Panjer class united – one formula for the Poisson, Binomial, and Negative Binomial distribution

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Abstract

This paper gives a formula representing all discrete loss distributions of the Panjer class (Poisson, Binomial, and Negative Binomial) in one. Further it provides an overview of the many Negative Binomial variants used by actuaries.

Keywords

Panjer class, discrete loss distribution, Negative Binomial

1 Introduction

The three well-known discrete loss distributions Poisson, Binomial, and Negative Binomial are closely related. First of all, they form the Panjer class. Secondly, the Poisson distribution is a limiting case of the two others, which finally have their origin in the modelling of Bernoulli trials. The probability (mass) functions of the three distributions look quite different, though.

In this paper we present a common representation of the distributions of the Panjer class.

Section 2 states classical parametrisations of the three distributions, adding a Binomial variant that is less customary but easier to compare to the other distributions. Section 3 presents a formula representing all three distributions.
To give some orientation in view of the confusing variety of parametrisations used in the actuarial world (especially for Negative Binomial) the appendix collects all those we have found in the literature.

2 Representations of the Panjer distributions

In order to see how the above distributions of loss numbers N are related we first collocate: probability function (pf) $p_k = P(X=k)$, probability generating function (pgf) $E(z^N)$, expected value E(N), and dispersion D(N) = Var(N)/E(N) of the three distributions.
Among the – not few – different parametrisations that can be found for them in the literature (for a discussion see the appendix) we particularly regard those using the expected value as a parameter, which in the following is denoted by $\lambda>0$. (We leave the degenerate case $\lambda=0$ aside, where all distributions coincide.) Note that in general insurance one often has models with $\lambda = v\theta$ where v is a measure of the size of the risk (or portfolio of risks) and θ the loss frequency per volume unit (see Mack [5, section 1.4.2]). For simplicity we will, however, always write λ.

- The Poisson distribution essentially has one common representation (P), using the expectation as the (only) parameter.
- For the Binomial distribution we state the classical parametrisation (B1) using the number of trials m and the probability of success p. We add another one (B2) where p is replaced by the expected value (of successes) $\lambda = mp$. $p < 1$ means $\lambda < m$.
- For the Negative Binomial distribution we first state the parametrisation (NB1) coming from Bernoulli trials, using the number of successes α (originally integer-valued but extendable to all positive real numbers) and the probability of success $p<1$. Then in (NB2) we replace p by the expectation (of failures) $\lambda = \alpha(1-p)/p$.

Regard Table 1.

Table 1

	pf	pgf	E(N)	D(N)
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{\lambda(z-1)}$	λ	1
B1	$\binom{m}{k}p^k(1-p)^{m-k}$	$(1-p+pz)^m$	mp	$1-p$
B2	$\binom{m}{k}\frac{\lambda^k(m-\lambda)^{m-k}}{m^m}$	$\left(1+\frac{\lambda}{m}(z-1)\right)^m$	λ	$1-\frac{\lambda}{m}$
NB1	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	$\frac{\alpha(1-p)}{p}$	$\frac{1}{p}$
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	λ	$1+\frac{\lambda}{\alpha}$

It seems that the traditional representations, namely B1 and NB1, somehow obscure the relationship between the three distributions. If we instead look at B2 and NB2 at least the pgfs look very similar, and here and in the formulae for the dispersion there is an obvious correspondence between α and m, or merely –m. This well-known correspondence (see Heckman & Meyers [2, sections 3 and 5]) will turn out to be the key of the common representation.

3 The all-in-one formula

Proposition: The formula

$$p_k = \left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}, \qquad k = 0, 1, 2, \dots \qquad (1)$$

describes the probability (mass) function of all distributions of the Panjer class. The parameter λ is the expected value and can take on all positive real numbers. The parameter α can take on the following values:

a) $\alpha \in \,]0; \infty[$: Negative Binomial.
b) $\alpha = \infty$, $\alpha = -\infty$: Poisson. (1) is well defined in this infinite case as the limits exist and coincide.
c) $\alpha \in \,]-\infty; -\lambda[\cap \mathbf{Z}$: Binomial. The parameter α here is restricted to integers –m satisfying m > λ.

The corresponding probability generating function is given by $E(z^N) = \left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$,

which again is well defined for infinite α.

Definition: We call the above parametrisation of the Panjer class distribution **Panjer United (PanU)**.

Proof: First we convert (1) into a known pf for each of the three cases.

a) We only have to rearrange the terms of NB2, noting that

$$\binom{\alpha+k-1}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(\alpha+i)\,, \quad \left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha} = \left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}, \quad \left(\frac{\lambda}{\alpha+\lambda}\right)^{k} = \lambda^k \prod_{i=0}^{k-1}\frac{1}{\alpha+\lambda}.$$

b) Recall that e = lim $(1+1/\alpha)^{\alpha}$ for $\alpha \to \infty$ and $\alpha \to -\infty$, therefore $\left(1+\frac{y}{\alpha}\right)^{-\alpha} = \left(\left(1+\frac{y}{\alpha}\right)^{\alpha/y}\right)^{-y}$ tends to e^{-y}. Hence the first factor in (1) equals $e^{-\lambda}$. Since the third factor equals 1 we are done.

c) If we set m := $-\alpha$ in B2 we get $\binom{m}{k}\frac{\lambda^k (m-\lambda)^{m-k}}{m^m} = \frac{1}{k!}\frac{\lambda^k(-\alpha-\lambda)^{-\alpha-k}}{(-\alpha)^{-\alpha}}\prod_{i=0}^{k-1}(-\alpha-i) =$

$$= \frac{\lambda^k}{k!}(-1)^k \frac{\alpha^{\alpha}}{(\alpha+\lambda)^{\alpha+k}}(-1)^k \prod_{i=0}^{k-1}(\alpha+i) = \left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$$

Note that (1) is well defined and valid even for k>m. In this case the p_k equal zero as the products $\prod_{i=0}^{k-1}(\alpha+i)$ contain the factor α+m=0.

The pgf formula is obvious for finite α, and for infinite α the reasoning is as in b) with y = $\lambda(z-1)$.

Finally, to see that we have a one-to-one correspondence of the parameters appearing in the usual representations of the Panjer class and in the PanU formula we only have to check that the restrictions for negative α coincide: The Binomial distribution has a positive integer m and λ is less than m. This translates to a negative integer α and to the condition $-\alpha > \lambda$, being exactly case c) of the Proposition.

Remark 1: If we regard the product to the right of (1) we can see quickly that it is not possible to extend the parameter space in the case of negative α: The first factor $\frac{\alpha+0}{\alpha+\lambda}$ must be positive, otherwise p_0 and p_1 would have different sign. Hence the denominator must be negative, i.e. $-\alpha > \lambda$. Now assume that α is not an integer. Then all factors $\frac{\alpha+k}{\alpha+\lambda}$, and with them all p_k, are non-zero. Thus the factors must be postive, otherwise p_k and p_{k+1} would have different sign. Hence all numerators $\alpha+k$ must be negative, but this is impossible as k is unlimited.

Corollary 1: With the above parametrisation the Panjer recursion reads $p_k = p_{k-1}\,(a + b/k)$ with $a = \frac{\lambda}{\alpha+\lambda}$, $b = \frac{(\alpha-1)\lambda}{\alpha+\lambda}$ and we have

$$E(N) = \lambda, \quad Var(N) = \lambda\left(1+\frac{\lambda}{\alpha}\right), \quad CV^2(N) = \frac{1}{\lambda}+\frac{1}{\alpha}, \quad D(N) = 1+\frac{\lambda}{\alpha}.$$

Note that again all formulae are well defined for infinite α.

Proof: From (1) we immediately get $p_k = p_{k-1}\frac{\lambda}{k}\frac{\alpha+k-1}{\alpha+\lambda} = p_{k-1}\left(\frac{\lambda}{\alpha+\lambda}+\frac{(\alpha-1)\lambda}{(\alpha+\lambda)k}\right)$

The following formulae are straightforward consequences of the Panjer recursion (see Schmidt [6, section 7.2]): $E(N) = \frac{a+b}{1-a}$, $Var(N) = \frac{a+b}{(1-a)^2}$, $CV^2(N) = \frac{1}{a+b}$, $D(N) = \frac{1}{1-a}$.

Plugging in $a+b = \frac{\alpha\lambda}{\alpha+\lambda}$, $1-a = \frac{\alpha}{\alpha+\lambda}$ yields the claimed results.

It is possible to define a Panjer United variant **PanU*** without the somewhat odd-looking infinite parameter values by replacing the parameter α by its inverse c = 1/α. This parameter was named "contagion" (see Heckman & Meyers [2, sections 3 and 5]) in order to give an intuitive meaning to deviations from the Poisson distribution: A higher (lower) probability of many losses was interpreted as positive (negative) contagion of losses. Here $0 < c < \infty$ is the Negative Binomial case, c = 0 corresponds to Poisson (no contagion) and negative c is the Binomial case with the quite intricate parameter restriction c = –1/m with integer m > λ > 0:

Corollary 2: The formula

$$p_k = \left(1+c\lambda\right)^{-1/c} \frac{\lambda^k}{k!} \prod_{i=0}^{k-1} \frac{1+ci}{1+c\lambda}, \qquad k = 0, 1, 2, \ldots \qquad (2)$$

describes the probability (mass) function of all distributions of the Panjer class. The parameter λ is the expected value and can take on all positive real numbers. The parameter c can take on the following values:

a) $c \in]0; \infty[$: Negative Binomial.
b) $c = 0$: Poisson. (2) is well defined as the limit of the first factor exists.
c) $c \in]-1/\lambda; 0[\cap \{1/z \mid z \in \mathbf{Z}\}$: Binomial.

The corresponding probability generating function is given by $E(z^N) = \left(1 - c\lambda(z-1)\right)^{-1/c}$, which again is well defined for all c as above.

The coefficients of the Panjer recursion are $a = \frac{c\lambda}{1+c\lambda}$, $b = \frac{(1-c)\lambda}{1+c\lambda}$ and we have

$E(N) = \lambda$, $Var(N) = \lambda + c\lambda^2$, $CV^2(N) = \frac{1}{\lambda} + c$, $D(N) = 1 + c\lambda$.

Remark 2: It is easy to calculate further quantities in the PanU representation, say higher moments. Note that we now have a common representation for the pgf $\left(1 - \frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$ coming originally from Negative Binomial (NB2) but being extendable to the whole Panjer class. The same extension is possible for the moments, since they are just values of the pgf and/or its derivatives, which have common formulae as well: The n-th derivative of the pgf equals $\lambda^n \left(1 - \frac{\lambda}{\alpha}(z-1)\right)^{-\alpha-n} \prod_{i=0}^{n-1} \left(1 + \frac{i}{\alpha}\right)$.
For finite α this is a straightforward calculation. (Note that this formula is correct for all integers n>0, even in the Binomial case where for n > m = –α the derivative equals zero.) The limit of the formula for $\alpha \to \pm\infty$ equals $\lambda^n e^{\lambda(z-1)}$, which is exactly the n-th derivative of the Poisson pgf. Hence the formula applies to all distributions of the Panjer class and it is clear without further calculations that e.g. the well-known skewness formula $E((N-E(N))^3) = \lambda\left(1 + \frac{\lambda}{\alpha}\right)\left(1 + \frac{2\lambda}{\alpha}\right)$ is not only valid for

positive α (NB2) but for all admissible values as in the Proposition, including the cases of negative skewness for Binomial distributions with $-\alpha/2 < \lambda < -\alpha$, i.e. $0.5 < p < 1$.

4 Conclusion

The "united" representation of the three Panjer distributions might not enable new mathematics to be done, but it is very instructive as it makes clearer how closely related and at the same time how different the three distributions are: Binomial and Negative Binomial appear very similar in the PanU representation, but they are in a way the opposite sides of a coin, being connected, or rather separated, by the limiting case Poisson.

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Appendix

In order to give an overview we enhance Table 1 by adding several variants of the Negative Binomial distribution, all being useful in certain areas but partly tricky to convert into each other. We start from the table provided by Mack (see in the following [5, section 1.4.2]) showing essentially three different ways of interpreting the distribution, all using α but having different second parameters:

- Bernoulli trial with probability p: NB1
- Expectation λ: NB2 (see also Johnson et al. [3] who dedicate their whole Chapter 5 to the Negative Binomial distribution)
- Poisson-Gamma: If the parameter of a Poisson distribution is Gamma distributed with density $\beta^{\alpha}\, x^{\alpha-1}e^{-\beta x}/\Gamma(\alpha)$ then as mixed distribution we get NB3, which inherits the parameters α and β from the Gamma distribution (see also e.g. Bühlmann & Gisler [1]).

Actuaries found two other useful second parameters:

- NB1b is a variant of the Bernoulli trial using q=1–p (see Johnson et al. [3], Schmidt [6]).
- NB4 has totally different sources. On one hand it is a Poisson-Gamma variant using ξ=1/β according to an alternative definition of the Gamma density having parameters α and the inverse of β (see Klugman et al. [4]). But it also comes about (see Johnson et al. [3]) when we apply the generalized binomial theorem (for non-integer exponents) to the term
$$1 = \left((1+\xi)-\xi\right)^{-\alpha} = \sum_{k=0}^{\infty}\binom{-\alpha}{k}(1+\xi)^{-\alpha-k}(-\xi)^k = \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\frac{\xi^k}{(1+\xi)^{\alpha+k}}.$$

The conversion of the parameters is as follows: $p = 1-q = \dfrac{\alpha}{\alpha+\lambda} = \dfrac{\beta}{1+\beta} = \dfrac{1}{1+\xi}$,

$$q = 1-p = \frac{\lambda}{\alpha+\lambda} = \frac{1}{1+\beta} = \frac{\xi}{1+\xi}, \qquad \lambda = \frac{\alpha(1-p)}{p} = \frac{\alpha q}{1-q} = \frac{\alpha}{\beta} = \alpha\xi,$$

$$\beta = \frac{p}{1-p} = \frac{1}{q}-1 = \frac{\alpha}{\lambda} = \frac{1}{\xi}, \qquad \xi = \frac{1}{p}-1 = \frac{q}{1-q} = \frac{\lambda}{\alpha} = \frac{1}{\beta}.$$

It will turn out from the overview that for any of these NB variants there is something it describes better (in a simpler way) than the others – but then it has more intricate formulae for other quantities that could be of interest. It seems that there is no "best" parametrisation for all actuarial needs, which probably is why so many different ones have been established.

See Table 2 containing 10 distributions (1 Poisson, 2 Binomial, 5 Negative Binomial, and the 2 all-in-one representations) providing for each: probability function, probability generating function, probability of no losses, expectation, variance, squared coefficient of variation, dispersion, and finally the parameters a and b of the Panjer recursion.

Remark: In this paper we have restricted ourselves to parametrisations using α or the inverse c. For completeness we mention two further parametrisations (see Johnson et al. [3]). They combine the expectation with one of the above second parameters:

- NB2/1b: λ together with q yields the pgf $\left(\dfrac{1-q}{1-qz}\right)^{\lambda\left(\frac{1}{q}-1\right)}$
- NB2/4: λ together with ξ yields the pgf $\left(1-\xi(z-1)\right)^{-\lambda/\xi}$

Table 2

	pf	pgf	P(N = 0)
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{\lambda(z-1)}$	$e^{-\lambda}$
B1	$\binom{m}{k}p^k(1-p)^{m-k}$	$(1-p+pz)^m$	$(1-p)^m$
B2	$\binom{m}{k}\frac{\lambda^k(m-\lambda)^{m-k}}{m^m}$	$\left(1+\frac{\lambda}{m}(z-1)\right)^m$	$\left(1-\frac{\lambda}{m}\right)^m$
NB1	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	p^α
NB1b	$\binom{\alpha+k-1}{k}(1-q)^\alpha q^k$	$\left(\frac{1-q}{1-qz}\right)^\alpha$	$(1-q)^\alpha$
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha$
NB3	$\binom{\alpha+k-1}{k}\frac{\beta^\alpha}{(1+\beta)^{\alpha+k}}$	$\left(1-\frac{z-1}{\beta}\right)^{-\alpha}$	$\left(\frac{\beta}{1+\beta}\right)^\alpha$
NB4	$\binom{\alpha+k-1}{k}\frac{\xi^k}{(1+\xi)^{\alpha+k}}$	$(1-\xi(z-1))^{-\alpha}$	$(1+\xi)^{-\alpha}$
PanU	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}$
PanU*	$(1+c\lambda)^{-1/c}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{1+ci}{1+c\lambda}$	$(1-c\lambda(z-1))^{-1/c}$	$(1+c\lambda)^{-1/c}$

Parameters: p, q ∈]0;1[. m positive integer. λ, α, β, ξ, c∈]0; ∞[. PanU admits more values for α, c.

E(N)	**Var(N)**	**CV²(N)**	**D(N)**	**a**	**b**	
λ	λ	$\frac{1}{\lambda}$	1	0	λ	**P**
mp	$mp(1-p)$	$\frac{1-p}{mp}$	$1-p$	$\frac{-p}{1-p}$	$\frac{(m+1)p}{1-p}$	**B1**
λ	$\lambda\left(1-\frac{\lambda}{m}\right)$	$\frac{1}{\lambda}-\frac{1}{m}$	$1-\frac{\lambda}{m}$	$\frac{-\lambda}{m-\lambda}$	$\frac{(m+1)\lambda}{m-\lambda}$	**B2**
$\frac{\alpha(1-p)}{p}$	$\frac{\alpha(1-p)}{p^2}$	$\frac{1}{\alpha(1-p)}$	$\frac{1}{p}$	$1-p$	$(\alpha-1)(1-p)$	**NB1**
$\frac{\alpha q}{1-q}$	$\frac{\alpha q}{(1-q)^2}$	$\frac{1}{\alpha q}$	$\frac{1}{1-q}$	q	$(\alpha-1)q$	**NB1b**
λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	**NB2**
$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta}\left(1+\frac{1}{\beta}\right)$	$\frac{1+\beta}{\alpha}$	$1+\frac{1}{\beta}$	$\frac{1}{1+\beta}$	$\frac{\alpha-1}{1+\beta}$	**NB3**
$\alpha\xi$	$\alpha\xi(1+\xi)$	$\frac{1+\xi}{\alpha\xi}$	$1+\xi$	$\frac{\xi}{1+\xi}$	$\frac{(\alpha-1)\xi}{1+\xi}$	**NB4**
λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	**PanU**
λ	$\lambda+c\lambda^2$	$\frac{1}{\lambda}+c$	$1+c\lambda$	$\frac{c\lambda}{1+c\lambda}$	$\frac{(1-c)\lambda}{1+c\lambda}$	**PanU***