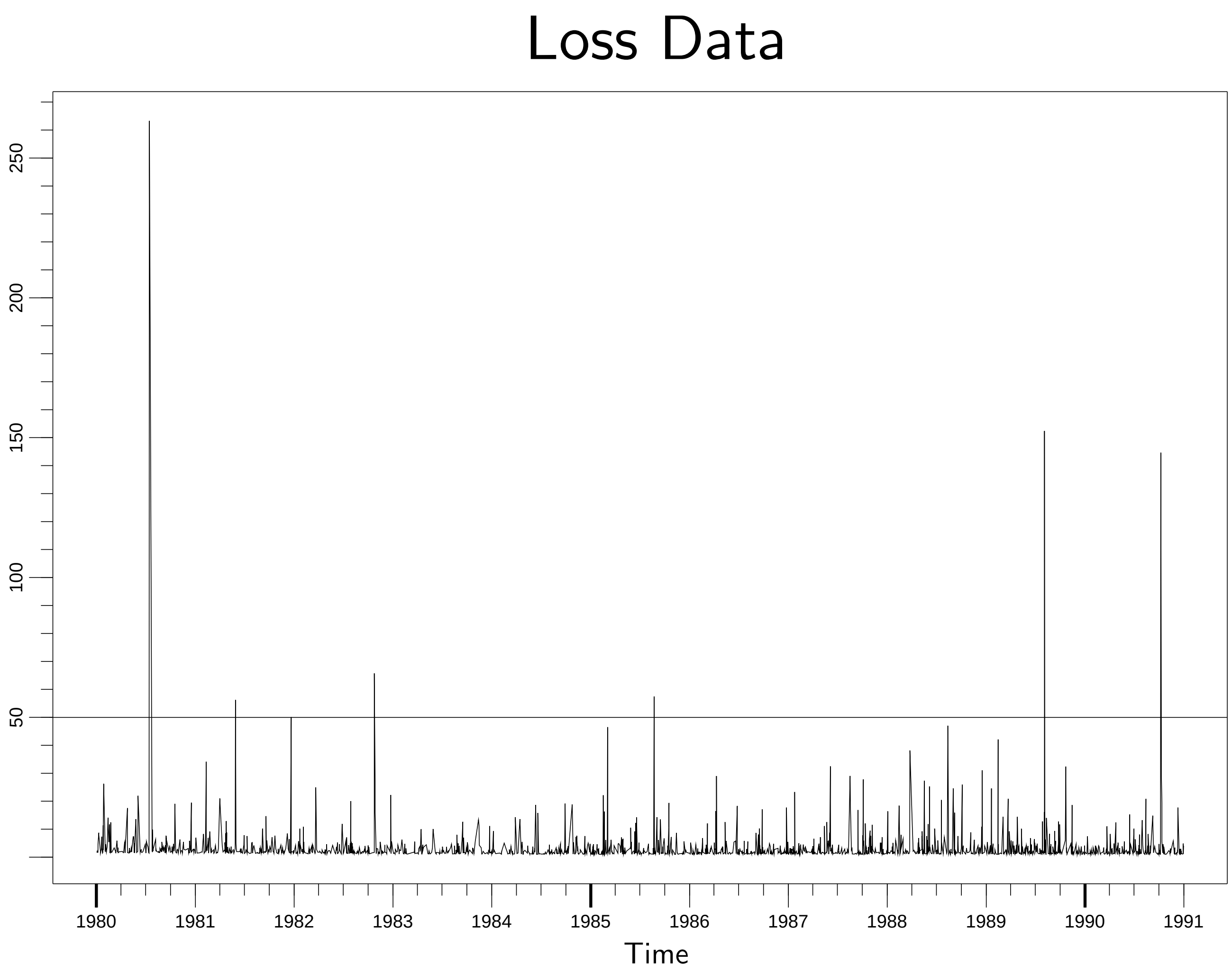
- "... The trading floor is quiet. But this masks their attempt at picking up the pieces with a new fund, JWM Partners. Now, Mr. Meriwether is preaching new gospel: World financial markets are bound to hit **extreme turbulences** again... Mr. Meriwether's crew, once bitten, also is betting on more liquid securities: "With globalisation increasing, you'll see more crises," he says. "Our whole focus is on the extremes now - **what's the worst that can happen to you in any situation** - because we never want to go through that again""

(John Meriwether, The Wall Street Journal, 21/8/2000)

B. Messages from the methodological frontier

- **Static case** (time fixed)
  - $d = 1$: Classical Extreme Value Theory (EVT)
    Peaks-over-threshold method (POT)
  - $d \geq 2$: Multivariate Extreme Value Theory (MEVT)
    Copulae
- **Dynamic case**
  - Extremes of stochastic processes in $d > 1$, only in rather special cases (Gaussian, Markov, ...)
  - Non-BSM models: Lévy driven price processes, incompleteness

C. The one-dimensional theory



Dimension 1: EVT - POT

- **Notation:** $M_n = \max(X_1, \ldots, X_n)$, $X_i \sim F$,
  $F_u(x) = P(X - u \leq x | X > u)$
- **Tail estimation:** $\overline{F}(x) = P(X > x) \approx \frac{N_u}{n} \, \overline{G}_{\hat{\xi}, \hat{\beta}}(x - u)$, $x \geq u$ where the excesses $(X_i - u)_+$ over the high threshold u can be approximated by a Generalized Pareto distribution (GPD):

$$G_{\xi,\beta}(x) = \begin{cases} 1 - (1 + \xi \, \frac{x}{\beta})^{-1/\xi} & \xi \neq 0 \\ 1 - e^{-x} & \xi = 0 \end{cases}$$

  where

$$\begin{array}{ll} x \geq 0 & \text{for } \xi \geq 0 \\ 0 \leq x \leq -1/\xi & \text{for } \xi < 0 \end{array}$$

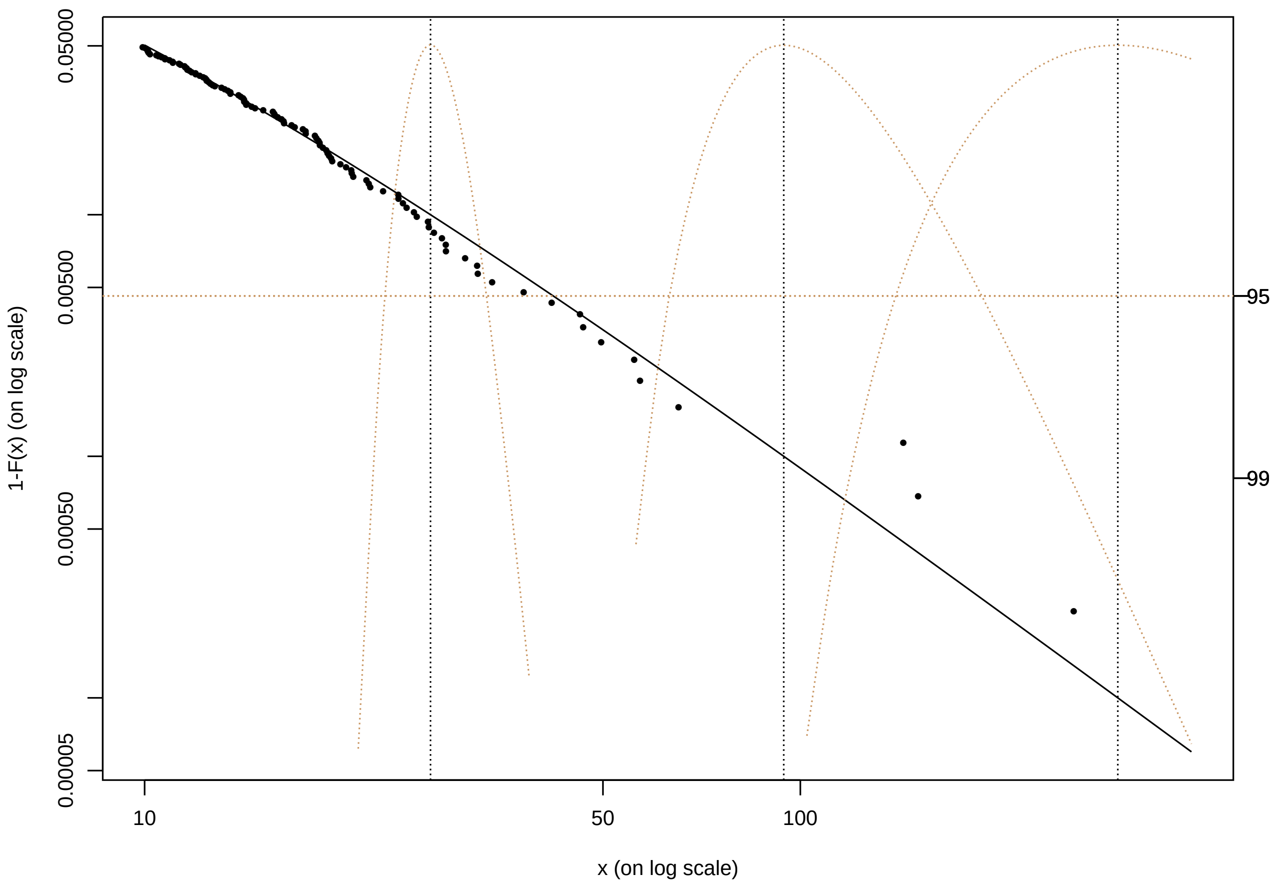
- **Tail conditions:** regular variation

Dimension 1: EVT - POT

- For $\xi > 0$,
$$F \in \mathsf{MDA}(H_\xi) \Longleftrightarrow \overline{F}(x) = x^{-1/\xi} L(x)$$
with L slowly varying. This means that for $x > 0$,
$$\frac{\overline{F}(tx)}{\overline{F}(t)} = \frac{P(X > tx)}{P(X > t)} \longrightarrow x^{-1/\xi},\ t \to \infty$$

- A graphical device for checking the above condition:
$$\textbf{plot} \quad \log(\overline{F}_n(x)) \quad \textbf{versus} \quad \log(x)$$
where F_n is the empirical distribution function of $(X_1, \ldots, X_n)$ and check for
(ultimate) linearity

Dimension 1: EVT - POT



p	Quantile	ES
99.00%	27.28	58.24
99.90%	94.33	191.53
99.99%	304.90	610.13

EVT software: EVIS (`www.math.ethz.ch/~mcneil`)

Dimension $d \geq 2$

So far the by now classical one-dimensional theory of extremes, but what about a more-dimensional theory?

- no standard ordering

- curse of dimensionality

- different approaches possible

- application dependent, . . .

D. Towards a multivariate theory

- **Ansatz 1:**

$$\mathbf{X}_i = (X_{i1}, \ldots, X_{id}) \quad i = 1, 2, \ldots, n$$

Denote componentwise maxima by

$$M_j^n = \max_{1 \le i \le n} X_{ij}, \quad j = 1, \ldots, d$$

Limit theory for $(\beta_{1n}^{-1}(M_1^n), \ldots, \beta_{dn}^{-1}(M_d^n))$, leading to

- **Ansatz 2:** spectral theory

- **Ansatz 3:** (A.A. Balkema and P. Embrechts): geometric (portfolio based) approach

Ansatz 2: Spectral Theory of Extremes

- Suppose that the d-dimensional random vector $\mathbf{X}$ has a regularly varying tail distribution, i.e., the tail behaviour of $\mathbf{X}$ is characterised by a tail index α and the limit

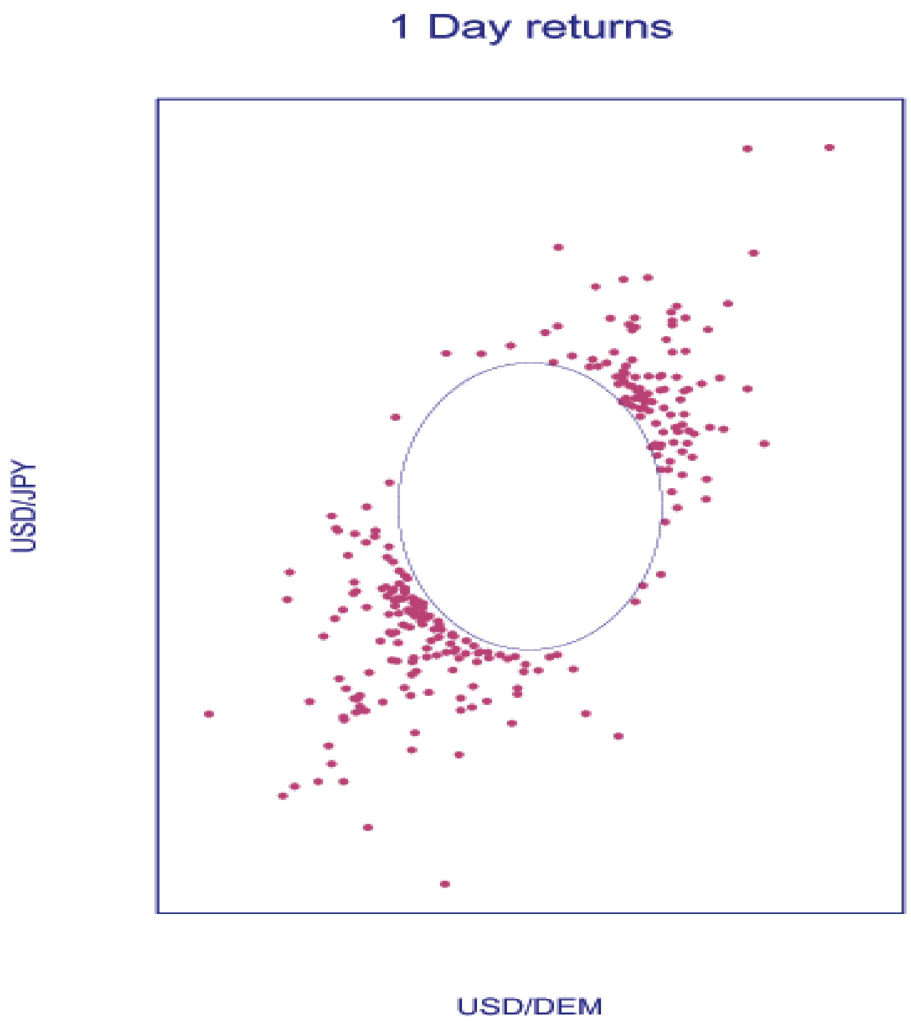
$$\frac{P(\| \mathbf{X} \| > tx, \mathbf{X}/ \| \mathbf{X} \| \in \cdot)}{P(\| \mathbf{X} \| > t)} \quad \xrightarrow{v} \quad x^{-\alpha} P(\mathbf{\Theta} \in \cdot),$$

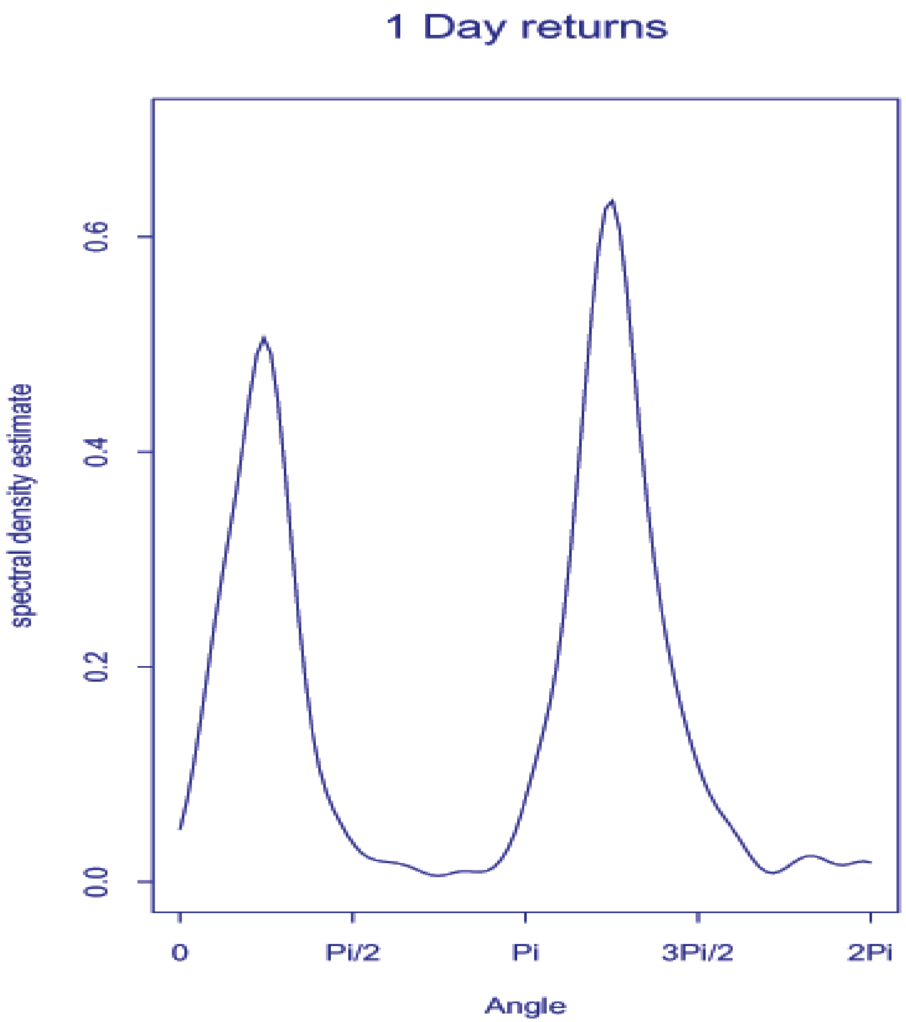
  where $x > 0$, $t \to \infty$, exists. The distribution function of $\mathbf{\Theta}$ is the spectral distribution of $\mathbf{X}$

- **Estimator:**

$$\hat{P}(\mathbf{\Theta} \in S) = \frac{1}{k_n} \sum_{i=1}^{n} \epsilon_{\mathbf{x}_i/\|\mathbf{x}\|_{k_n,n}}(V(S))$$

  where $V(S) = \{\mathbf{x} \in \mathbb{S}_{+}^{d-1} : \mathbf{x}/ \| \mathbf{x} \| \in S\}$


1 Day returns
USD/JPY
USD/DEM


1 Day returns
spectral density estimate
0.0
0.2
0.4
0.6
0
Pi/2
Pi
3Pi/2
2Pi
Angle

E. High risk scenarios

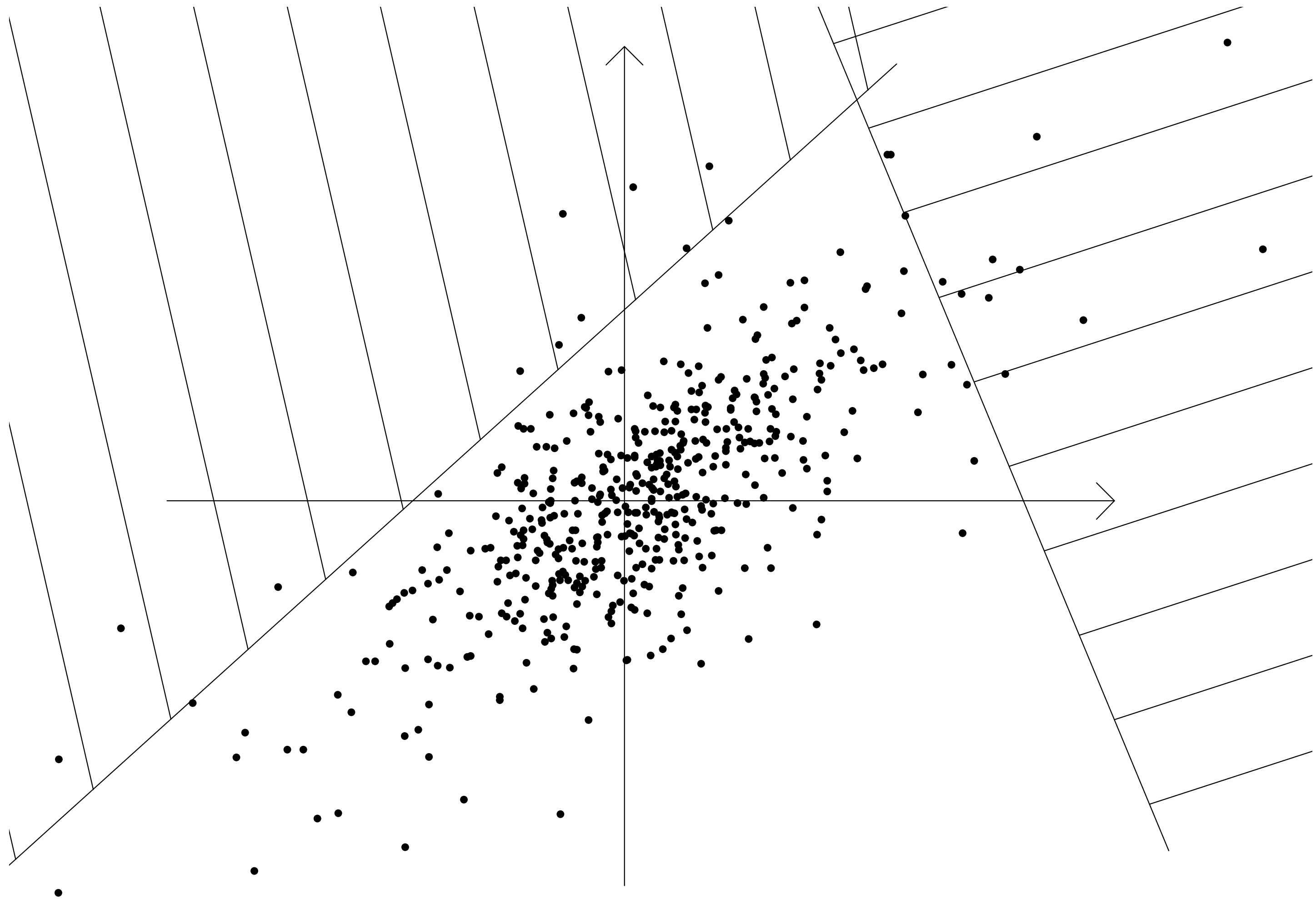
Goal: develop a theory which yields a model for the conditional distribution of the vector of (all) market variables given that the market (an index, say) hits a rare (extreme) event
$\equiv$ extreme market scenario, or high risk scenario

Main problem: any theory proposed will crucially depend on the probabilistic translation of "the market hits a rare event"
(a more-dimensional POT-theory; several alternative approaches exist: Resnick, Tajvidi, . . .)

The basic ingredients of our more geometric approach are:

- bland data
- rare event = hitting remote hyperplane (isotropically)
- high risk scenarios
- models with rotund level sets

Bland data and remote hyperplanes:



High risk scenarios

$$\mathbf{X} = (X_1, \ldots, X_d)$$

H : a hyperplane in $\mathbb{R}^d$

remote: $P(\mathbf{X} \in H) = \alpha > 0$, small

$\mathbf{X}^H$: vector with conditional df that $\{\mathbf{X} \in H\}$

β_H : affine maps

The problem:

$$\mathbf{W}_H = \beta_H^{-1}(\mathbf{X}^H) \Rightarrow \mathbf{W} \quad \text{for } 0 < P(\mathbf{X} \in H) \to 0$$

- find all $\mathbf{W}$ non-degenerate
- given a $\mathbf{W}$, which dfs (for $\mathbf{X}$) are attracted to $\mathbf{W}$

The solution:

partial, but covering most relevant cases for practical applications

Models with rotund level sets

Suppose $0 \in D \subset \mathbb{R}^d$, a bounded, open, convex set (smooth).
The (unique) gauge function $n_D : \mathbb{R}^d \to [0, \infty)$ satisfies:

$$D = \{n_D < 1\},\, n_D(r\,\mathbf{z}) = r\,n_D(\mathbf{z}), \mathbf{z} \in \mathbb{R}^d, r \geq 0$$

(If D is the unit ball, then n_D is the Euclidean norm)

Definition: A rotund set in $\mathbb{R}^d$ is a bounded, open, convex set which contains the origin and whose gauge function n_D is C^2 on $\mathbb{R}^d \setminus \{0\}$. In addition, the second derivative of n_D^2 is positive definite in each point $\mathbf{z} \neq \mathbf{0}$.

Remark: rotundity of D is equivalent to ∂D being a compact C^2 manifold with positive curvature in each point (think of egg-shaped sets)

The standard multivariate GPDs

$$\mathbf{w} = (u_1, \ldots, u_{d-1}, v) \in \mathbb{R}^{d-1} \times [0, \infty), \quad h = d - 1$$

$$g_\tau(\mathbf{w}) = \begin{cases} c_1(\tau) \left((1+\tau v)^2 + \tau \, \mathbf{u}^T \mathbf{u} \right)^{-1/2\tau - d/2} & \tau > 0 \\ (2\pi)^{-h/2} \exp\{-(v + \mathbf{u}^T \mathbf{u}/2)\} & \tau = 0 \\ c_2(\tau) \left(1 + \tau v + \tau \, \mathbf{u}^T \mathbf{u}/2 \right)_+^{-1/\tau - 1 - h/2} & -2/h < \tau < 0 \end{cases}$$

(called Pareto-parabolic high risk limit distributions)

and domain of attraction results can be given including:

- multivariate normal and t-distributions
- several elliptical distributions
- hyperbolic distributions
- and models in a "neighborhood" of the above

F. Conclusion

- The above yields the first steps towards a new theory
- Many problems remain:
  - full characterization of $\{\mathbf{W}\}$ and $\{\mathrm{DA}(\mathbf{W})\}$
  - rates of convergence
  - statistical estimation
- Comparison with alternative interpretations of "high or extreme risk"
- Going from variables (processes) to log-variables (-processes)
- Dynamic models
- What are good/useful (very) high-dimensional models in finance
- Towards real applications
- A lot more work is needed