Bonus systems in an open portfolio*

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Abstract

In this paper we study Bonus systems in an open portfolio, i.e.we consider that a policyholder can transfer his policy from our to another insurance company and vice-versa. We make use of non-homogeneous Markov chains to model the system and show, under quite fair assumptions, that the stationary distribution is independent of the market shares, and is very easily calculated.

Keywords

Bonus systems; Markov chains; stationary distribution; open model; closed model;

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1 Introduction

Bonus systems can be developed using the theory of Markov chains. This has already been done by (Loimaranta 1972), (Norberg 1976), (Borgan, Hoem, and Norberg 1981), (Gilde and Sundt 1989) and (Lemaire 1995).

In this paper we follow very close the work of (Norberg 1976), (Borgan, Hoem, and Norberg 1981) and (Gilde and Sundt 1989) and generalize their work to cope with the portuguese situation.

There is a big variety of bonus scales in Portugal. Each company can have its own tariff model, including experience rating systems. There exist significant moves among companies, explained in part by the aggressiveness of the market, but also by the lack of data transmission among insurance companies. It is possible to leave a company and to declare to another one that it is the first motor insurance policy that is being bought. As a consequence the policyholders placed in a severe class will tend to leave the company. Hence, although there is a starting class for the drivers buying (or declaring to buy) an automobile policy for the first time, there are some policyholders that are placed in some other class depending on the record of claims reported to the former company or on the commercial aggressiveness policy of the insurance company.

Considering the portuguese situation, which should be common to other countries, we tried to model the system to include transfers between the insurance company and the rest of the market. We will name this model by "open model" as opposed to the model presented in section 2, which will be named by "closed model".

We assume that the structure of the transfers is the same along the different periods.

In section 2 we summarize the results of (Norberg 1976), (Borgan, Hoem, and Norberg 1981) and (Gilde and Sundt 1989), in section 3 we present the "open model " and in section 4 we give an example using both models.

2 Bonus Systems and Homogeneous Markov Chains

The papers of (Norberg 1976), (Borgan, Hoem, and Norberg 1981) and (Gilde and Sundt 1989) assume that a bonus system can be dealt under the framework of homogeneous Markov chains.

In their papers a bonus system is supposed to be an experience rating system such that:

1. The insurance periods are of equal duration (generally one year).

2. The policies in the portfolio are divided into a finite number of classes, numbered from 1 to K. A policy stays in one and the same class through one insurance period.

3. The one-period premium is $b(j)$ for all the policies in the portfolio in class j. The vector $\mathbf{b}=(b(1),\ldots,b(K))$ is called the bonus scale.

4. All policies are placed in the same initial class, say k, in the first period.

5. The transition rules are such that the bonus class of a policyholder in any period can be determined as a function of the bonus class and the number of claims in the preceding period, which implies that the system is Markovian.

The transition rules are represented by a $K \times K$ matrix $\mathbf{T}$, whose entry $T(i,j)$ is the set of claim numbers leading from class i to class j. When the length period is given, the bonus system is defined by the triplet $S = (\mathbf{T}, \mathbf{b}, k)$.

It is assumed that $\mathbf{T}$ is such that the Markov chain is irreducible (i.e. all the classes communicate), and that there exists an "elite" class with the property that a policy in that class remains there after a claim free period, which implies that the Markov chain is aperiodic.

Let X_n be the total claim amount of a single risk in period n, chosen at random from an automobile portfolio of an insurance company. It is assumed that $\{X_n\}$ are i.i.d. random variables and that each of them has a compound distribution, with claim numbers M_n and claim severities $\{Y_{nj}\}_{j=1,\ldots}$, which are supposed to be i.i.d. and independent of M_n. The value of θ, the accident proneness of the risk, is assumed to be picked out of the collective and it is regarded as the outcome of a positive random variable Θ with distribution function $U(.)$. It is assumed that X_n depends on θ only through M_n.

Let $Z_{S,n}$ denote the bonus class in period n for a given policy. Then, for fixed $\Theta = \theta$, $\{Z_{S,n}\}_{n=1,\ldots,\infty}$ is a homogeneous Markov chain, with state space $\{1,2,\ldots,K\}$, and with one-step transition probability matrix $\mathbf{P}_{T,\theta} = [p_{T,\theta}(i,j)]$. For a given $\Theta = \theta$, the distribution of $Z_{S,n}$ is

$$p_{S,\theta}^{(n)}(j) = \Pr_\theta(Z_{S,n} = j), \;\; j = 1,2,\ldots,K,$$

which depends on S though $\mathbf{T}$ and k. As the Markov chain is finite, irreducible and aperiodic it has a limiting distribution $\boldsymbol{\pi}=[\pi_{T,\theta}(j)]$

$$\pi_{T,\theta}(j) = \lim_{n\longrightarrow\infty} p_{S,\theta}^{(n)}(j), \;\; j=1,\ldots,K, \tag{1}$$

which is the stationary distribution, i.e. the left normalized eigenvector associated to the unit eigenvalue of the matrix $\mathbf{P_{T,\theta}}$.

The unconditional distribution of $Z_{S,n}$ is

$$p_S^{(n)}(j) = \Pr\{Z_{S,n} = j\} = \int_0^\infty p_{S,\theta}^{(n)}(j)dU(\theta), \;\; j=1,2,\ldots,K. \tag{2}$$

If Z_T is a random variable with conditional distribution, for each value of θ, equal to the limit distribution of $Z_{S,n}$, that is

$$\Pr_\theta\{Z_T = j\} = \pi_{T,\theta}(j), \;\; j=1,\ldots,K,$$

then the unconditional distribution of Z_T is

$$\pi_T(j) = \Pr\{Z_T = j\} = \int_0^\infty \pi_{T,\theta}(j)dU(\theta), \;\; j=1,\ldots,K.$$

If it is considered that the expected value of the claim severity is 1, which implies that this expected value is chosen as monetary unit, and if the claim number distribution is parametrized in such a way that its expected value is θ, then the value $b_T(j)$, that minimizes the mean square error $Q_0(S) = \mathrm{E}\{[\mathrm{E}(X^*|\theta) - b_T(Z_T)]^2\}$ is $b_T(j) = b_T^A(j)$ satisfying

$$\begin{aligned} b_T^A(j) = \mathrm{E}[\mathrm{E}(X^*|\theta)|Z_T=j] &= \frac{\int_0^\infty \mathrm{E}(X^*|\theta)\pi_{T,\theta}(j)dU(\theta)}{\pi_T(j)} \\ &= \frac{\int_0^\infty \theta\pi_{T,\theta}(j)dU(\theta)}{\pi_T(j)}, \;\; j=1,\ldots,K, \end{aligned} \tag{3}$$

where X^* is identically distributed with X_n. Equation (3) defines the optimal bonus scale for the specific transition rules $\mathbf{T}$. It does not depend on the initial class k, because an asymptotic criterion was used. This is the bonus scale proposed by (Norberg 1976).

(Borgan, Hoem, and Norberg 1981) generalized (Norberg 1976) work by introducing a set of nonnegative weights w_n, $n = 0,2,\ldots$, with sum equal 1, and by measuring the performance of S by a weighted average of the form

$$Q(S) = \sum_{n=0}^\infty w_n Q_n(S) = \sum_{n=0}^\infty w_n \int_0^\infty \sum_{j=1}^K [\mathrm{E}(S_n|\theta) - b(j)]^2 p_{S,\theta}^{(n)}(j)dU(\theta). \tag{4}$$

w_n represents the weight given to period n, $n = 1, 2, \ldots$ and w_0 represents the weight to give to the stationary distribution.

The minimizer of (4) is $b(j) = b^B_{T,k}(j)$,

$$b^B_{T,k}(j) = \frac{\sum_{n=0}^{\infty} w_n \int_0^{\infty} \mathrm{E}(S_n|\theta) p^{(n)}_{S,\theta}(j) dU(\theta)}{p_S(j)}, \tag{5}$$

with

$$p_S(j) = \sum_{n=0}^{\infty} w_n \int_0^{\infty} p^{(n)}_{S,\theta}(j) dU(\theta). \tag{6}$$

The solution is now dependent of the starting class k, with exception of the solution when $w_0 = 1$ and $w_n = 0$, for $n > 0$, which corresponds to (3).

(Gilde and Sundt 1989) by imposing a liner scale, i.e. by minimizing (4) subject to constraints of the form $b(j) = a + bj$, $j = 1, \ldots, K$, to smooth the irregularities of the former scales, got that the optimal solution is $b^L_{T,k}(j) = a^L + b^L j$ with

$$\begin{cases} b^L = \dfrac{\sum_{j=1}^{s} j b^B_{T,k}(j) p_S(j) - \sum_{j=1}^{s} j p_S(j) \sum_{j=1}^{s} b^B_{T,k}(j) p_S(j)}{\sum_{j=1}^{s} j^2 p_S(j) - \left(\sum_{j=1}^{s} j p_S(j)\right)^2} \\ a^L = \sum_{j=1}^{s} b^B_{T,k}(j) p_S(j) - b_L \sum_{j=1}^{s} j p_S(j). \end{cases} \tag{7}$$

3 The open model

Let us consider that the collective is now the set of all the drivers with an insurance policy in a given country. Let q_0, $0 < q_0 < 1$, be the market share at period 0, of the portfolio of a given insurance company, and let us assume that this share varies from period $n - 1$ to period n according to a rate r_n, i.e.

$$q_n = q_{n-1}(1 + r_n),$$

Consequently,

$$q_n = q_0 \prod_{j=1}^{n} (1 + r_j) = q_0 \exp\left(\sum_{j=1}^{n} \ln(1 + r_j)\right). \tag{8}$$

The system can now be modelized by means of a non-homogenous Markov chain with $K+1$ states, where state labeled $K+1$ refers to the outside world of the company. The first K states correspond to the bonus classes.

We assume that the policies, when entering the portfolio, are placed in the different classes of bonus according to the line vector $\mathbf{v} = [v_i]$, where v_i is the probability for a new policy to be placed in class i, $i = 1, 2, \ldots, K$, once it enters the system. We consider that the probability function $\mathbf{v}$ is independent of θ, but this assumption could be relaxed.

Let $R_m(i)$ be such that $R_m(i) = j$, with $i, j = 1, \ldots, K$, if the policy is transferred from class i to class j when m claims have been reported in one period of time. For each $i = 1, \ldots, K$, let $d_{m,\theta}(i)$ be the conditional probability given θ, that a policyholder which was in class i in the preceding period and declares m accidents in the period, exits the company (we assume that all the exits are made at the end of the period) and let $d_{S,\theta}(i)$ be the conditional probability given θ that a policyholder which was in class i in the preceding period exits the company. Let $p_\theta(m)$ be the probability that risk θ declares m accidents in a period of time. Then

$$d_{S,\theta}(i) = \sum_{m=0}^{\infty} p_\theta(m) d_{m,\theta}(i) \tag{9}$$

A reasonable assumption would be $d_{m,\theta}(i)$ independent of m ($= d_{S,\theta}(i)$), that would reflect the situation (probability close to reality) that a policyholder needs time to realize that his premium has increased a lot, when it happens.

The system, S, at a given period of time has now to reflect the transition rules $\mathbf{T}$, the vector $\mathbf{b}$ of premiums, the market share at the time period, the vector $\mathbf{v}$ of entrances and the probabilities $d_{m,\theta}(i)$.

Let $\mathbf{P}_{T,\theta} = [p_{T,\theta}(i,j)]$ be the $K \times K$ transition probability matrix in the "closed model" (as in Section 2) and let $\mathbf{A}_{S,\theta}^{(n,n+1)} = [a_{S,\theta}^{(n,n+1)}(i,j)]$ be the $(K+1) \times (K+1)$ transition probability matrix from period n to period $n+1$.

Let

$$L_T(i,j) = \{l : R_l(i) = j\},$$

then, it is straightforward to see that

$$a_{S,\theta}^{(n,n+1)}(i,j)=\sum_{l\in L_T(i,j)} p_\theta(l)\,(1-d_{l,\theta}(i)),\; i,j=1,...,K. \tag{10}$$

As $a_{S,\theta}^{(n,n+1)}(i,j)$ does not depend on n whenever $i,j=1,...,K$, we will drop the superscript $(n,n+1)$ and use just $a_{S,\theta}(i,j)$, for $i,j=1,...,K$. Let $\mathbf{A}^*_{S,\theta}=[a_{S,\theta}(i,j)]_{i,j=1,...,K}$. The elements $a_{S,\theta}^{(n,n+1)}(i,K+1)$ are equal to $d_{S,\theta}(i)$ for $i=1,...,K$.

Let $\mathbf{e}_{S,\theta}^{(n)}=[e_{S,\theta}^{(n)}(i)]$ be a line vector whose entries represent the probability that, in period n, risk θ is in state i, for $i=1,\ldots,K+1$. Given our assumption on the market share rates we have that $e_{S,\theta}^{(n+1)}(K+1)=1-q_{n+1}=1-q_n(1+r_{n+1})$, for $n=0,1,\ldots$ which implies, by conditioning on the state of the system at period n, that

$$1-q_{n+1}=a_{S,\theta}^{(n,n+1)}(K+1,K+1)(1-q_n)+\sum_{j=1}^{K} e_{S,\theta}^{(n)}(j)d_{S,\theta}(j)$$

from where it follows that

$$a_{S,\theta}^{(n,n+1)}(K+1,K+1)=\frac{1-q_n(1+r_{n+1})-\sum_{j=1}^{K} e_{S,\theta}^{(n)}(j)d_{S,\theta}(j)}{1-q_n}. \tag{11}$$

Then the other entries $a_{S,\theta}^{(n,n+1)}(K+1,j)$, $j=1,...,K$, are easily calculated according to

$$a_{S,\theta}^{(n,n+1)}(K+1,j)=\left(1-a_{S,\theta}^{(n,n+1)}(K+1,K+1)\right)v_j,\;\; j=1,\ldots,K. \tag{12}$$

The matrix $\mathbf{A}_{T,\theta}^{(n,n+1)}$ can then be written by blocks as

$$\mathbf{A}_{S,\theta}^{(n,n+1)}=\begin{bmatrix} \mathbf{A}^*_{S,\theta} & \mathbf{d}'_{S,\theta} \\ (1-a_{S,\theta}^{(n,n+1)}(K+1,K+1))\mathbf{v} & a_{S,\theta}^{(n,n+1)}(K+1,K+1) \end{bmatrix}$$

where $\mathbf{d}'_{S,\theta}$ is a column vector, with $\mathbf{d}'_{S,\theta}=[d_{S,\theta}(i)]_{i=1,...,K}$.

Considering that $q_n=q_0\prod_{j=1}^{n}(1+r_j)$ it is straightforward to prove the lemma that follows.

Lemma 1 *Let*

$$\mathbf{P}_{S,\theta}^{*(n,n+1)}=\begin{bmatrix} \widetilde{\mathbf{P}}_{S,\theta} & \mathbf{0} \\ (1-l^{(n,n+1)})\mathbf{v} & l^{(n,n+1)} \end{bmatrix}, \tag{13}$$

with $\widetilde{\mathbf{P}}_{S,\theta} = [\widetilde{\mathbf{p}}_{S,\theta}(i,j)]_{i,j=1,\dots,K}$, *where*

$$\widetilde{\mathbf{p}}_{S,\theta}(i,j) = a_{S,\theta}(i,j) + d_{S,\theta}(i)v(j),\ i,j = 1,\dots,K, \tag{14}$$

and

$$l^{(n,n+1)} = \frac{1 - q_0 \prod_{j=1}^{n+1}(1+r_j)}{1 - q_0 \prod_{j=1}^{n}(1+r_j)}, \tag{15}$$

then the transition probability matrix $\mathbf{A}_{S,\theta}^{(n,n+1)}$ *is equivalent to the matrix* $\mathbf{P}_{S,\theta}^{*(n,n+1)}$, *in the sense that*

$$\mathbf{e}_\theta^{(n+1)} = \mathbf{e}_\theta^{(n)} \mathbf{A}_{S,\theta}^{(n,n+1)} = \mathbf{e}_\theta^{(n)} \mathbf{P}_{S,\theta}^{*(n,n+1)},\ n = 0,1,\dots \tag{16}$$

If $0 < q_n < 1,\ n = 0,1,\dots$ then $\mathbf{P}_{S,\theta}^{*(n,n+1)}$ is a Markov matrix provided that $r_j \geqslant 0,\ j = 1,2,\dots$.If $r_{n+1} < 0$, then $l^{(n,n+1)} > 1$, but (16) still holds.

This result allows us to say that

$$\mathbf{e}_{S,\theta}^{(n)} = \mathbf{e}_\theta^{(0)} \mathbf{A}_{S,\theta}^{(0,n)} = \mathbf{e}_\theta^{(0)} \mathbf{P}_{S,\theta}^{*(0,n)};\ \ n = 1,2,\dots \tag{17}$$

and consequently

$$\lim_{n\longrightarrow\infty} \mathbf{e}_{S,\theta}^{(n)} = \mathbf{e}_\theta^{(0)} \lim_{n\longrightarrow\infty} \mathbf{A}_{S,\theta}^{(0,n)} = \mathbf{e}_\theta^{(0)} \lim_{n\longrightarrow\infty} \mathbf{P}_{S,\theta}^{*(0,n)}. \tag{18}$$

The matrix $\mathbf{P}_{S,\theta}^{*(0,n)}$ is easily calculated and is

$$\mathbf{P}_{S,\theta}^{*(0,n)} = \begin{bmatrix} \widetilde{\mathbf{P}}_{S,\theta}^n & \mathbf{0} \\ \mathbf{x}_{S,\theta}^{(0,n)} & l^{(0,n)} \end{bmatrix}, \tag{19}$$

with

$$\mathbf{x}_{S,\theta}^{(0,n)} = \frac{q_0}{1-q_0} \mathbf{v} \sum_{j=1}^{n} \left(r_j \prod_{i=1}^{j-1} (1+r_i) \widetilde{\mathbf{P}}_{S,\theta}^{n-j} \right), \tag{20}$$

and

$$l^{(0,n)} = \frac{1 - q_0 \prod_{j=1}^{n}(1+r_j)}{1-q_0} \tag{21}$$

We are interested in applying Norberg's asymptotic criterion to determine the optimal bonus scale in the open model. We need to know the limiting distribution (18). Before we calculate it, we will state the following lemma which will be used later.

Lemma 2

$$1 + \sum_{j=1}^{k} r_j \prod_{i=1}^{j-1} (1+r_i) = \prod_{j=1}^{k} (1+r_j),\ k = 1,2,\dots \tag{22}$$

Hence if $\sum_{j=1}^{\infty} ln(1+r_j)$ *is an absolutely convergent series then* $\sum_{j=1}^{\infty} r_j \prod_{i=1}^{j-1}(1+r_i)$ *is also absolutely convergent.*

Proof. *(22) is valide for* $k=1$ *by definition. Let us assume that is valid for* $k=n$ *. Then it is also valid for* $k=n+1$*,since*

$$\begin{aligned} 1+\sum_{j=1}^{n+1} r_j \prod_{i=1}^{j-1}(1+r_i) &= 1+r_{n+1}\prod_{i=1}^{n}(1+r_i)+\sum_{j=1}^{n} r_j \prod_{i=1}^{j-1}(1+r_i) \\ &= r_{n+1}\prod_{i=1}^{n}(1+r_i)+\prod_{j=1}^{n}(1+r_j) = \\ &= (1+r_{n+1})\prod_{j=1}^{n}(1+r_j)=\prod_{j=1}^{n+1}(1+r_j) \end{aligned}$$

Theorem 1 *Provided that* r_n, $n=1,2,...$, *are such that* $0<q_n=q_0\prod_{j=1}^{n}(1+r_j)<1$, $\forall n$ *and that* $\lim_{n\to\infty} q_0\prod_{j=1}^{n}(1+r_j)$ *exists and is in the open interval (0,1), then the long run distribution of the policyholders among the* K *classes of the Bonus system is independent of the market shares and is given by the stationary distribution of the matrix* $\widetilde{\mathbf{P}}_{S,\theta}$, *i.e. is given by the vector* $\widetilde{\boldsymbol{\pi}}_{S,\theta}=[\widetilde{\pi}_{T,\theta}(i)]_{i=1,...,K}$ *satisfying the system of equations*

$$\begin{cases} \widetilde{\boldsymbol{\pi}}_{S,\theta}=\widetilde{\boldsymbol{\pi}}_{S,\theta}\widetilde{\mathbf{P}}_{S,\theta} \\ \sum_{i=1}^{K}\widetilde{\pi}_{S,\theta}(i)=1 \end{cases}, \tag{23}$$

what is to say it is given by the normalised left eigenvector of the matrix $\widetilde{\mathbf{P}}_{T,\theta}$ *associated to the eigenvalue 1.*

Proof. Note that $\widetilde{\mathbf{P}}_{S,\theta}$, whose entries are given by (14), is calculated as a function of the transition probability matrix in the closed model $\mathbf{P}_{T,\theta}$, of $d_{S,\theta}(i)$ given by (9) and of the vector $\mathbf{v}$. Hence that matrix and its eigenvectors depend on all these quantities. Note however that an entry of $\widetilde{\mathbf{P}}_{S,\theta}$ is zero if and only if the respective entry of $\mathbf{P}_{T,\theta}$ is zero. Then, if the rules are such that the Markov chain in the closed model is finite, irreducible and aperiodic, the same happens to the Markov matrix $\widetilde{\mathbf{P}}_{S,\theta}$, and we can guaranty using the Jordan canonical form, see for instance (Cox and Miller 1965), that there exists a matrix $\mathbf{B}_{S,\theta}$ such that $\widetilde{\mathbf{P}}_{S,\theta}=\mathbf{B}_{S,\theta}^{-1}\mathbf{J}_{S,\theta}\mathbf{B}_{S,\theta}$, where $\mathbf{B}_{S,\theta}$ is the matrix of the left eigenvectors of $\widetilde{\mathbf{P}}_{S,\theta}$ and $\mathbf{J}_{S,\theta}$ is the Jordan matrix

$$\mathbf{J}_{S,\theta}=\begin{bmatrix} \mathbf{J}_1(1)=1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m_2}(\lambda_2) & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{J}_{m_l}(\lambda_l) \end{bmatrix}, \tag{24}$$

where $\lambda_1 = 1, \lambda_2, ..., \lambda_l$, are the eigenvalues of $\widetilde{\mathbf{P}}_{S,\theta}$ with multiplicity $1, m_2, ..., m_l$, respectively, and are such that $|\lambda_i| < 1$ for $i = 2, ..., l$ and with

$$\mathbf{J}_{m_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix} = \lambda_i \mathbf{I}_{m_i} + \mathbf{M}_{m_i}, \tag{25}$$

where $\mathbf{I}_{m_i}$ is the identity matrix $(m_i \times m_i)$ and $\mathbf{M}_{m_i}$ is a $(m_i \times m_i)$ matrix with

$$\mathbf{M}_{m_i} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{26}$$

Then

$$\begin{aligned} \mathbf{x}_{S,\theta}^{(0,n)} &= \mathbf{v}\tfrac{q_0}{1-q_0} \textstyle\sum_{j=1}^{n} r_j \widetilde{\mathbf{P}}_{S,\theta}^{n-j} \prod_{i=1}^{j-1}(1+r_i) \\ &= \mathbf{v}\tfrac{q_0}{1-q_0} \mathbf{B}_{S,\theta}^{-1} \left(\textstyle\sum_{j=1}^{n} r_j \mathbf{J}_{S,\theta}^{n-j} \prod_{i=1}^{j-1}(1+r_i)\right) \mathbf{B}_{S,\theta} \end{aligned} \tag{27}$$

We will prove now that, under our assumptions on the market shares,

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} r_j \mathbf{J}_{S,\theta}^{n-j} \prod_{i=1}^{j-1}(1+r_i) \right) = \sum_{j=1}^{n} r_j \prod_{i=1}^{j-1}(1+r_i) \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{28}$$

We just have to note that for $n - j \geqslant m_i$,

$$\mathbf{J}_{m_i}^{n-j}(\lambda_i) = \lambda_i^{n-j}\mathbf{I}_{m_i} + \binom{n-j}{1}\lambda_i^{n-j-1}\mathbf{M}_{m_i} + \dots + \binom{n-j}{m_i-1}\lambda_i^{n-j-m_i+1}\mathbf{M}_{m_i}^{m_i-1},$$

and that, if $|\lambda_i| < 1$, $\sum_{j=1}^{n} r_j \mathbf{J}_{m_i}^{n-j}(\lambda_i) \prod_{i=1}^{j-1}(1+r_i)$ has as limit the null matrix, because all its elements go to 0 when n goes to infinity. That all its elements go to zero can be shown because

$$\sum_{j=1}^{n} \lambda_i^{n-j} r_j \prod_{i=1}^{j-1}(1+r_i) = \sum_{k+j=n} \lambda_i^k r_j \prod_{i=1}^{j-1}(1+r_i) \tag{29}$$

can be regarded as the general term of Cauchy product of the series $\sum_{k=1}^{\infty} \lambda_i^k$ (which is absolutely convergent) and the series $\sum_{j=1}^{\infty} r_j \prod_{i=1}^{j-1}(1+r_i)$ (which is also absolutely convergent, because of lemma 2). As the Cauchy product of two absolutely convergent series is still convergent, then its general term has to vanish when n goes to infinity.

Hence, taking limits to both sides of (27) and using (28)we get that

$$\lim_{n\to\infty} \mathbf{x}_{S,\theta}^{(0,n)} = \frac{q_0}{1-q_0}\sum_{j=1}^{\infty} r_j \prod_{i=1}^{j-1}(1+r_i)\mathbf{v}\widetilde{\mathbf{\Pi}}_{S,\theta}, \tag{30}$$

where $\widetilde{\mathbf{\Pi}}_{S,\theta}$ is the limit of the Markov matrix $\widetilde{\mathbf{P}}_{S,\theta}^{n}$, that is to say it has all the lines equal to the stationary distribution $\widetilde{\boldsymbol{\pi}}_{S,\theta}$. Hence (30) is equivalent to

$$\begin{aligned}\lim_{n\to\infty} \mathbf{x}_{S,\theta}^{(0,n)} &= \tfrac{q_0}{1-q_0}\textstyle\sum_{j=1}^{\infty} r_j \prod_{i=1}^{j-1}(1+r_i)\widetilde{\boldsymbol{\pi}}_{S,\theta} = \\ &= \tfrac{1}{1-q_0}(q_\infty - q_0)\widetilde{\boldsymbol{\pi}}_{S,\theta}\end{aligned} \tag{31}$$

by Lemma 2, with $q_\infty = q_0 \prod_{i=1}^{\infty}(1+r_i)$ to represent the long run market share of the company.

Then

$$\lim_{n\to\infty} e_{S,\theta}^{(n)}(i) = q_\infty \widetilde{\pi}_{S,\theta}(i),\ i = 1,...,K, \tag{32}$$

and hence the probability that θ is in class i of the bonus system provided that it is in the company is $\widetilde{\pi}_{S,\theta}(i)$.

Hence, if we wish to calculate the optimal bonus scale in the open model, when Norberg's asymptotic criterion is used, we only have to calculate the matrix $\widetilde{\mathbf{P}}_{S,\theta}$, according to (14) and apply (3) (or (7) with $w_0 = 1$ if we wish a linear scale).

If we wish to calculate the optimal bonus scale in the open model, using the generalization proposed in (Borgan, Hoem, and Norberg 1981) we need to forecast the market shares of the company in the future, to calculate $\mathbf{e}_{S,\theta}^{(n)}$ according to (17) and to use (5) with $p_{S,\theta}^{(n)}(j)$ substituted by $e_{S,\theta}^{(n)}(j)/q_n$. The same substitution is made in (7) when we wish to apply the linear scale with some positive w_n for some $n \geqslant 1$.

4 An Example

Let us consider the Swiss bonus system, which is a system with 22 classes, numerated in an increasing order of dangerousness, with entrance in class 10 (for a first automobile insurance policy), and with the following transition rules for a policy in class i: for each claim free year the policy goes to class $\max(i-1,1)$ and with m claims it goes to $\min(i+4m,22)$.

We assume that the number of claims M_n, conditional on $\Theta = \theta$, is Poisson distributed with parameter θ, and that $\Theta = \theta$ is distributed according to the discrete structure function in Table 1.

Table 1: The structure function

θ	0.0050	0.0165	0.0310	0.0485	0.0690	0.0925	0.1190	0.1485	0.1810	0.2165
$u(\theta)$	0.2142	0.1368	0.1185	0.1039	0.0898	0.0761	0.0630	0.0509	0.0401	0.0307
θ	0.2550	0.2965	0.3410	0.3885	0.4390	0.4925	0.5490	0.6105	0.6845	0.8000
$u(\theta)$	0.0231	0.0169	0.0120	0.0084	0.0057	0.0038	0.0024	0.0016	0.0011	0.0010

We chose the $d_{m,\theta}(i)$'s independent of m and θ. Those figures together with distribution $\mathbf{v}$, are presented in Table 2.

Table 2: The vector $\mathbf{v}$

i	1	2	3	4	5	6	7	8	9	10	11
$v(i)$	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.79	0.005
$d_{m,\theta}(i)$	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.095	0.11	0.125	0.14
i	12	13	14	15	16	17	18	19	20	21	22
$v(i)$	0.005	0.005	0.005	0.002	0.002	0.002	0.002	0.001	0.001	0	0
$d_{m,\theta}(i)$	0.155	0.17	0.185	0.2	0.22	0.24	0.26	0.28	0.30	0.32	0.34

Table 3 gives the results for the open and closed models. The first comments suggested by Table 3 is that the two long run distributions are quite different. Taking the open model as close to reality, we can say that the closed model overvaluates the probabilities of the extreme classes. As a consequence, Norberg's asymptotic criterion, when applied to the closed model, under prices the extremes (both the elite and the worst classes). When the linear scale is used, we obtain a smaller premium in all the classes for the open model with exception of class 1 and 2.

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Table 3: Long run distributions and optimal scales

	Closed Model			Open Model		
Class j	$\pi_T(j)$	$b_T^A(j)$	$b_T^L(j)$	$\pi_S(j)$	$b_S^A(j)$	$b_S^L(j)$
1	0.6901	0.0395	0.0413	0.5573	0.0418	0.0426
2	0.0284	0.0852	0.0558	0.0355	0.0828	0.0561
3	0.0310	0.0884	0.0703	0.0391	0.0871	0.0695
4	0.0339	0.0916	0.0848	0.0437	0.0922	0.0830
5	0.0373	0.0951	0.0993	0.0499	0.0983	0.0964
6	0.0138	0.1283	0.1138	0.0336	0.1083	0.1099
7	0.0133	0.1343	0.1283	0.0365	0.1144	0.1233
8	0.0125	0.1415	0.1429	0.0405	0.1221	0.1368
9	0.0113	0.1507	0.1574	0.0461	0.1322	0.1502
10	0.0085	0.1699	0.1719	0.0526	0.1448	0.1637
11	0.0082	0.1789	0.1864	0.0114	0.1870	0.1771
12	0.0079	0.1894	0.2009	0.0112	0.2007	0.1906
13	0.0076	0.2016	0.2154	0.0104	0.2169	0.2040
14	0.0073	0.2159	0.2300	0.0084	0.2349	0.2175
15	0.0075	0.2284	0.2445	0.0043	0.2416	0.2309
16	0.0078	0.2424	0.2590	0.0041	0.2580	0.2444
17	0.0084	0.2580	0.2735	0.0036	0.2766	0.2578
18	0.0092	0.2753	0.2880	0.0029	0.2949	0.2713
19	0.0104	0.2941	0.3025	0.0018	0.2976	0.2847
20	0.0122	0.3156	0.3171	0.0019	0.3254	0.2982
21	0.0148	0.3401	0.3316	0.0021	0.3636	0.3116
22	0.0187	0.3682	0.3461	0.0029	0.4040	0.3251

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