Subjective Risk Measures: Bayesian Predictive Scenarios Analysis

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Abstract

In this paper we study the methods of risk measurement. First, we introduce a conditional risk measure and prove that it is a coherent risk measure. Then using Bayesian statistics idea a subjective risk measure is defined. In some special cases, closed form solutions can be obtained. The credibility idea can be fitted to our model too.

Key Words: Coherent risk measure, subjective risk measure, Bayesian analysis, credibility theory, risk interval, conditional risk measure, scenario analysis, Bühlman estimators, global investment.

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§1 Introduction

Risk measurement is one of the most important issues in financial and insurance industries. In financial industry, volatility is a commonly used risk measure. Ruin probability has been used by insurance industries for many years. Recently, value at risk (VaR) has become very popular. Developed by the commercial bank J.P. Morgan, VaR is an attempt to provide a single number summarizing the total risk in a portfolio of financial assets. For an introduction, see J.P. Morgan's RiskMetrics - Technical Document. For a survey see the paper by Duffie and Pan (1997). In Artzner, Delbaen, Eber and Heath (1998), both market and non-market risks have been studied. A set of four desirable properties for measures of risk is presented and justified. A risk measure which satisfies the four properties is called a coherent risk measure. It has been pointed out in Artzner, Delbaen, Eber and Heath (1998) that value at risk does not satisfy all of the four properties. Motivated by the paper of Artzner et al. (1998), Cvitanic and Karatzas (1998) have studied the dynamic measures of risk. Such a measure of risk is also discussed by Follmer and Leukert (1998).

Risk can be defined as an exposure to uncertainty, (see Holton (1997)). Different people may have different view on the uncertainty, even the same person, if he/she looks at the uncertainty from different point of view, he/she may obtain different conclusions. Therefore risk is a subjective thing. We should include the subjective view when we model the risk. In this paper, motivated by Holton (1997), we extend the model of Artzner et al. (1998) by including a subjective view in our model. Our goal is to construct a model for risk measurement which captures both the objective market data and the subjective view of the risk trader. We first define a conditional risk measure and prove that it is a coherent risk measure. Then we define a Bayesian risk measure, in this case the properties of monotonicity and subadditivity become meaningless, so we cannot say that the Bayesian risk measure is a coherent risk measure, but it does satisfy the other properties. In some special cases, we can obtain closed form solutions (i.e. we are able to obtain an explicit expression for the risk measure). Using the credibility approach, we can relax the stringent assumptions in the conjugate prior cases. No assumption for the model distribution need to be imposed. In this case, we are able to obtain an alternative expression for the subjective risk measure. Also, the subjective risk measure can be applied to measure both the financial risk and insurance risk. Finally, we will give an alternative way to capture the random effect of foreign exchange(FX) rates. A modification of Bayesian risk measure is introduced. The idea of this paper can be applied to value at risk (VaR). We shall discuss subjective VaR in a seperate paper.

§2 Conditional risk measure

In this section, we define a risk measure for the $(n+1)$-st period's net worth of a portfolio based on the information (or market data) up to time n. As more information comes, the risk measure can be updated sequentially. In this way, we can mark to market by adjusting the risk measure in the daily balance sheet.

Let Ω be the set of all states of nature. $\mathcal{F}$ be a σ-algebra on Ω. We equip our sample space $(\Omega, \mathcal{F})$ with a filtration $\{\mathcal{F}_n\}$ (i.e. $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\mathcal{F}_n$ is a sub-σ-field of $\mathcal{F}$ for all n). Here, $\mathcal{F}_n$ represents the market information (data) up to time n. $\mathbb{P}(A|\mathcal{F}_n)$ be the conditional probability of an event $A \subseteq \Omega$ given the market information up to time n. (i.e. given $\mathcal{F}_n$) For our purpose here, we assume it is the physical probability. P_n is the family of all conditional probability measures $\mathbb{P}(\cdot|\mathcal{F}_n)$. (i.e. the set of all "scenarios" defined based on the given information up to time n.) We assume that Ω is known and the conditional probability measure on Ω is not subjected to a general agreement.

Let ΔX_i be the change in market value (measured in terms of the domestic currency of the risk trader) of a portfolio during the i-th period, where $\Delta X_i : \Omega \to \mathbb{R}$ for i=1,2,...,n,n+1. $\mathcal{G}_{n+1}$ be the set of all functions $\Delta X_{n+1} : \Omega \to \mathbb{R}$ (i.e. the set of all risks during the $(n+1)$-st period). Note that $\{\Delta X_i\}$ is adapted to $\{\mathcal{F}_i\}$.

Then, the conditional risk measure is a function $\rho_{P_n}(\cdot|\mathcal{F}_n) : \mathcal{G}_{n+1} \to \mathbb{R}$ defined as

$$\rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n) = \sup\left\{ -E_{\mathbb{P}}\left(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\right)\Big|\mathbb{P} \in P_n\right\} \quad \text{for all } \Delta X_{n+1} \in \mathcal{G}_{n+1}$$

Here, r_n is the return at time $(n+1)$ per unit domestic price investing in a reference instrument at time n. $\rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n)$ measures the risk of a portfolio at time $(n+1)$

given the market information up to time n. It takes into account the worst "scenario" among the set of all possible "scenarios" given the market information up to time n.

In the following, we will prove that $\rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n)$ is a coherent risk measure. We check the four properties given in Artzner, Delbaen, Eber and Heath (1998)

(1) ρ_{P_n} satisfies the translation invariance property. For $\alpha_n \in \mathbb{R}$

$$\begin{aligned}
\rho_{P_n}(\Delta X_{n+1} + \alpha_n r_n|\mathcal{F}_n) &= \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1} + \alpha_n r_n}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} \\
&= \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big) - \alpha_n\Big|\mathbb{P} \in P_n\Big\} \\
&= \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} - \alpha_n \\
&= \rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n) - \alpha_n
\end{aligned}$$

(2) ρ_{P_n} satisfies positive homogeneity property

$$\begin{aligned}
\rho_{P_n}(\lambda \Delta X_{n+1}|\mathcal{F}_n) &= \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\lambda \Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} \\
&= \lambda\, sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} \\
&= \lambda \rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n) \quad \forall \lambda \geq 0\,.
\end{aligned}$$

(3) ρ_{P_n} satisfies monotonicity property

If $\Delta X_{n+1} \leq \Delta Y_{n+1}$ (i.e. $\Delta X_{n+1}(\omega) \leq \Delta Y_{n+1}(\omega)\ \forall\, \omega \in \Omega$), then

$$\begin{aligned}
-E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big) &\geq -E_{\mathbb{P}}\Big(\frac{\Delta Y_{n+1}}{r_n}\Big|\mathcal{F}_n\Big) \quad \forall\, \mathbb{P} \in P_n \\
\Rightarrow \quad \rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n) &= \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} \\
&\geq \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta Y_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} \\
&= \rho_{P_n}(\Delta Y_{n+1}|\mathcal{F}_n)
\end{aligned}$$

(4) ρ_{P_n} satisfies subadditivity property

$$\rho_{P_n}(\Delta X_{n+1} + \Delta Y_{n+1}|\mathcal{F}_n) = \sup\Big\{ -E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1} + \Delta Y_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\}$$

$$
\begin{aligned}
&= \sup\Big\{ - \Big[E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big) + E_{\mathbb{P}}\Big(\frac{\Delta Y_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big]\Big|\mathbb{P} \in P_n\Big\} \\
&\leq \sup\Big\{ - E_{\mathbb{P}}\Big(\frac{\Delta X_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} + \\
&\quad \sup\Big\{ - E_{\mathbb{P}}\Big(\frac{\Delta Y_{n+1}}{r_n}\Big|\mathcal{F}_n\Big)\Big|\mathbb{P} \in P_n\Big\} \\
&= \rho_{P_n}(\Delta X_{n+1}|\mathcal{F}_n) + \rho_{P_n}(\Delta Y_{n+1}|\mathcal{F}_n)
\end{aligned}
$$

Therefore, ρ_{P_n} is a coherent risk measure.

§3 Bayesian risk measure

In this section, we construct a risk measure by including the subjective view in our model and using the Bayesian ideas. By introducing the Bayesian predictive distribution to our risk measure, it can capture both the subjective view of the risk traders and the information provided by the market data. This is a special case of the conditional risk measure. Instead of considering the family of all conditional probability measures on Ω given $\mathcal{F}_n$, we restrict the family to the set of Bayesian predictive distributions. In this case, both the properties of subadditivity and monotonicity cannot be defined.

We first give some notations. Let ΔX_i be the change in market value (measured in terms of the domestic currency of the risk trader) of a portfolio during the i-th period (i.e. the i-th period net worth of a portfolio). The random variable Θ is the risk characteristic of a risk trader. We also define the vector $\underline{\Delta x_n}$ as $(\Delta x_1, \ldots, \Delta x_n)$ which represents the market data up to time n. We impose the following assumptions:

(1) $\Delta X_i|\Theta = \theta$ is conditionally independent and identically distributed with common distribution $F(x|\theta)$ (Note that $F(x|\theta)$ is the sampling distribution).

(2) $\pi(\theta)$ is the prior density of the risk characteristic Θ chosen by the risk trader subjectively.

Then, the posterior density of Θ given $\underline{\Delta x_n}$ is calculated by the Bayes' formula as follows:

$$
\pi(\theta|\underline{\Delta x_n}) = C \prod_{i=1}^{n} f(\Delta x_i|\theta)\pi(\theta)
$$

(where C is a normalization constant.)

The predictive distribution of the next period's net worth of the portfolio (i.e. the $(n+1)$-st period's net worth of the portfolio ΔX_{n+1}) given the data $\underline{\Delta x_n}$ can be calculated as follows:

$$\begin{aligned}
F_{\Delta X_{n+1}|\underline{\Delta x_n}}(x) &= P(\Delta X_{n+1} \le x|\underline{\Delta x_n}) \\
&= E(1(\Delta X_{n+1} \le x)|\underline{\Delta x_n}) \\
&= E(E(1(\Delta X_{n+1} \le x)|\Theta, \underline{\Delta x_n})|\underline{\Delta x_n}) \\
&= E(P(\Delta X_{n+1} \le x|\Theta, \underline{\Delta x_n})|\underline{\Delta x_n}) \\
&= E(P(\Delta X_{n+1} \le x|\Theta)|\underline{\Delta x_n}) \\
&= \int P(\Delta X_{n+1} \le x|\theta)\pi(\theta|\underline{\Delta x_n})\,d\theta \\
&= \int F(x|\theta)\pi(\theta|\underline{\Delta x_n})\,d\theta \\
&= C\int F(x|\theta)\left[\prod_{i=1}^{n} f(\Delta x_i|\theta)\right]\pi(\theta)\,d\theta
\end{aligned}$$

This expression depends on $\pi(\theta)$ which is chosen subjectively by the trader.

Suppose a group of risk traders $\mathcal{T}$ involve in choosing predictive densities. Without loss of generality, we can assume that each trader in the group $\mathcal{T}$ chooses exactly one predictive distribution and their predictive distributions are all different from each other. Let P_n be a family of all predictive distributions chosen by the risk traders in the group $\mathcal{T}$. Then, we can write P_n as $\{F^v_{\Delta X_{n+1}|\underline{\Delta x_n}}(x)|v \in \mathcal{T}\}$, where $F^v_{\Delta X_{n+1}|\underline{\Delta x_n}}(x)$ is chosen by the risk trader $v \in \mathcal{T}$ through the model density and the prior density. Also, $F^v_{\Delta X_{n+1}|\underline{\Delta x_n}}(x)$ includes the objective part through the market data $\underline{\Delta x_n}$ up to time n.

Then, we define "the subjective (Bayesian) risk measure" on the next period's net worth ΔX_{n+1} given the market data $\underline{\Delta x_n}$ as

$$\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n}) = \sup\{-E_{F^v}(\Delta X_{n+1}/r_n|\underline{\Delta x_n})|v \in \mathcal{T}\}$$

(where we write $F^v = F^v_{\Delta X_{n+1}|\underline{\Delta x_n}}(x)$.)

Suppose C_n is the cash amount in the investor's account at the current time n. If $\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})$ is positive, then $\max(\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n}) - C_n, 0)$ can be interpreted as the call margin added to the investor's account and invested in a reference instrument with return r_n in order to support the maximum expected loss in the portfolio at the $(n+1)$-st period. If $\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})$ is negative, then $\min(-\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n}), C_n)$ is interpreted as the cash amount that can be withdrawn from the investor's current account so that it can still support the maximum expected loss in the portfolio at the next period. $\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})$ is the most conservative (or safe) measure of risk agreed by all the people in the group $\mathcal{T}$.

Instead of summarizng the risk of a portfolio by a single number, we introduce an risk interval for risk measurement. We impose the risk limits or the call margin limits as follows:

$$\left[\inf\left\{-E_{F^v}\left(\tfrac{\Delta X_{n+1}}{r_n}|\underline{\Delta x_n}\right)|v \in \mathcal{T}\right\},\quad \sup\left\{-E_{F^v}\left(\tfrac{\Delta X_{n+1}}{r_n}|\underline{\Delta x_n}\right)|v \in \mathcal{T}\right\}\right]$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$$

$$\text{lower limit} = a_n \qquad\qquad \text{upper limit} = b_n = \rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})$$

If both a_n and b_n are negative, then $\min(-a_n, C_n)$ is the largest amount that can be withdrawn from the investor's account and $\min(-b_n, C_n)$ is the smallest amount that can be withdrawn so that the cash amount in the account can still be sufficient to cover the expected loss of the portfolio at the next period. Suppose both a_n and b_n are non-negative. If the margin call is less than $\min(a_n - C_n, 0)$, then it is unlikely to be sufficient for supporting the expected loss of the portfolio in the next period. If the margin call is greater than $\max(b_n - C_n, 0)$, then it is considered as more than sufficient to cover the maximum expected loss in the next period. Each risk trader can decide the cash amount added to his/her current account and invest in a reference instrument with return rate r_n within the range $[\min(a_n - C_n, 0), \max(b_n - C_n, 0)]$ by considering his/her own financial situation. The risk manager can set the margin call requirement for the portfolio based on the range $[\min(a_n - C_n, 0), \max(b_n - C_n, 0)]$, the financial situations of their clients and himself, the competitive condition in the market.

Remarks:

(1) r_n is close to 1 if the length of the period is short. The length of the time period is usually one day. r_n is considered to be the return rate of an overnight riskless instrument.

(2) We have pointed out that even the same person may have different views about the uncertainty in the financial market. He/She may choose several "scenarios" to evaluate the risk of his/her portfolio. In this case, we interpret $\mathcal{T}$ as the index set of the family of all predictive distributions chosen by the person. He/She can generate a set of "scenarios" through choosing a set of prior densities. Also, if the investor thinks that the central banks are interventing in the market and this may drive the extreme movement of the market value of his/her portfolio, then he/she can set the prior mean of Θ as an extreme value in order to capture the effect of central bank intervention. For more detailed treatment and application of extreme value theory, see the book of Embrechts et. al.(1997).

Before end of this section, we would like to point out that the subjective (Bayesian) risk measure satisfies both the properties of translation invariance and positive homogeneity. Also, the Bayesian risk measures introduced in the later sections satisfy both of these properties. However, the property of translation invariance becomes meaningless for the Bayesian risk measure. We will discuss this issue later. Now, we would like to state the interpretations derived from the properties of translation invariance and positive homogeneity.

(1) Translation invariance:
If we add an amount of capital α_n into the current account/portfolio and invest it in a riskless instrument with return rate r_n, then we reduce the risk of the portfolio at the next period by the same amount α_n. If we let α_n be $\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})$, then the property of translation invariance becomes "$\rho_{P_n}(\Delta X_{n+1}+r_n\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})|\underline{\Delta x_n}) = 0$". This means that if we add $\max(\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n}) - C_n, 0)$ to the investor's current account/portfolio and invest it in a riskless instrument with return rate r_n at the n-th period, then the risk measure for the portfolio at the $(n+1)$-st period is non-negative.

(2) Positive Homogeneity:

If the next period's net worth is multiplied by a factor λ, then its Bayesian risk measure is multiplied by the same factor λ.

§4 A special case (normal-normal conjugate prior case)

In this section, the normality assumption is imposed for both the prior and model densities. The risk traders can choose the prior means and variances based on their subjective views. Also, they can choose the sampling variance based on the estimates from the market data. A closed form solution is obtained in this case.

Remarks:

(1) Although the density function of the domestic price for a foreign security may not be normal, the normality assumption for the domestic price of the whole portfolio is acceptable provided that the portfolio consists of sufficiently large number of securities.

(2) The normality assumption for the market value of the portfolio can still be acceptable even though the portfolio consists of some non-linear financial instruments (e.g. stock option, bond option, FX swap,etc.)

Note that $\mathcal{T}$ is a finite set in reality. Suppose $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$, m is a finite positive integer. Model for the trader T_j (for $j = 1, 2, \ldots, m$) is given by:

$$\Theta \sim N(\mu_j, a_j)$$

$$\Delta X_i | \Theta = \theta \overset{\text{i.i.d.}}{\sim} N(\theta, v_j) \text{ for } i = 1, 2, \ldots, n, n+1$$

where we first assume v_j is known for simplicity (this assumption can be released by introducing a prior for v_j, see the later part of this section). Then, after some calculations, we have that the predictive distribution of the trader T_j denoted by $F^{T_j}_{\Delta X_{n+1}|\underline{\Delta x_n}}(x)$ is

$$N\left(\frac{\frac{n\overline{\Delta x_n}}{v_j} + \frac{\mu_j}{a_j}}{\frac{n}{v_j} + \frac{1}{a_j}}, \ (\frac{n}{v_j} + \frac{1}{a_j})^{-1} + v_j\right)$$

. Therefore,

$$
\begin{aligned}
E_{F^{T_j}}(\Delta X_{n+1}/r_n|\underline{\Delta x_n}) &= \frac{1}{r_n}E_{F^{T_j}}(\Delta X_{n+1}|\underline{\Delta x_n}) \\
&= \frac{1}{r_n}[(\frac{n}{v_j}+\frac{1}{a_j})^{-1}\frac{n}{v_j}\overline{\Delta x_n}+(\frac{n}{v_j}+\frac{1}{a_j})^{-1}\frac{1}{a_j}\mu_j]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n}) = \sup\Big\{&\frac{-1}{r_n}[(\frac{n}{v_j}+\frac{1}{a_j})^{-1}\frac{n}{v_j}\overline{\Delta x_n}+(\frac{n}{v_j}+\frac{1}{a_j})^{-1}\frac{1}{a_j}\mu_j] \\
&\text{for } j=1,2,\ldots,m\Big\}.
\end{aligned}
$$

As long as we can find the predictive means for all traders $T_j \in \mathcal{T}$, the exact numerical value of $\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})$ can be found. It is simple if m is not too large. Also, we can define the lower and upper risk limits as follows:

$$
\rho_{P_n}^{\inf}(\Delta X_{n+1}|\underline{\Delta x_n}) = \inf\Big\{-\frac{1}{r_n}[(\frac{n}{v_j}+\frac{1}{a_j})^{-1}\frac{n}{v_j}\overline{\Delta x_n}+(\frac{n}{v_j}+\frac{1}{a_j})^{-1}\frac{1}{a_j}\mu_j] \mid j=1,2,\ldots,m\Big\}
$$

$$
\rho_{P_n}^{\sup}(\Delta X_{n+1}|\underline{\Delta x_n}) = \rho_{P_n}(\Delta X_{n+1}|\underline{\Delta x_n})
$$

So, we have a risk interval $[\rho_{P_n}^{\inf}(\Delta X_{n+1}|\underline{\Delta x_n}),\ \rho_{P_n}^{\sup}(\Delta X_{n+1}|\underline{\Delta x_n})]$.

Now, we give an extension of the model by considering the precision of the sampling density as a random variable and choosing Gamma distribution as its prior distribution. A closed form solution can be obtained in this case.

Model for trader T_j (for$j=1,2,\ldots,m$), in this case, is given by:

$$
\Theta|\tau \sim N(\mu_j, 1/a_j\tau), \quad \tau \sim G_a(\alpha_j,\beta_j)
$$

$$
\Delta X_i|\Theta,\tau \overset{\text{i.i.d.}}{\sim} N(\Theta, 1/\tau)
$$

After some calculations, we know that the predictive density of $\Delta X_{n+1}|\underline{\Delta x_n}$ is a t-density with mean

$$
\frac{a_j}{a_j+n}\mu_j+\frac{n}{a_j+n}\overline{\Delta X_n} = (1-z_j)\mu_j+z_j\overline{\Delta X_n},
$$

where $z_j = \frac{n}{a_j+n}$.

Usually, the prior parameters $\mu_j, a_j, \alpha_j, \beta_j$ are chosen by the risk trader T_j based on prior information/experience and their subjective view. In practice, it is difficult to set the parameters α_j and β_j based on the prior information provided by the market data. Since the predictive mean does not depend on the parameters α_j and β_j, the risk measure does not depend on them either. However, we can use a_j to indicate the variability of the prior information. If the variability of the prior information is large, then a_j is small. Hence, z_j is large. This implies that more weight is given to the market data. The risk trader T_j will incline to use the market data to estimate ΔX_{n+1}. In this way, z_j can be interpreted as a credibility factor placed on the market data. Also, if the risk trader T_j has strong confidence for his/her prior guess, he/she can set a_j as a large value in order to make the prior variance of $\Theta|\tau$ small. Again, we define the risk limits as follows:

$$\rho_{P_n}^{\text{sup}}(\Delta X_{n+1}|\underline{\Delta x_n}) = \sup\{ -\frac{1}{r_n}[(1-z_j)\mu_j + z_j\overline{\Delta x_n}]|j = 1, 2, \ldots, m\}$$

and

$$\rho_{P_n}^{\text{inf}}(\Delta X_{n+1}|\underline{\Delta x_n}) = \inf\{ -\frac{1}{r_n}[(1-z_j)\mu_j + z_j\overline{\Delta x_n}]|j = 1, 2, \ldots, m\}$$

where $z_j = \frac{a_j}{n+a_j}$. $[\rho_{P_n}^{\text{inf}}(\Delta X_{n+1}|\underline{\Delta x_n}), \rho_{P_n}^{\text{sup}}(\Delta X_{n+1}|\underline{\Delta x_n})]$ is the risk interval.

§5 Credibility theory approach

Sometimes, it is difficult to find the predictive mean $E_{F^{T_j}}(\Delta X_{n+1}|\underline{\Delta x_n})$, especially when the predictive distribution F^{T_j} is not known (i.e. not a conjugate-prior case). In this section, we employ "Bülmann least square model" in credibility theory to approximate $E_{F^{T_j}}(\Delta X_{n+1}|\underline{\Delta x_n})$ as a linear combination of the past market data $\underline{\Delta x_n}$ and prior means.

Assumptions:

(1) $E(\Delta X_i|\Theta) = \Theta$ and $\text{Var}(\Delta X_i|\Theta) = v_{jn}$ for $i = 1, 2, \ldots, n, n+1$. v_{jn} can be estimated by the trader T_j from the volatility (or variation) of the market data up to time n.

(2) $\Delta X_i|\Theta$ are assumed to be conditionally independent for $i = 1, 2, \ldots, n, n+1$ with common mean Θ and variance v_{jn}.

(3) $E(\Theta) = \mu_j$ and $\mathrm{Var}(\Theta) = a_j$. They summarize the prior information about the risk characteristic Θ assumed by the subjective view of the risk trader $T_j \in \mathcal{T}$.

Here, we do not assume any specific form of probability distributions for Θ and $\Delta X_i|\Theta$. So, $F^{T_j}_{\Delta X_{n+1}|\underline{\Delta x_n}}(x)$ is not known. Hence, the closed form expression of $E_{F^{T_j}}(\Delta X_{n+1}|\underline{\Delta X_n})$ cannot be calculated. We will approximate $E_{F^{T_j}}(\Delta X_{n+1}|\underline{\Delta X_n})$ by the linear combination $\alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} \Delta X_k$ (i.e. We need to determine $\alpha_k^{T_j}$ for $k = 0, 1, 2, \ldots, n$) under the view of the risk trader $T_j \in \mathcal{T}$.

$$
\begin{aligned}
\text{Let } Q^{T_j} &= E(E(\Delta X_{n+1}|\underline{\Delta X_n}) - \alpha_0^{T_j} - \sum_{k=1}^{n} \alpha_k^{T_j} \Delta X_k)^2 \\
\frac{\partial Q^{T_j}}{\partial \alpha_0^{T_j}} &= -2E(E(\Delta X_{n+1}|\underline{\Delta X_n}) - \alpha_0^{T_j} - \sum_{k=1}^{n} \alpha_k^{T_j} \Delta X_k) = 0 \\
&\Rightarrow E(E(\Delta X_{n+1}|\underline{\Delta X_n})) = \alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} E(\Delta X_k) \\
&\Rightarrow E(\Delta X_{n+1}) = \alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} E(\Delta X_k) \\
\frac{\partial Q^{T_j}}{\partial \alpha_r^{T_j}} &= -2\alpha_r^{T_j} E[\Delta X_r (E(\Delta X_{n+1}|\underline{\Delta X_n}) - \alpha_0^{T_j} - \sum_{k=1}^{n} \alpha_k^{T_j} \Delta X_k)] = 0 \\
&\Rightarrow E[\Delta X_r E(\Delta X_{n+1}|\underline{\Delta X_n})] = \alpha_0^{T_j} E(\Delta X_r) + \sum_{k=1}^{n} \alpha_k^{T_j} E(\Delta X_k \Delta X_r) \\
&\qquad \text{for} \quad r = 1, 2, \ldots, n\,. \\
&\Rightarrow E[E(\Delta X_{n+1} \Delta X_r|\underline{\Delta X_n})] = \alpha_0^{T_j} E(\Delta X_r) + \sum_{k=1}^{n} \alpha_k^{T_j} E(\Delta X_k \Delta X_r) \\
&\Rightarrow E(\Delta X_{n+1} \Delta X_r) = \alpha_0^{T_j} E(\Delta X_r) + \sum_{k=1}^{n} \alpha_k^{T_j} E(\Delta X_k \Delta X_r) \\
&\qquad \text{for} \quad r = 1, 2, \ldots, n\,.
\end{aligned}
$$

Now,

$$
E(\Delta X_i) = E(E(\Delta X_i|\Theta)) = E(\Theta) = \mu_j
$$

$$
\begin{aligned}
E(\Delta X_k \Delta X_r) &= E(E(\Delta X_k \Delta X_r | \Theta)) \\
&= E(E(\Delta X_k|\Theta) E(\Delta X_r|\Theta)) \\
&\quad \text{(for } k \neq r \text{ and by conditional independent.)} \\
&= E(\Theta^2) = \mathrm{Var}(\Theta) + E^2(\Theta) \\
&= a_j + \mu_j^2
\end{aligned}
$$

$$
\begin{aligned}
E(\Delta X_i^2) &= E(E(\Delta X_i^2|\Theta)) \\
&= E(\mathrm{Var}(\Delta X_i|\Theta) + E^2(\Delta X_i|\Theta)) \\
&= E(v_{jn} + \Theta^2) \\
&= v_{jn} + E(\Theta^2) \\
&= v_{jn} + \mathrm{Var}(\Theta) + E^2(\Theta) \\
&= v_{jn} + a_j + \mu_j^2
\end{aligned}
$$

So, we have the following $(n+1)$ equations:

$$
\mu_j = \alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} \mu_j \tag{1}
$$

$$
a_j + \mu_j^2 = \alpha_0^{T_j} \mu_j + (a_j + \mu_j^2) \sum_{k=1}^{n} \alpha_k^{T_j} + v_{jn} \alpha_r^{T_j}, \text{ for } r = 1, 2, \ldots, n \tag{2}
$$

From (2),

$$
n(a_j + \mu_j^2) = n\alpha_0^{T_j} \mu_j + [n(a_j + \mu_j^2) + v_{jn}] \sum_{k=1}^{n} \alpha_k^{T_j} \tag{3}
$$

From (1) and (3),

$$
\begin{aligned}
-n\mu_j^2 + n(a_j + \mu_j^2) &= [n(a_j + \mu_j^2) + v_{jn} - n\mu_j^2] \sum_{k=1}^{n} \alpha_k^{T_j} \\
\Rightarrow \quad na_j &= [n(a_j + \mu_j^2 - \mu_j^2) + v_{jn}] \sum_{k=1}^{n} \alpha_k^{T_j} \\
\Rightarrow \sum_{k=1}^{n} \alpha_k^{T_j} &= (na_j + v_{jn})^{-1} na_j \\
\Rightarrow \quad \alpha_k^{T_j} &= (na_j + v_{jn})^{-1} a_j \qquad \text{for} \quad k = 1, 2, \ldots, n \; .
\end{aligned}
$$

From (1),

$$\begin{aligned} \mu_j &= \alpha_0^{T_j} + \mu_j(na_j + v_{jn})^{-1} na_j \\ \Rightarrow \quad \alpha_0^{T_j} &= \mu_j\Big(1 - \frac{na_j}{na_j + v_{jn}}\Big) = \Big(\frac{v_{jn}}{na_j + v_{jn}}\Big)\mu_j \end{aligned}$$

So,

$$\begin{aligned} E_{F^{T_j}}(\Delta X_{n+1} | \underline{\Delta x_n}) &\approx \alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} \Delta x_k \\ &= \Big(\frac{v_{jn}}{na_j + v_{jn}}\Big)\mu_j + \Big(\frac{na_j}{na_j + v_{jn}}\Big)\overline{\Delta x_n} \\ &= (1 - z_{jn})\mu_j + z_{jn}\overline{\Delta x_n} \end{aligned}$$

where the credibility factor placed on the market data chosen by the risk trader $T_j \in \mathcal{T}$, which is denoted by z_{jn}, equals $\frac{na_j}{na_j + v_{jn}}$.

Remarks:

(1) $(1 - z_{jn})\mu_j + z_{jn}\overline{\Delta x_n}$ is also a good estimator for both $E(\Delta X_{n+1} | \Theta) = \Theta$ and ΔX_{n+1} used by the risk trader $T_j \in \mathcal{T}$.

(2) z_{jn} can be updated sequentially by updating the values of n and v_{jn}.

(3) As the variability of the market data up to time n becomes large, v_{jn} becomes large and z_{jn} becomes small. So, we put a small weight to the market data.

So,

$$E_{F^{T_j}}(\frac{\Delta X_{n+1}}{r_n} | \underline{\Delta x_n}) \approx \frac{1}{r_n}[(1 - z_{jn})\mu_j + z_{jn}\overline{\Delta x_n}]$$

Now, suppose that $\mathcal{T} = \{T_1, \ldots, T_m\}$. Then, we approximate the lower and upper risk limits as follows:

$$\begin{aligned} \rho_{P_n}^{\inf}(\Delta X_{n+1} | \underline{\Delta x_n}) &\approx \inf\Big\{ -\frac{1}{r_n}[(1 - z_{jn})\mu_j + z_{jn}\overline{\Delta x_n}] \,\Big|\, j = 1, 2, \ldots, m\Big\} = a_n \\ \rho_{P_n}^{\sup}(\Delta X_{n+1} | \underline{\Delta x_n}) &\approx \sup\Big\{ -\frac{1}{r_n}[(1 - z_{jn})\mu_j + z_{jn}\overline{\Delta x_n}] \,\Big|\, j = 1, 2, \ldots, m\Big\} = b_n \end{aligned}$$

Therefore, $[a_n, b_n]$ is the approximate risk interval. Since we do not impose any assumption for the density function of the change in the market value of a portfolio, the approximate risk measure can be applied to evaluating the risk of the portfolio, in terms of domestic currency of the risk trader, which consists of both the foreign financial securities and non-linear instruments.

§6 Generalized Bayesian premium calculation (Bayesian "scenarios" analysis in credibility theory)

In credibility theory, we are interested in calculating a premium for the $(n+1)$-st period of a policyholder given the policyholder's claim experience in the first n periods. Suppose we have a set of manual rates. Each of them corresponds to one type of the past policyholders. Now, we want to calculate the premium charge to a policyholder at the beginning of the $(n+1)$-st period based on the set of manual rates and the policyholder's past claim records up to the n-th period.

Let ΔX_i be the claim amount of a policyholder during the i-th period.

Let $\mathcal{T}$ be the set of manual rates in an insurance company. We have m manual rates in the insurance company. (i.e. $\mathcal{T} = \{T_1, \dots, T_m\}$)

For each manual rate $T_j \in \mathcal{T}$, we would like to calculate (if it is a conjugate-prior case)/approximate (if $E_{F^{T_j}}(\Delta X_{n+1} | \underline{\Delta x_n})$ is difficult to be calculated) $E_{F^{T_j}}(\Delta X_{n+1} | \underline{\Delta x_n})$. As in the previous section, $E_{F^{T_j}}(\Delta X_{n+1} | \underline{\Delta x_n})$ is approximated by

$$\alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} \Delta x_k$$

In this way, we get a set of approximations for $E_{F^{T_j}}(\Delta X_{n+1} | \underline{\Delta x_n})$ as follows:

$$\alpha_0^{T_j} + \sum_{k=1}^{n} \alpha_k^{T_j} \Delta x_k = (1 - z_{j^n})\mu_j + z_{j^n} \overline{\Delta x_n}, \qquad \text{for} \quad j = 1, 2, \dots, m .$$

where the credibility factor z_{j^n} is $\frac{na_j}{na_j + v_{j^n}}$. Then, we define the maximum premium charged for a policyholder for the $(n+1)$-st period as

$$\rho_{P_n}^{\sup}(\Delta X_{n+1} | \underline{\Delta x_n}) = \sup\{\frac{1}{r_n}[(1 - z_{j^n})\mu_j + z_{j^n} \overline{\Delta x_n}] | j = 1, 2, \dots, m\}$$

Here, we use r_n to denote the investment income received by an insurance company at the $(n+1)$-st period per unit price invested in a riskless instrument at the n-th period.

As r_n gets larger, the insurance company can charge a smaller amount of premium for the same risk. Also, we define the minimum premium charged for a policyholder for the $(n+1)$-st period as

$$\rho_{P_n}^{\inf}(\Delta X_{n+1}|\underline{\Delta x_n}) = \inf\{\frac{1}{r_n}[(1-z_{j^n})\mu_j + z_{j^n}\overline{\Delta x_n}]|j=1,2,\ldots,m\}$$

So, $[\rho_{P_n}^{\inf}(\Delta X_{n+1}|\underline{\Delta x_n}), \rho_{P_n}^{\sup}(\Delta X_{n+1}|\underline{\Delta x_n})]$ is the range of premiums that should be charged for a policyholder by an insurance company. The insurance company can decide the amount of premium charged for a policyholder within the range $[\rho_{P_n}^{\inf}(\Delta X_{n+1}|\underline{\Delta x_n})$, $\rho_{P_n}^{\sup}(\Delta X_{n+1}|\underline{\Delta x_n})]$ based on its financial situation, and its competitors in the market. Note that both $\rho_{P_n}^{\inf}$ and $\rho_{P_n}^{\sup}$ are non-negative in this situation. In practice, it is not sensible to separate a premium into two parts, the risky and riskless parts. So, the property of translation invariance cannot be defined.

§7 Application of Bayesian risk measure to evaluate insurance risk

In this section, we apply the Bayesian risk measure to the measurement of insurance risk. Instead of using the traditional method of ruin probability, we capture the subjective view by using the Bayesian predictive distribution. We define a risk measure which can be updated sequentially so that it can fit into the daily balance sheet of an insurance company. The Bühlman credibility estimators are applied to find an approximate for the risk measure. First, we give some notations.

Suppose we have a set of manual rates (or a group of insurers) $\mathcal{T} = \{T_1, \ldots, T_m\}$. Let x denote the initial capital of an insurance company, P_i denote the aggregate amount of premium received by the insurance company during the i-th period, S_i denote the aggregate amount of claim paid by the insurance company during the i-th period, I_i denote the total investment income received by the insurance company during the i-th period, U_i denote the surplus of the insurance company at the end of the i-th period.

$$U_i = x + \sum_{i=1}^{n} P_i - \sum_{i=1}^{n} S_i + \sum_{i=1}^{n} I_i$$

. Let $\Delta U_i = U_i - U_{i-1}$, X_{ji} denote the amount of j-th premium during the i-th period, Y_{ji} denote the amount of j-th claim during the i-th period, N_{1i} denote the number of premiums payment received during the i-th period, N_{2i} denote the number of claims paid during the i-th period.

We assume the following sampling densities for the insurance data.

$N_{1i}|\ (\Lambda_1 = \lambda_1) \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda_1)$, $N_{2i}|\ (\Lambda_2 = \lambda_2) \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda_2)$, $X_{ji}|\ (\Delta_1 = \delta_1) \overset{\text{i.i.d.}}{\sim} \exp(1/\delta_1)$, $Y_{ji}|\ (\Delta_2 = \delta_2) \overset{\text{i.i.d.}}{\sim} \exp(1/\delta_2)$, $I_i|\ (\Theta = \theta,\ \tau = \tau_0) \overset{\text{i.i.d.}}{\sim} N(\theta, {\tau_0}^{-1})$, for $i = 1, 2, \ldots, n, n+1$.

We also assume the following prior densities for the unknown parameters.

$\Lambda_1 \sim G_a(\alpha_1^{T_k}, \beta_1^{T_k})$, $\Lambda_2 \sim G_a(\alpha_2^{T_k}, \beta_2^{T_k})$, $\Delta_1 \sim IG(\gamma_1^{T_k}, \eta_1^{T_k})$, $\Delta_2 \sim IG(\gamma_2^{T_k}, \eta_2^{T_k})$, $\Theta|\ (\tau = \tau_0) \sim N(\mu^{T_{k1}}/a^{T_k}\tau_0)$, $\tau \sim G_a(r^{T_k}, s^{T_k})$, where we assume that Λ_k and Δ_k are independent for $k = 1, 2$.

Note that the prior parameters can be chosen from the past manual rate $T_k \in \mathcal{T}$ or chosen by the insurer $T_k \in \mathcal{T}$ subjectively.

Let c be the minimum capital requirement for an insurance company to continue their business. We define the risk measure as follows:

$$
\begin{aligned}
& \rho_{P_n}(\Delta U_{n+1} | \underline{P}_n, \underline{S}_n, \underline{I}_n, \min_{1 \le i \le n} U_i \ge c) \\
= \ & \sup\{ -E_{F_P^k}(P_{n+1}|P_1, \ldots, P_n) + E_{F_S^k}(S_{n+1}|S_1, \ldots, S_n) - E_{F_I^k}(I_{n+1}|I_1, \ldots, I_n) \\
& \quad |k = 1, 2, \ldots, m\}
\end{aligned}
$$

We would like to apply the Bühlman credibility estimates to approximate $E_{F_P^k}(P_{n+1}|P_1, \ldots, P_n)$ as follows:

$$
\begin{aligned}
K^* &= \frac{E(2\Lambda_1 \Delta_1^2)}{\text{Var}(\Lambda_1 \Delta_1)} = \frac{2\beta_1^{T_k}(\eta_1^{T_k} - 2)}{\alpha_1^{T_k} + \eta_1^{T_k} - 2} \\
z^{T_k} &= \frac{n}{n + K^*} = \text{the credibility factor} .
\end{aligned}
$$

The prior mean $= E(\Lambda_1 \Delta_1) = \left(\frac{\alpha_1^{T_k}}{\beta_1^{T_k}}\right)\left(\frac{\gamma_1^{T_k}}{\eta_1^{T_k} - 2}\right) = \frac{\alpha_1^{T_k}\gamma_1^{T_k}}{\beta_1^{T_k}(\eta_1^{T_k} - 2)}$.

So,

$$\begin{aligned}
& E_{F_P^k}(P_{n+1}|P_1,\ldots,P_n) \\
\approx\ & (1-z^{T_k})(\text{mean of prior}) + z^{T_k}(\text{mean of observed data}) \\
=\ & \frac{K^*}{n+K^*}\Big(\frac{\alpha_1^{T_k}\gamma_1^{T_k}}{\beta_1^{T_k}(\eta_1^{T_k}-2)}\Big) + \Big(\frac{n}{n+K^*}\Big)\Big(\frac{\sum_{i=1}^n P_i}{n}\Big) \\
=\ & \frac{2\alpha_1^{T_k}\gamma_1^{T_k} + (\alpha_1^{T_k}+\eta_1^{T_k}-2)\sum_{i=1}^n P_i}{2\beta_1^{T_k}(\eta_1^{T_k}-2) + (\alpha_1^{T_k}+\eta_1^{T_k}-2)n}
\end{aligned}$$

Similarly, we approximate $E_{F_S^k}(S_{n+1}|S_1,\ldots,S_n)$ as follows:

$$E_{F_S^k}(S_{n+1}|S_1,\ldots,S_n) \approx \frac{2\alpha_2^{T_k}\gamma_2^{T_k} + (\alpha_2^{T_k}+\eta_2^{T_k}-2)\sum_{i=1}^n S_i}{2\beta_2^{T_k}(\eta_2^{T_k}-2) + (\alpha_2^{T_k}+\eta_2^{T_k}-2)n}$$

Also, $E_{F_I}^k(I_{n+1}|I_1,\ldots,I_n)$ can be calculated exactly as follows:

$$E_{F_I}^k(I_{n+1}|I_1,\ldots,I_n) = \frac{a^{T_k}}{a^{T_k}+n}\mu^{T_k} + \frac{n}{a^{T_k}+n}\Big(\frac{\sum_{i=1}^n I_i}{n}\Big)$$

Therefore,

$$\begin{aligned}
& \rho_{P_n}(\Delta U_{n+1}|\underline{P}_n, \underline{S}_n, \underline{I}_n \min_{1\le i\le n} U_i \ge c) \\
\approx\ & \sup\Big\{ -\frac{2\alpha_1^{T_k}\gamma_1^{T_k} + (\alpha_1^{T_k}+\eta_1^{T_k}-2)\sum_{i=1}^n P_i}{2\beta_1^{T_k}(\eta_1^{T_k}-2) + (\alpha_1^{T_k}+\eta_1^{T_k}-2)n} + \frac{2\alpha_2^{T_k}\gamma_2^{T_k} + (\alpha_2^{T_k}+\eta_2^{T_k}-2)\sum_{i=1}^n S_i}{2\beta_2^{T_k}(\eta_2^{T_k}-2) + (\alpha_2^{T_k}+\eta_2^{T_k}-2)n} \\
& -\Big(\frac{a^{T_k}}{a^{T_k}+n}\Big)\mu^{T_k} - \Big(\frac{n}{a^{T_k}+n}\Big)\Big(\frac{\sum_{i=1}^n I_i}{n}\Big)\Big| k = 1,2,\ldots,m\Big\}
\end{aligned}$$

Remarks:

(1) The risk measure can be updated sequentially so that it can serve as a tool of risk measurement in the daily balance sheet of an insurance company.

(2) Apart from viewing it as a risk measure, we can consider its negative part as a performance index of an insurance company.

§8 A modification of Bayesian risk measure for the global financial market

In this section, we give another way to capture the random effect of foreign exchange(FX) rates in our model. A modification of Bayesian risk measure with the random effect of FX rate included explicitly is defined and its closed form expression can be obtained. The Bayesian risk measure is expected to give an indication for the investment risk in the global financial market. We consider a portfolio consisting of both the domestic and foreign financial securities. For investment in the foreign securities, we not only subject to the risk due to the movement of the value of the foreign securities. But also, we subject to the risk from the movement of FX rates.

Suppose the global market in our model consists of the securities from the domestic country and k foreign countries. Let m_0 be the number of domestic securities and m_i be the number of securities from the i-th foreign country, for $i = 1, 2, ..., k$. Then, we denote a vector of the changes in the market values (measured in terms of the i-th foreign currency) per unit of all the securities from the j-th country during the r-th period as $\Delta Y_r^{(i)} = (\Delta Y_{1r}^{(i)}, \Delta Y_{2r}^{(i)}, \ldots, \Delta Y_{m_i,r}^{(i)})$, for $i = 0, 1, 2, \ldots, k$ and $r = 1, 2, \ldots, n, n+1$. f_{it} denotes the domestic price at time t of one unit of the i-th foreign currency, for i = 1, 2, ..., k

We would like to impose the following assumptions:

(1) The market values of the securities within each country are correlated.

(2) The market values of the securities in different countries are independent.

(3) The exchange rates are independent.

(4) The exchange rates are independent of the market values of the securities.

(5) The portfolio can be updated only at the end of each period or at the beginning of each period. It cannot be changed during each period. Due to the presence of transaction costs and taxes in the real trading situation, this is a more realistic assumption.

Our model is constructed as follows:

$$\underline{\Delta Y}_r^{(i)} = \underline{\Theta}^{(i)} \Delta t + \underline{\varepsilon}_r^{(i)} \sqrt{\Delta t}; \quad \text{for} \quad i = 0, 1, 2, \ldots, k$$

where Δt is the time length of each period. $\underline{\varepsilon}_r^{(i)} \sim N(\underline{0}_{mi}, \Sigma_i^{-1})$. Note that $\underline{0}_{mi}$ is a m_i-dimensional vector and the $(m_i \times m_i)$-matrix Σ_i^{-1} is known. The prior distribution of the random vector $\underline{\Theta}^{(i)}$ is $N(\underline{\mu}_{is}, \mathcal{T}_{is}^{-1})$, where $\underline{\mu}_{is}$ and $\mathcal{T}_{is}^{-1}$ can be chosen by the trader T_s subjectively. So $\underline{\Delta Y}_r^{(i)}|\underline{\Theta}^{(i)} = \underline{\theta}^i$, for $r = 1, 2, \ldots, n, n+1$, are conditionally independent and identically distributed with common distribution

$$N(\underline{\mu}_{is}\Delta t, \Sigma_i^{-1}); \quad \text{for} \quad i = 0, 1, 2, \ldots, k$$

After some calculations, we know that the predictive distribution of $\underline{\Delta Y}_{n+1}^{(i)}$ given $(\underline{\Delta Y}_1^{(i)}, \ldots, \underline{\Delta Y}_n^{(i)})$ is a multivariate normal distribution with mean vector $(\mathcal{T}_{is}+n\Delta t\Sigma_i)^{-1}$ $(\mathcal{T}_{is}\underline{\mu}_{is}\Delta t) + (\mathcal{T}_{is}+n\Delta t\Sigma_i)^{-1}(n\Delta t\Sigma_i)\,\overline{\underline{\Delta Y}}^{(i)}$, where $\overline{\underline{\Delta Y}}^{(i)} = \sum_{r=1}^{n}\underline{\Delta Y}_r^{(i)}/n$. (Some of these calculation can be found in Siu and Yang (1998))

Let $\underline{\alpha}_r^{(i)}$ be a m_i-dimensional vector($\alpha_{1r}^{(i)}$, $\alpha_{2r}^{(i)}$, …, $\alpha_{m_i,r}^{(i)}$) , for i = 0, 1, 2, …, k. where $\alpha_{jr}^{(i)}$ represents the number of units of the j-th security in the i-th foreign country at the end of the r-th period (i.e. at the beginning of the $(r+1)$-st period). If $\alpha_{jr}^{(i)}$ is equal to zero, then the j-th security in the i-th foreign country does not include in the portifolio at the end of the r-th period. If $\alpha_{jr}^{(i)}$ is less than zero, then $-\alpha_{jr}^{(i)}$ is interpreted as the number of units short for the j-th security in the i-th foreign country at the end of the r-th period. Some countries would introduce some policies (regulations) to restrict the activities of short selling. In this case, we impose some constraints to the values of $\alpha_{jr}^{(i)}$.

Let $\Delta X_r^{(j)}$ be the change in the domestic price of investing in a portfolio of securities in the i-th country during the r-th period. Then $\Delta X_r^{(j)} = f_{ir}\sum_{j=1}^{mi}\alpha_{j,r-1}^{(i)}\Delta y_{jr}^{(i)}$ for i = 1, 2, …, k and for r = 1, 2, …, n, n+1. Also, we assume that the process f_{it} satisfies a mean-reverting model (the Vasicek Model) as follows:

$$df_{it} = a^{T_s}(b^{T_s} - f_{it})dt + \sigma^{T_s}dW_{it}$$

for $i = 1, 2, \ldots, k$, where a^{T_s}, b^{T_s} and σ^{T_s} are non-negative constants chosen by the trader T_s subjectively.

Let $\mathcal{F}_n^{f_i}$ be the σ-algebra generated by the process f_{it} up to time n. (i.e. $\mathcal{F}_n^{f_i} = \sigma\{f_{it}|0 \leq$

$t \leq n\}$) After some calculations, we obtain

$$E_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i}) = f_{in}e^{-a^{T_s}\Delta t} + b^{T_s}(1 - e^{-a^{T_s}\Delta t})$$

$$\sigma_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i}) = \frac{\sigma^{T_s}}{\sqrt{2a^{T_s}(1 - e^{-2a^{T_s}\Delta t})}}$$

Let ΔX_{n+1} be the change in the market value (measured in terms of the domestic currency) of a global portfolio during the $(n+1)$-st period. Then $\Delta X_{n+1} = \sum_{i=0}^{k} \Delta X_{n+1}^{(i)}$. Let $\underline{\Delta Y}^{(i)}$ be $(\underline{\Delta Y}_1^{(i)}, \underline{\Delta Y}_2^{(i)}, \ldots, \underline{\Delta Y}_n^{(i)}.)^T$ for $i = 0, 1, 2, \ldots, k$. (i.e. It is a $(n \times m_i)$-matrix). Then, we define the risk measure as follows:

$$\begin{aligned}
&\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta Y}^{(0)}, \underline{\Delta Y}^{(1)}, \ldots, \underline{\Delta Y}^{(k)}, \mathcal{F}_n^{f_1}, \ldots, \mathcal{F}_n^{f_k}) \\
= &\frac{1}{r_n} sup\{-E_{T_s}[(\underline{\alpha}_n^{(0)})^T \underline{\Delta Y}_{n+1}^{(0)}|\Delta Y^{(0)}] - \sum_{i=1}^{k}[E_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i}) \\
&+ I(-E_{T_s}[(\underline{\alpha}_n^{(i)})^T \underline{\Delta Y}_{n+1}^{(i)}|\Delta Y^{(i)}])\sigma_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i})]E_{T_s}[(\underline{\alpha}_n^{(i)})^T \underline{\Delta Y}_{n+1}^{(i)}|\Delta Y^{(i)}]|T_s \in \mathcal{T}\}
\end{aligned}$$

Write

$$M_{sn}^{(i)} = (\underline{\alpha}_n^{(i)})^T[(\mathcal{T}_{is} + n\Delta t\Sigma_i)^{-1}(\mathcal{T}_{is}\underline{\mu}_{is}\Delta t) + (\mathcal{T}_{is} + n\Delta t\Sigma_i)^{-1}(n\Delta t\Sigma_i)\overline{\underline{\Delta y}}^{(i)}]$$

Then,

$$\begin{aligned}
&\rho_{P_n}(\Delta X_{n+1}|\underline{\Delta Y}^{(0)}, \underline{\Delta Y}^{(1)}, \ldots, \underline{\Delta Y}^{(k)}, \mathcal{F}_n^{f_1}, \ldots, \mathcal{F}_n^{f_k}) \\
&= \frac{1}{r_n} sup\{-M_{sn}^{(0)} - \sum_{i=1}^{k}[f_{i,n}e^{-a^{T_s}\Delta t} + b^{T_s}(1 - e^{-a^{T_s}\Delta t}) \\
&\quad + I(-M_{sn}^{(i)})\frac{\sigma^{T_s}}{\sqrt{2a^{T_s}(1 - e^{-2a^{T_s}\Delta t})}}]M_{sn}^{(i)}|T_s \in \mathcal{T}\}
\end{aligned}$$

where

$$I(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}$$

Also, we can define the risk limits as follows:

$$\begin{aligned}
&\rho_{P_n}^{\inf}(\Delta X_{n+1}|\underline{\Delta Y}^{(0)}, \underline{\Delta Y}^{(1)}, \ldots, \underline{\Delta Y}^{(k)}, \mathcal{F}_n^{f_1}, \ldots, \mathcal{F}_n^{f_k}) \\
&= \frac{1}{r_n} inf\{-M_{sn}^{(0)} - \sum_{i=1}^{k}[f_{i,n}e^{-a^{T_s}\Delta t} + b^{T_s}(1 - e^{-a^{T_s}\Delta t})
\end{aligned}$$

$$+I(M_{sn}^{(i)})\frac{\sigma^{T_s}}{\sqrt{2a^{T_s}(1-e^{-2a^{T_s}\Delta t})}}]M_{sn}^{(i)}|T_s \in \mathcal{T}\}$$

and

$$\rho_{P_n}^{\sup}(\Delta X_{n+1}|\underline{\Delta Y}^{(0)}, \underline{\Delta Y}^{(1)}, \ldots, \underline{\Delta Y}^{(k)}, \mathcal{F}_n^{f_1}, \ldots, \mathcal{F}_n^{f_k})$$
$$= \rho_{P_n}(\Delta X_{n+1}|\underline{\Delta Y}^{(0)}, \underline{\Delta Y}^{(1)}, \ldots, \underline{\Delta Y}^{(k)}, \mathcal{F}_n^{f_1}, \ldots, \mathcal{F}_n^{f_k})$$

Remarks:

(1) In order to obtain a conservative risk measure, we multiply $M_{sn}^{(i)}$ by $[E_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i}) + \sigma_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i})]$ if $M_{sn}^{(i)}$ is negative and by $E_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i})$ if $M_{sn}^{(i)}$ is positive, for $i = 1, 2, \ldots, k$. To achieve this, we introduce the indicator function $I(-M_{sn}^{(i)})$. However, the introduction of $I(-M_{sn}^{(i)})$ makes the risk measure not satisfy the property of translation invariance.

(2) To define the lower risk limit, we multiply $M_{sn}^{(i)}$ by $E_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i})$ if $M_{sn}^{(i)}$ is positive and by $[E_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i}) + \sigma_{T_s}(f_{i,n+1}|\mathcal{F}_n^{f_i})]$ if $M_{sn}^{(i)}$ is negative, for $i = 1, 2, \ldots, k$. To achieve this, we introduce the indicator function $I(M_{sn}^{(i)})$.

§9 Conclusion and Further Researches

We have introduced a conditional risk measure which can be updated sequentially as more information is obtained. Then, a Bayesian risk measure is defined as a special case of the conditional risk measure in order to capture the subjective view of the risk trader. For a conjugate-prior case, we can obtain a closed form expression for the Bayesian risk measure. For a non-conjugate-prior case, we use the Bühlman credibility estimator to approximate the Bayesian risk measure. Also, the Bayesian risk measure can be applied to measure both the financial risks and insurance risks. We have illustrated how to apply the Bayesian risk measure to calculate premiums and to measure the risks of an insurance company. Finally, we give an alternative method to capture the impacts of random foreign exchange rates on global investments in our model. A modification of the Bayesian risk measure is introduced.

For further developments of our model, we suggest that different priors can be used in order to have a more realistic model. As long as they are conjugate-prior cases, we can obtain the closed form solutions. Otherwise, we can use some simulation techniques such as Markov Chain Monte Carlo Method to approximate the Bayesian risk measure. The disadvantage, however, of this method is that it is time-consuming if the rate of convergence is very slow. If the data set is too large and the updating of the risk measure is too frequent, the computer may not have enough capacity to handle the calculations. Finally, we can study the problem of robustness of the Bayesian risk measure with respect to the change in prior distributions. This idea is similar to performing stress test of our risk measure with respect to the extreme market movement. If the risk measure is robust, then we get a risk interval with a small length. Otherwise, we get a wide risk interval.

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