# Decomposing total risk of a portfolio into the contributions of individual assets

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**Abstract.** Analysing the concentration risk in the portfolio is one of the important issues for the risk management of financial institutions. Among several measures proposed in order to quantify the concentration risk, we consider the risk contribution (RC) of asset $j$, defined by $RC_j \equiv a_j \partial R_p / \partial a_j$, where $a_j$ is the holding amount of asset and $R_p$ is the total risk of the portfolio. $RC_j$ can satisfy the additivity; the sum of RCs of all assets are equal to the total risk of the portfolio $R_p$. We can select many famous risk measures as $R_p$ such as standard deviation, Value at Risk (VaR) and Expected Shortfall (ES), however, the accurate and robust estimation of RC is very difficult. One of the hopeful methods is "hybrid method." In this method, assuming that the future prices of individual assets are conditionally independent with respect to the risk factors, we generate some scenarios of the risk factors by the Monte Carlo simulation, and calculate the conditional distribution of the future value of the portfolio and RCs of individual assets analytically by using the saddlepoint approximation. The unconditional estimates are obtained as the expectation of conditionals. The hybrid method gives much more reliable estimates of VaR and RCs to VaR than the ordinary Monte Carlo simulation, however, the accuracy of the estimates of ES and RCs to ES is not so good. In this article, we summarize the hybrid method in more general setting, and propose a more accurate estimation method of ES and RCs for ES. Since this new method is based on the universal mathematical relation between VaR and ES, it can be applicable to many risk evaluation models. We show some numerical examples to confirm some merits of our method.

**Keywords:** risk management, marginal risk contribution, VaR (Value at Risk), ES (Expected Shortfall), conditional independence, saddlepoint approximation.

# 1 Introduction

The risk measurement of a portfolio consisting of many financial instruments is very important for the financial institutions. Its importance grows more and more in these days, especially after the worldwide financial crisis because this crisis showed that the previous risk management methods could not work well in the spread of the crisis all over the world.

One of the most important roles of the risk measurement is estimating the potential loss of a portfolio, monitoring and reporting them to the investors periodically. The other one is the concentration risk analysis of the portfolio, that is, measuring how much each asset contributes to the total risk in the portfolio. The first step of the concentration risk analysis is estimating the volumes of the risk of individual assets (or subportfolios), of course, with taking the diversification effect into consideration. Some risk measures have been proposed for the concentration risk analysis, however, it is very difficult to estimate such risk measures accurately and robustly, especially by the Monte Carlo simulation, which is often used in the practical risk evaluation models.

The risk contribution is one of the risk measures for the concentration risk analysis, and its significant property is that the additivity of the risk contributions is satisfied if they can be defined well. Here, the additivity means that the sum of the risk contributions of individual assets included in a portfolio is equal to the total risk of the portfolio. However, its robust and accurate estimation is also difficult.

In 2001, Martin et al.[8] derived an approximated formula for the risk contribution of each asset to the portfolio's VaR (Value at Risk) by the saddlepoint method, and showed some numerical examples. Based on their research, Muromachi[10] proposed a new framework of risk evaluation models by assuming the conditional independence and using the saddlepoint method, and he showed that much more reliable estimates of the risk contributions for VaR and ES (Expected Shortfall) could be obtained by this new framework than those obtained by the ordinary Monte Carlo simulation. However, according to Muromachi[11], the estimates of ES and risk contributions for ES are not so reliable than those of VaR and risk contributions for VaR. Since the ES is one of the coherent risk measures (Artzner et al.[1]), the accurate and robust estimation of them is important for the theoretically desirable risk management, therefore, we have been seeking more reliable estimation methods.

In this article, we propose a new estimation method of ES and the risk contributions for ES. The proposed calculation method is based on a universal mathematical relation between

VaR and ES, and uses a saddlepoint approximation in order to calculate ES and its risk contributions easily and quickly. This article is organized as follows. In Section 2 we show a generalized hybrid method and some useful expressions; the approximated formulas for the conditional distribution and the risk contributions for VaR and ES. In order to estimate ES and risk contributions for ES, two methods are introduced; the method proposed by Muromachi[10] (old method) and the new one proposed in this article. In Section 3, we presents some numerical examples and compare the estimates by the old and new methods, and Section 4 concludes this article.

# 2 Hybrid method

This section describes the framework of what we call the hybrid method in more general setting than that described in Muromachi[10]. There exist two approaches for evaluating the potential loss of a portfolio: the simulation approach, and the analytical approach. The hybrid method uses both to calculate the potential loss of a portfolio. It is an extended method of the conditionally independent default model proposed by Martin et al.[7].

## 2.1 The setting

We consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $P$ is the physical probability measure. Suppose that there are $n$ assets, and that the price of the $j$–th asset per share or face value at time $t$ is denoted by $X_j(t)$. Here, $t = 0$ means the present, and the risk horizon is $T$, $T > 0$. Consider a portfolio $\pi$, where $a_j$ denotes the holding amount of the $j$–th asset in the portfolio. Then the time $t$ price of the portfolio $\pi$ is $X(t) = \sum_{j=1}^{n} a_j X_j(t)$.

We use a "sub-filtration approach" called in credit risk modeling. Let $w_j(t)$, $j = 1, \ldots, m$, denote the basic factors and let

$$\mathbf{W}(t) \quad = \quad (w_1(t), \ldots, w_m(t)), \quad t \geq 0.$$

A filtration $\mathcal{G}_t$ is defined as a filtration generated by the process $\mathbf{W}(t)$, that is, $\mathcal{G}_t = \sigma(\mathbf{W}(s), 0 \leq s \leq t)$ [1]. Here, the basic factors are defined as stochastic variables such that $X_j(t|\mathcal{G}_T)$, $0 \leq t \leq T$, $j = 1, \cdots, n$ become conditionally independent when $\mathcal{G}_T$ are given. Here, the conditional independence with respect to $\mathcal{G}_T$ means that $X_j(t)$, $0 \leq t \leq$

[1] Hereafter, we assume that all the filtrations including $\mathcal{G}$ satisfy the usual conditions.

$T$, $j = 1, \cdots, n$ are independent given $\mathcal{G}_T$. On the other hand, the filtration generated by the processes except for $\mathbf{W}(t)$ is denoted by $\mathcal{H}_t$, and the filtration $\mathcal{F}$ is defined as the minimum filtration including $\mathcal{G} \cup \mathcal{H}$, i.e., $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ for any $t \in \mathcal{R}^+$.

Next, we define some statistics explicitly. Suppose that $X$ is a stochastic variable which shows the future price of a portfolio, and $F_X(x)$ denotes its distribution function, and $f_X(x)$ is its density function if it exists. In this article, the $100\alpha$ percentile of $X$, denoted by $Q_X(\alpha)$, $0 < \alpha < 1$, is defined by [2]

$$Q_X(\alpha) \equiv \inf \{x | F_X(x) \geq \alpha\}, \tag{1}$$

and the VaR ( Value at Risk ) with the confidence level $100\alpha$ % is defined by

$$\mathrm{VaR}_X(\alpha) \equiv c - Q_X(1-\alpha) \tag{2}$$

where $c$ is the reference value, for example, today's price. Moreover, the expected shortfall (abbreviated by ES) with the confidence level $100\alpha$ % is defined by

$$\mathrm{ES}_X(\alpha) \equiv \frac{1}{1-\alpha} \int_0^{1-\alpha} \mathrm{VaR}_X(p) dp. \tag{3}$$

When $F_X(x)$ is strictly increasing in $x$, these can be expressed more easily; that is, we obtain

$$Q_X(\alpha) = \{x | F_X(x) = \alpha\} \tag{4}$$

$$\mathrm{ES}_X(\alpha) = c - \mathrm{TCE}_X(Q_X(1-\alpha)) \tag{5}$$

where TCE is the tail conditional expectation defined by

$$\mathrm{TCE}_X(x) \equiv E[X | X \leq x]. \tag{6}$$

Notice that these definitions are used in this article, and other definitions might be used in other articles [3]. Hereafter, we assume that $F_X(x)$ is strictly increasing in $x$.

## 2.2 Estimation of the density and the distribution functions

By assumption, since the conditional prices $X_j(T|\mathcal{G}_T)$, $j = 1, \cdots, n$ is conditionally independent given $\mathcal{G}_T$, the conditional moment generating function $M_X(s|\mathcal{G}_T)$ of $X(T|\mathcal{G}_T)$ is given

[2] Definition (1) is a general expression for the inverse function of the distribution function, which can be denoted by $F_X^{-1}(\alpha)$.

[3] Some people call ES (expected shortfall) in this article as CVaR (conditional Value at Risk) or TCE (tail conditional expectation).

by

$$M_X(s|\mathcal{G}_T) = E\left[e^{sX(T|\mathcal{G}_T)}\right] = E\left[e^{s\sum_{j=1}^{n} a_j X_j(T|\mathcal{G}_T)}\right] = \prod_{j=1}^{n} E\left[e^{s a_j X_j(T|\mathcal{G}_T)}\right].$$

Here we implicitly assume that the conditional moment generating function $M_X(s|\mathcal{G}_T)$ exists. Since $M_X(s|\mathcal{G}_T)$ is the Laplace transformation [4] of the conditional density function $f_X(u|\mathcal{G}_T)$ of $X(T|\mathcal{G}_T)$, we obtain by the inverse Laplace transformation,

$$f_X(x|\mathcal{G}_T) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M_X(s|\mathcal{G}_T)e^{-xs}ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{K_X(s|\mathcal{G}_T)-xs}ds, \qquad (7)$$

where $K_X(s|\mathcal{G}_T) = \log M_X(s|\mathcal{G}_T)$ is the conditional cumulant generating function of $X(T|\mathcal{G}_T)$, and $\sigma$ is the real value such that the integral in (7) exists.

In order to evaluate (7), we use the saddlepoint approximation method, which is very famous in engineering science. In the saddlepoint method, the integral in the complex domain is approximated by the contribution from the curvilinear integral near the saddlepoint. A detailed description is available in texts on engineering science or statistical science such as Jensen[4]. In this article, we adopt the saddlepoint approximation with an asymptotic expansion derived in Daniels[2]. Using the approximation [5], we obtain from (7),

$$f_X(x|\mathcal{G}_T) \simeq \frac{e^{K_X(\bar{s}|\mathcal{G}_T)-\bar{s}x}}{\sqrt{2\pi K_X^{(2)}(\bar{s}|\mathcal{G}_T)}}\left[1+\frac{1}{8}\lambda_{(4)}(\bar{s}|\mathcal{G}_T)-\frac{5}{24}\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)\right], \qquad (8)$$

where $\bar{s}$ is the saddlepoint of $J_X(s|\mathcal{G}_T) \equiv K_X(s|\mathcal{G}_T) - xs$, which means $dJ_X(\bar{s}|\mathcal{G}_T)/ds = 0$, $K_X^{(n)}(s|\mathcal{G}_T)$ is the $n$–th order derivative of $K_X(s|\mathcal{G}_T)$, and $\lambda_{(r)}(s|\mathcal{G}_T) \equiv K_X^{(r)}(s|\mathcal{G}_T)/(K_X^{(2)}(s|\mathcal{G}_T))^{r/2}$. Since $K_X(s|\mathcal{G}_T)$ is a convex function, the saddlepoint $\bar{s}$ is unique and can be searched numerically with ease. We refer to the approximated formula (8) as the first–order approximation, and the formula obtained by ignoring the terms of $\lambda_{(3)}(\bar{s}|\mathcal{G}_T)$ and $\lambda_{(4)}(\bar{s}|\mathcal{G}_T)$ in (8) as the zero–order one.

Integrating the density function $f_X(v)$ from $v = -\infty$ to $v = x$, and using the saddlepoint approximation method with an asymptotic expansion, we obtain the approximated formula for the conditional distribution function as

$$F_X(x|\mathcal{G}_T) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{M_X(s|\mathcal{G}_T)e^{-xs}}{s}ds$$

[4]The direction on the contour in the integral is inverse to the standard definition of the Laplace transformation.

[5]According to Jensen[4], there exist two approximation formulas: for the continuous variable and for the discrete one. In this article we use the former because of its simpler form.

$$
\begin{aligned}
\simeq \quad & e^{K_X(\bar{s}|\mathcal{G}_T)-\bar{s}x+\frac{1}{2}\hat{z}^2}\Bigg[\{1-\Phi(\hat{z})\}\Bigg\{1+\frac{1}{6}\lambda_{(3)}(\bar{s}|\mathcal{G}_T)\hat{z}^3 \\
& \qquad +\left(\frac{1}{24}\lambda_{(4)}(\bar{s}|\mathcal{G}_T)\hat{z}^4+\frac{1}{72}\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)\hat{z}^6\right)\Bigg\} \\
& +\phi(\hat{z})\Bigg\{-\frac{1}{6}\lambda_{(3)}(\bar{s}|\mathcal{G}_T)(\hat{z}^2-1) \\
& \qquad -\left(\frac{1}{24}\lambda_{(4)}(\bar{s}|\mathcal{G}_T)(\hat{z}^3-\hat{z})+\frac{1}{72}\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)(\hat{z}^5-\hat{z}^3+3\hat{z})\right)\Bigg\}\Bigg] \qquad (9)
\end{aligned}
$$

for $x \leq E[X(T|\mathcal{G}_T)]$, where $\hat{z} \equiv \sqrt{\bar{s}^2 K_X^{(2)}(\bar{s}|\mathcal{G}_T)}$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the distribution and density functions of the standard normal distribution, respectively [6]. Corresponding to the Edgeworth expansion, we call the approximated formula (9) the second–order approximation, while the formula obtained by ignoring the terms of $\lambda_{(4)}(\bar{s}|\mathcal{G}_T)$ and $\lambda_{(3)}^2(\bar{s}|\mathcal{G}_T)$ is called the first–order approximation, and the formula obtained by ignoring all $\lambda_{(r)}(\bar{s}|\mathcal{G}_T)$ terms is the zero–order approximation.

Since the approximated values for the conditional distribution, $f_X(x|\mathcal{G}_T)$ and $F_X(x|\mathcal{G}_T)$, are obtained above, the approximated values for the unconditional distribution can be obtained by using the chain rule of the expectation. That is, the unconditional density and distribution functions of $X(T)$ are given by

$$
f_X(x) \quad = \quad E\left[f_X(x|\mathcal{G}_T)\right] \qquad (10)
$$

and

$$
F_X(x) \quad = \quad E\left[F_X(x|\mathcal{G}_T)\right]. \qquad (11)
$$

Here, we can replace $f_X(x|\mathcal{G}_T)$ in (10) and $F_X(x|\mathcal{G}_T)$ in (11) with the approximations (8) and (9), respectively.

In summary, we obtain the approximated values of the unconditional distribution, $f_X(x)$ and $F_X(x)$, by the following procedures:

1. Generate many sample paths of the basic factors from $t=0$ to $t=T$, $\{\mathbf{W}(s),\ 0 \leq s \leq T\}$. The Monte Carlo simulation is a powerful candidate to generate sample paths, however, other methods can be used.

2. Evaluate $f_X(x|\mathbf{W}(s),\ 0 \leq s \leq T)$ and $F_X(x|\mathbf{W}(s),\ 0 \leq s \leq T)$ given $\mathbf{W}(s),\ 0 \leq s \leq T$ approximately by using (8) and (9).

[6] The approximated formula for $x > E[X(T|\mathcal{G}_T)]$ is obtained by replacing $\lambda_{(3)}(\bar{s}|\mathcal{G}_T)$ with $-\lambda_{(3)}(\bar{s}|\mathcal{G}_T)$.

3. Calculate the right hand sides of (10) and (11) by using all sample paths to obtain the approximated values of $f_X(x)$ and $F_X(x)$.

When the density and the distribution functions are obtained, the VaRs with arbitrary confidence levels can be calculated easily.

Whatever the joint distribution of the basic factors is, the Monte Carlo simulation can be used in order to generate sample paths in Procedure 1, and then the calculation in Procedure 3 is done numerically. Thus, we refer to our method as hybrid, which means the hybrid of the Monte Carlo simulation in Procedure 1 and the analytical approximation formulas in Procedure 2.

On the other hand, when the joint distribution is described as a combination of some distributions such as the normal distributions and the $\chi^2$ distributions, we can apply the numerical quadratures to take the expectation in Procedure 3. For example, when the basic factors are normally distributed, the Gaussian quadratures can be used. In such cases, the best sample paths generated in Procedure 1 are already known in order to make the quadratures work most efficiently.

## 2.3 Estimation of the expected shortfall

Due to the assumption that the distribution function of $X$ is strictly increasing, we have

$$\mathrm{ES}_X(\alpha) \;=\; c - \mathrm{TCE}_X(Q_X(1-\alpha)).$$

Therefore, we focus on estimating $\mathrm{TCE}_{X(T)}(x)$, which is written as follows:

$$\mathrm{TCE}_{X(T)}(x) \;\equiv\; E[X(T)|X(T) \le x] = \frac{1}{P\{X(T) \le x\}} \int_{-\infty}^{x} v f_{X(T)}(v) dv. \tag{12}$$

Additionally assuming that $X(T) \ge 0$, *a.s.* and $E[X(T)] > 0$ [7], from (12), we have

$$\mathrm{TCE}_{X(T)}(x) \;=\; \frac{E[X(T)]}{P\{X(T) \le x\}} F_h(x), \tag{13}$$

where $F_h(x)$ is the distribution function of $X(T)$ under the equivalent probability measure $P_h$, under which the density function of $X(T)$ is given by

$$h_{X(T)}(x) \;\equiv\; \frac{x f_X(x)}{E[X(T)]}. \tag{14}$$

[7] These assumptions can be applied if $X(T)$ is finitely bounded below. When there exists a constant $d$ such that $X(T) \ge d$, *a.s.*, we define $Y(T) \equiv X(T) - d$ and then apply the following discussion to $Y(T)$ instead of $X(T)$.

Using the approximated formula (9) for $F_h(x)$ in (13), we can obtain the approximated formula for $\mathrm{TCE}_{X(T)}(x)$. Since the moment generating function of $X(T)$ under the probability measure $P_h$ is given by $M_h(x) = M'_{X(T)}(x)/M'_{X(T)}(0)$, the approximated formula for $F_h(x)$ can be calculated in the same way as (9) easily.

## 2.4 Risk contributions

First, we describe the risk contribution briefly. VaR and ES are used to measure the overall risk of a portfolio in financial institutions in these days, but in order to understand the risk profile of the portfolio, it is necessary to know how much each asset (or subportfolio) in the portfolio contributes to the total risk of the portfolio. One of the common measures of this contribution is the sensitivity of the total risk of the portfolio to an infinitesimal change in asset allocation. From this point of view, various researchers studied about the properties of the sensitivity, for example, Litterman[6], Tasche[12] and Hallerbach[3].

Let $R_p$ be a risk measure of a portfolio, for example, standard deviation, VaR and ES. The risk contribution of asset $j$, hereafter denoted by $RC_j$, is defined as the sensitivity to an infinitesimal change of the holding amount of the $j$–th asset in the portfolio $(\partial R_p/\partial a_j)$, multiplied by holding amount $(a_j)$, that is,

$$RC_j \equiv a_j \frac{\partial R_p}{\partial a_j}. \tag{15}$$

Notice that $\partial R_p/\partial a_j$ does not always exist. However, if $\partial R_p/\partial a_j$ exists, and additionally if the risk measure $R_p$ satisfies the first–order positive homogeneity [8], then the following equation is always satisfied:

$$\sum_{j=1}^{n} RC_j = R_p. \tag{16}$$

Equation (16) is directly derived by the Euler's theorem for the homogeneous function, therefore, its equality is guaranteed mathematically.

Previous studies show clear expressions of risk contributions for some risk measures such as VaR and ES. Some of them are shown below. See Tasche[12] in detail for the derivation

[8] A function $f$ is called as $n$-th order positive homogeneous when

$$f(\lambda a_1, \cdots, \lambda a_m) = \lambda^n f(a_1, \cdots, a_m), \qquad \lambda > 0$$

is satisfied.

and the necessary conditions. Soppose $X = \sum_{j=1}^{n} a_j X_j$. For example, the risk contribution of $j$-th asset for the standard deviation is given by

$$RC_j^{SD} \equiv a_j \frac{\partial \mathrm{SD}_X}{\partial a_j} = a_j \frac{\mathrm{Cov}(X, X_j)}{\mathrm{SD}_X}$$

if $\mathrm{SD}_X > 0$, where $\mathrm{SD}_X$ is the standard deviation of $X$ and $\mathrm{Cov}(X, X_j)$ is the covariance between $X$ and $X_j$. Notice that $RC_j^{SD}$ is given as a simple form. On the other hand, the risk contribution of $j$-th asset for the VaR with confidence level $\alpha$ is given by

$$\begin{aligned} RC_j^{VaR}(\alpha) &\equiv a_j \frac{\partial \mathrm{VaR}_X(\alpha)}{\partial a_j} = a_j \left[ \frac{\partial c}{\partial a_j} - \frac{\partial Q_X(1-\alpha)}{\partial a_j} \right] \\ &= a_j \frac{\partial c}{\partial a_j} - a_j E\left[ X_j | X = Q_X(1-\alpha) \right], \end{aligned} \tag{17}$$

if it exists. Moreover, assuming that all the distribution functions of $X_j$, $j = 1, \cdots, n$ are strictly increasing, for simplicity, the risk contribution of $j$-th asset for the ES with confidence level $\alpha$ is given by

$$RC_j^{ES}(\alpha) \equiv a_j \frac{\partial \mathrm{ES}_X(\alpha)}{\partial a_j} = a_j \frac{\partial c}{\partial a_j} - a_j E\left[ X_j | X \leq Q_X(1-\alpha) \right], \tag{18}$$

if it exists.

We condider the risk contribution for the VaR. The second term in (17) is difficult to calculate because this conditional expectation must be taken on condition that the total future price of the portfolio $X$ is constant; in general, if all the future prices $X_j$, $j = 1, \cdots, n$ are not redundant, the conditional expectation is calculated on a $(n-1)$-dimensional hyperplane in the $n$-dimensional space. As you see easily, it is very difficult to calculate the expectation, especially by the Monte Carlo simulation. So, we use another method to estimate the risk contribution.

## 2.5 Estimating the risk contributions for VaR

Consider a VaR with the confidence level $\alpha$, $0 < \alpha < 1$, in our conditionally independent setting. Given $\mathcal{G}_T$, the conditional distribution function is given by

$$F_X(x|\mathcal{G}_T) = \frac{1}{2\pi i} \int_{\sigma - i_\infty}^{\sigma + i_\infty} \frac{e^{K_X(s|\mathcal{G}_T) - sx}}{s} ds,$$

where the contour of the integral is on the imaginary axis and runs to the right of the origin to avoid the pole. Then, the unconditional distribution function is given by

$$F_X(x) = E\left[ \frac{1}{2\pi i} \int_{\sigma - i_\infty}^{\sigma + i_\infty} \frac{e^{K_X(s|\mathcal{G}_T) - sx}}{s} ds \right]. \tag{19}$$

Differentiating (19) with respect to $a_j$, $j = 1, \ldots, n$ and assuming the exchangeability between the differentiation and the integration, we have

$$\frac{\partial}{\partial a_j} F_X(x) = E\left[\frac{1}{2\pi i}\int_{\sigma - i_\infty}^{\sigma + i_\infty} \left\{\frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j} - s\frac{\partial x}{\partial a_j}\right\} \frac{e^{K_X(s|\mathcal{G}_T) - sx}}{s} ds\right]. \tag{20}$$

Keeping $F_X(x)$ constant in (20), we obtain the contribution [9] of the $j$-th asset to $\alpha$–percentile $Q_X(\alpha)$, denoted by $RC_j^{Q_X}(\alpha)$, as

$$\begin{aligned} RC_j^{Q_X}(\alpha) &\equiv a_j \frac{\partial Q_X(\alpha)}{\partial a_j} = a_j \frac{E\left[\int_{\sigma - i_\infty}^{\sigma + i_\infty} \frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j} \frac{e^{K_X(s|\mathcal{G}_T) - sQ_X(\alpha)}}{s} ds\right]}{E\left[\int_{\sigma - i_\infty}^{\sigma + i_\infty} e^{K_X(s|\mathcal{G}_T) - sQ_X(\alpha)} ds\right]} \\ &= a_j \frac{E\left[\frac{1}{2\pi i}\int_{\sigma - i_\infty}^{\sigma + i_\infty} \frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j} \frac{e^{K_X(s|\mathcal{G}_T) - sQ_X(\alpha)}}{s} ds\right]}{f_X(Q_X(\alpha))}. \end{aligned} \tag{21}$$

(21) is a slight modification of the expression derived by Martin et al.[8]. Formally, the integral in (21) can be obtained if $K_X(s)$ is replaced by

$$K_M(s|\mathcal{G}_T) \equiv K_X(s|\mathcal{G}_T) + \log(\partial K_X(s|\mathcal{G}_T)/\partial a_j) - \log s$$

in (7). Notice that $K_M(s|\mathcal{G}_T)$ is not the cumulant generating function yet, however, the saddlepoint approximation can be used to give an approximate in (21). Assuming that a reference value $c$ in (17) is differentiable with respect to $a_j$, we obtain

$$RC_j^{VaR}(\alpha) = \frac{\partial c}{\partial a_j} - a_j \frac{E\left[\frac{1}{2\pi i}\int_{\sigma - i_\infty}^{\sigma + i_\infty} \frac{\partial K_X(s|\mathcal{G}_T)}{\partial a_j} \frac{e^{K_X(s|\mathcal{G}_T) - sQ_X(\alpha)}}{s} ds\right]}{f_X(Q_X(\alpha))}. \tag{22}$$

## 2.6 Estimating the risk contributions for ES

Next, consider a ES with the confidence level $\alpha$, $0 < \alpha < 1$. Here we assume that the differentiation with respect to $a_j$ and the integration are exchangeable.

Given $\mathcal{G}_T$, since the tail conditional expectation $\mathrm{TCE}_{X(T)}(x|\mathcal{G}_T)$ is given by (13), we have

$$\begin{aligned} \frac{\partial}{\partial a_j} \mathrm{TCE}_{X(T)}(x) &= \frac{\partial}{\partial a_j}\left(\frac{E[X(T)]}{F_{X(T)}(x)} F_h(x)\right) \\ &= \frac{1}{F_{X(T)}(x)}\left[F_h(x)E[X_j(T)] + E[X(T)]\frac{\partial F_h(x)}{\partial a_j}\right] - \frac{E[X(T)]F_h(x)}{(F_{X(T)}(x))^2}\frac{\partial F_{X(T)}(x)}{\partial a_j}. \end{aligned}$$

[9] Although $Q_X(\alpha)$ does not mean a risk directly, we call $RC_j^{Q_X}(\alpha)$ a contribution to $Q_X(\alpha)$ simply.

Using the following relation

$$\frac{\partial}{\partial a_j}F_h(x) = \frac{\partial}{\partial a_j}E[F_h(x|\mathcal{G}_T)] = E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j} - s\frac{\partial x}{\partial a_j}\right\}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}ds\right],$$

where $K_h(s|\mathcal{G}_T)$ is the cumulant generationg function of $X(T)$ under $P_h$, we have

$$\begin{aligned}\frac{\partial}{\partial a_j}\mathrm{TCE}_{X(T)}(x) &= \frac{F_h(x)E[X_j(T)]}{F_{X(T)}(x)} + \frac{E[X(T)]}{F_{X(T)}(x)}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}ds\right] \\ &\quad -\frac{E[X(T)]}{F_{X(T)}(x)}\frac{\partial x}{\partial a_j}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{K_h(s|\mathcal{G}_T)-sx}ds\right] - \frac{E[X(T)]F_h(x)}{(F_{X(T)}(x))^2}\frac{\partial F_{X(T)}(x)}{\partial a_j} \\ &= \frac{F_h(x)E[X_j(T)]}{F_{X(T)}(x)} + \frac{E[X(T)]}{F_{X(T)}(x)}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}ds\right] \\ &\quad -\frac{xf_X(x)}{F_{X(T)}(x)}\frac{\partial x}{\partial a_j} - \frac{E[X(T)]F_h(x)}{(F_{X(T)}(x))^2}\frac{\partial F_{X(T)}(x)}{\partial a_j}.\end{aligned}$$

Therefore, considering that $F_{X(T)}(Q_{X(T)}(1-\alpha)) = 1-\alpha$ is constant, we obtain

$$\begin{aligned}\frac{\partial}{\partial a_j}\mathrm{ES}_{X(T)}(\alpha) &\equiv \left.\frac{\partial}{\partial a_j}\mathrm{TCE}_{X(T)}(Q_{X(T)}(1-\alpha))\right|_{\alpha=constant} \\ &= \frac{F_h(Q_{X(T)}(1-\alpha))E[X_j(T)]}{\alpha} \\ &\quad +\frac{E[X(T)]}{\alpha}E\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\partial K_h(s|\mathcal{G}_T)}{\partial a_j}\frac{e^{K_h(s|\mathcal{G}_T)-sx}}{s}ds\right] \\ &\quad -\frac{Q_{X(T)}(1-\alpha)f_X(Q_{X(T)}(1-\alpha))}{\alpha}\frac{\partial Q_{X(T)}(1-\alpha)}{\partial a_j}.\end{aligned} \tag{23}$$

The first and third terms in (23) are obtained before, and the second term is calculated by the same method as that used in (21).

As described in above, the explicit expression (23) is obtained of the risk contribution for ES. However, due to my numerical results shown later, (23) does not give us reliable and stable estimates. Moreover, in fact, (13) does not give us so reliable estimates. Therefore, we propose another estimation method of ES and the risk contribution for ES.

The concept of another method is very simple. As described in (3), the following relation exists;

$$\mathrm{ES}_X(\alpha) \equiv \frac{1}{1-\alpha}\int_0^{1-\alpha}\mathrm{VaR}_X(p)dp.$$

If we assume the exchangeability between the derivative with respect to $a_j$ and the integral with respect to $p$, we obtain

$$\frac{\partial}{\partial a_j}\mathrm{ES}_X(\alpha) = \frac{1}{1-\alpha}\int_0^{1-\alpha}\frac{\partial \mathrm{VaR}_X(p)}{\partial a_j}dp = \frac{1}{(1-\alpha)a_j}\int_0^{1-\alpha}RC_j^{VaR}(p)dp. \tag{24}$$

Since our preliminary results show that our method gives good estimates for $\mathrm{VaR}_X(p)$ and $RC_j^{VaR}(\alpha)$, it is expected that the combination of (24) and (22) would give us good estimates for $RC_j^{ES}(\alpha)$. Similarly, the combination of (3) and the estimation method of the distribution function $F_X(x)$ described in section 2.2 would give good estimates for $\mathrm{ES}_X(\alpha)$.

# 3 Numerical examples

In this section we provides numerical examples for estimating VaR, ES, and their risk contributions based on the distribution of the future value of a portfolio. As a risk evaluation model, we use Kijima and Muromachi model[5] (KM Model). KM model is a synthetic risk evaluation model of a portfolio, which means that the market and the credit risk can be evaluated simultaneously and synthetically. See Kijima and Muromachi[5] for details.

The bond portfolio consists of 100 corporate discount bonds with maturity 5 years and zero recovery rates. Each bond is issued from different firms, and these bonds have various credit ratings; Aaa-rated 10 bonds, Aa-rated 10 bonds, A-rated 10 bonds, Baa-rated 10 bonds, Ba-rated 30 bonds, and B-rated 30 bonds. The face values of bonds in Aaa, Aa, A, and Baa are 3, 6, 9, $\cdots$, 30, and those in Ba and B are 1, 2, 3, $\cdots$, 30.

The parameters used here are almost the same as those in Kijima and Muromachi[5]; for example, parameters of the initial forward rate curves for credit ratings, the stochastic differential equations describing the future changes of the instantaneous default-free short rate and hazard rates of individual bonds. If you want to know the parameterization in this model and their values in the calculation, see Kijima and Muromachi[5].

We do a Monte Carlo simulation with 500,000 scenarios and calculate VaRs, ESs and their risk contributions with many confidence levels. Hereafter, we use the results as reference values, and compare the estimates by the hybrid method with the reference values. Especially, the estimates of ESs and their risk contributions are calculated by two hybrid methods; by the old hybrid method proposed in Muromachi[10] and the new method proposed in this article.

## 3.1 VaR and ES

First, we compare the VaRs and ESs of the future portfolio value $X(T)$. Figure 1 shows the curves of VaRs and ESs with many confidence levels estimated by two or three methods.

The two black lines are the estimates by the Monte Carlo simulation; the thin one is the VaR curve, and the thick one is the ES curve. The two red lines, with "(IS-H)" [10], are the estimates by the old hybrid method; the thin one is the VaR curve, and the thick one is the ES curve, similarly. And, the blue line, with "(from VaR)" [11], is the estimates of ESs by the new hybrid method. This figure shows some useful results. The old hybrid method gives us very close estimates to those by the Monte Carlo simulation, while the estimates by the old method is rather different from the simulation results. However, the estimates by the new hybrid method are very close to the simulation results. Figure 2 shows the estimates of ESs in the high confidence level area (over 98%) in Figure 1. This figure shows the results described above clearly.

In order to show the results more clearly, Figure 3 shows the estimated diffferences between the VaRs and ESs with the same confidence levels. Apparently, the estimates from the old hybrid method have a tendency that they are smaller than the estimates from other methods. The new hybrid method "(from VaR)" gives us much more close values to the simulation results than the old one "(IS-H)".

## 3.2 Additivity of risk contributions

Next, we check whether the additivity of the estimated risk contributions is satisfied accurately or not. Here, the additivity means that the sum of the risk contributions of all assets is equal to the total risk of the portfolio.

Our numerical example shows that the risk contributions for VaR satisfies the additivity with the order 0.01. Therefore, in Figure 4, we compare (1) the differences between the estimated VaRs and ESs and (2) the differences between the sums of the estimated risk contributions for VaRs and ESs. The three lines, the black, the red and the blue, are the same as those in Figure 3, and the two dotted lines are the differences between the risk contributions for VaRs and ESs; the dotted green line, "total RC (before)", is the differences estimated from the old hybrid method, and the dotted yellow line, "total RC (new)", is the differences estimated from the new hybrid method. Figure 4 clearly shows that the new hybrid method gives us the estimates of RCs for ES which satisfy the additivity with the same order as the RCs for VaR in a wide range, on the other hand, that the additivity is

[10] The notation "(IS-H)" means the hybrid method with importance sampling techniques. This is not a simple hybrid method. See Muromachi[10] in detail.

[11] The notation "(from VaR)" comes from (3).

not satisfied in the old hybrid method, and their sums are near to the biased estimates of ES - VaR obtained from the old hybrid method.

However, the risk contributions estimated from the new hybrid method do not satisfy the additivity in all ranges. Figure 5 shows that the "total RC (new)" line differs from the "from VaR" line clearly in the high confidence levels, for example, over 99.9%. At present, we do not understand the reason clearly. More detailed analyses must be needed in order to clarify the reasons.

# 4 Concluding remarks

In this article, we summarize the hybrid method for estimating the risk measures such as VaR and ES and the risk contributions for the risk measures by assuming the conditional independence. Due to our preliminary numerical examples, the estimated values of VaRs, ESs and their risk contributions are much more reliable than the estimates by the ordinary Monte Carlo simulations. Especially, the estimated risk contributions for ES from the new hybrid method proposed in this article satisfies the additivity more accurately than those from the old hybrid method. However, the additivity is not satisfied in all confidence levels, for example, it is violated in the high confidence levels.

Since the concentration risk analysis becomes more important than before in financial institutions, more accurate and robust estimation methods of the risk contributions are needed in practice. It is necessary to do much more researches in this area.

# References

[1] Artzner, P., F. Delbaen, J.M. Eber and D. Heath (1999), "Coherent measures of risk," *Mathematical Finance*, **9**, 203–228.

[2] Daniels, H. (1987), "Tail probability approximations," *International Statistical Review*, **55**, 37–48.

[3] Hallerbach W. (2002), "Decomposing portfolio value-at-risk : a general analysis," *Journal of Risk*, **5**, No.2, 1–18.

[4] Jensen, J. (1995), *Saddlepoint Approximations*, Oxford Science Publications.

[5] Kijima, M. and Y. Muromachi (2000), "Evaluation of credit risk of a portfolio with stochastic interest rate and default processes," *Journal of Risk*, **3**, 5–36.

[6] Litterman, R. (1997), "Hot spots and hedges (I)," *Risk*, **10**, No.3, 42–45.

[7] Martin, R., K. Thompson, and C. Browne (2001a), "Taking to the saddle," *RISK*, June, 91–94.

[8] Martin, R., K. Thompson, and C. Browne (2001b), "VaR: who contributes and how much?" *RISK*, August, 99–102.

[9] Moody's Investors Service (1997), "Historical default rates of corporate bond issuers, 1920-1996."

[10] Muromachi, Y. (2004), "A conditional independence approach for portfolio risk evaluation," *Journal of Risk*, **7** (1), 27–53.

[11] Muromachi, Y. (2005), "Credit risk evaluation models for a portofolio," *Doctoral thesis*, Kyoto University, in Japanese.

[12] Tasche, D. (1999), "Risk contributions and performance measurement," *Working paper*, Zentrum Mathematik (SCA), TU München.

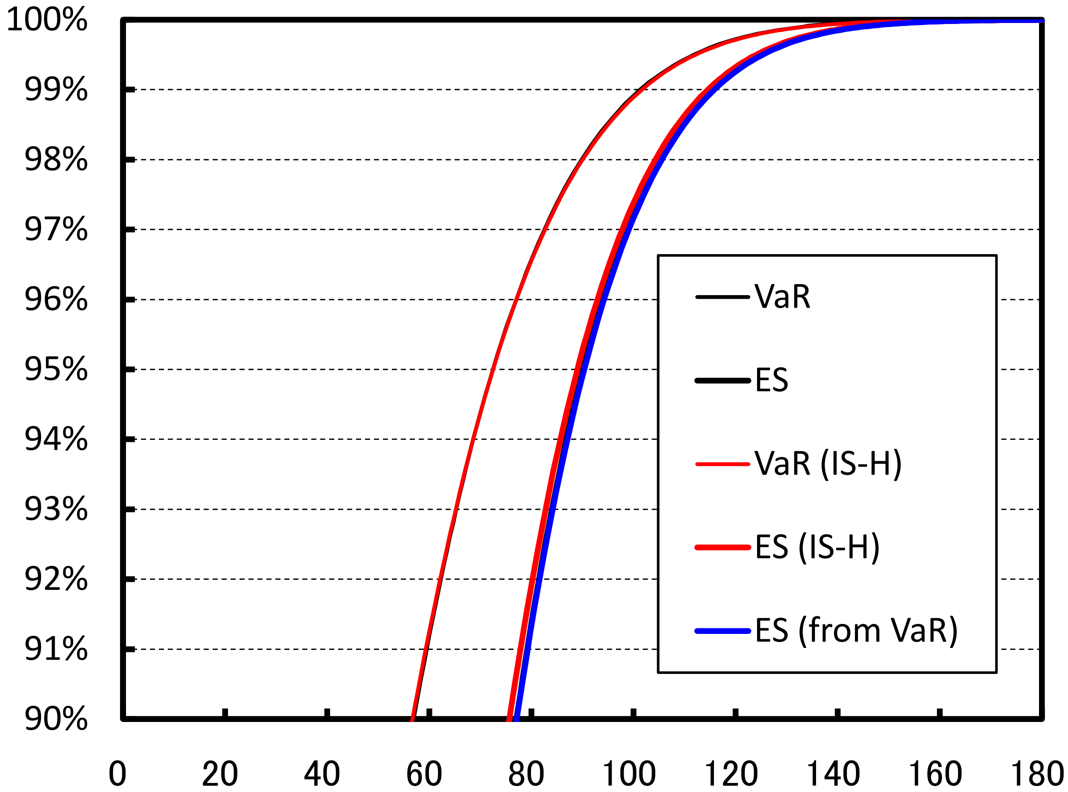


Figure 1. Estimates of Value at Risk and Expected Shortfall

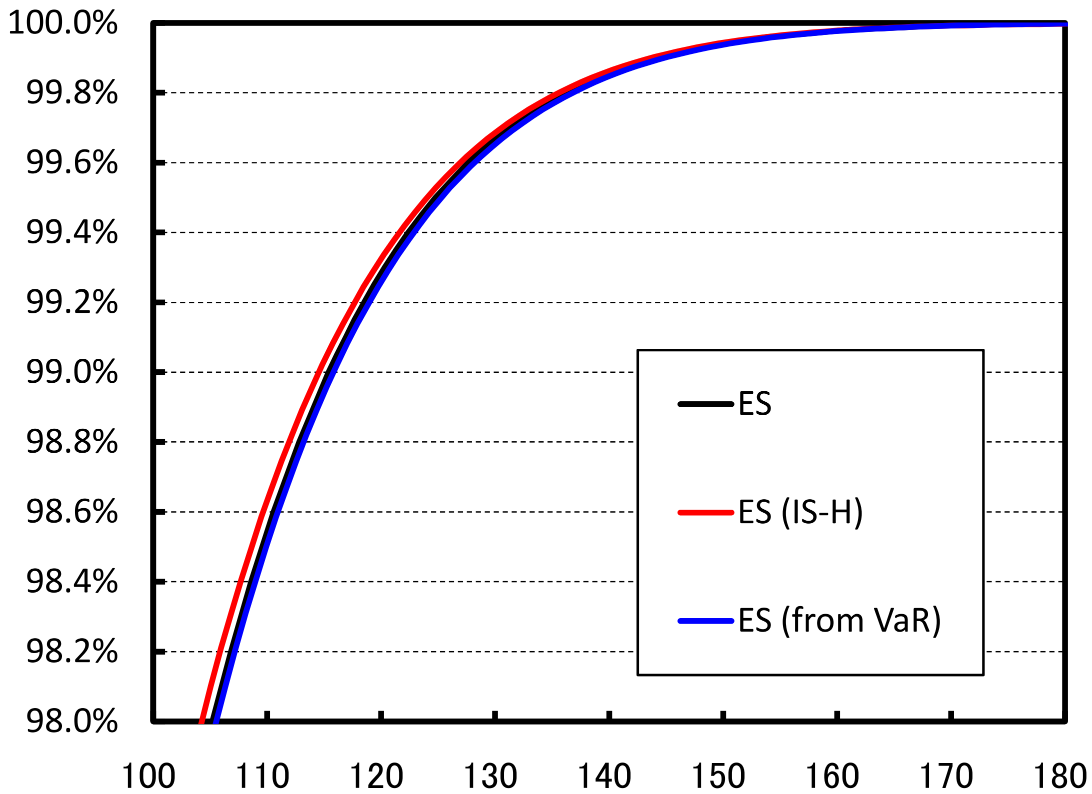


Figure 2. Estimates of Expected Shortfall

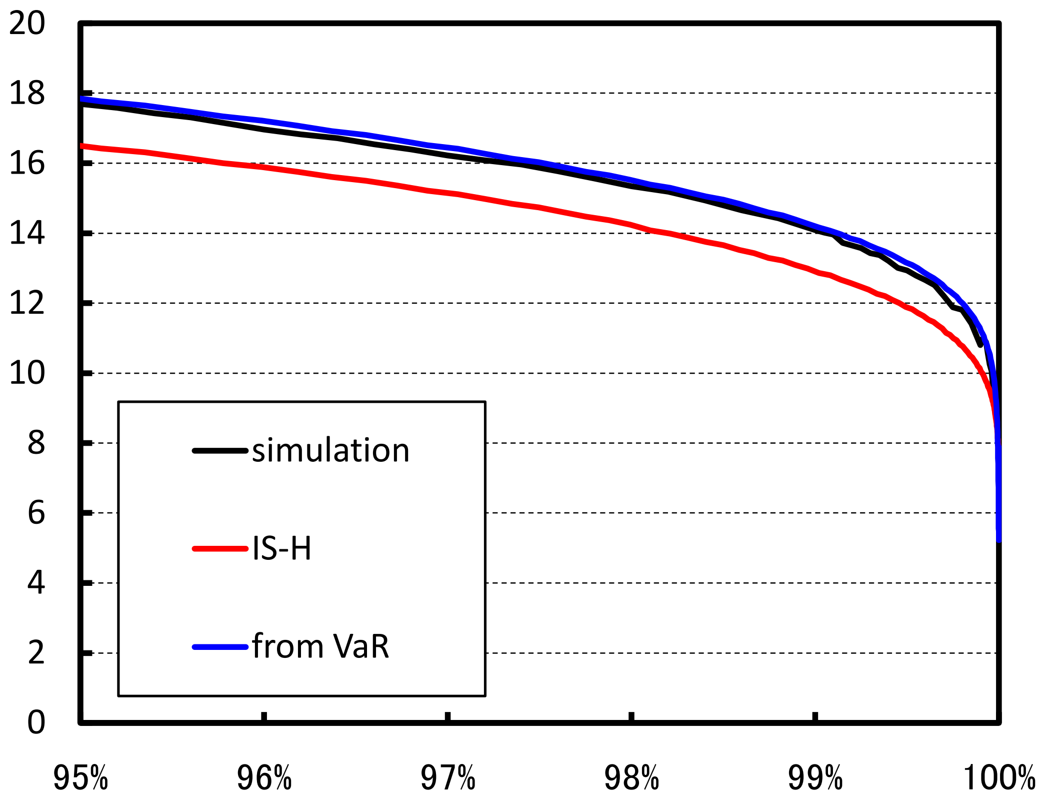


Figure 3. Difference between VaR and ES

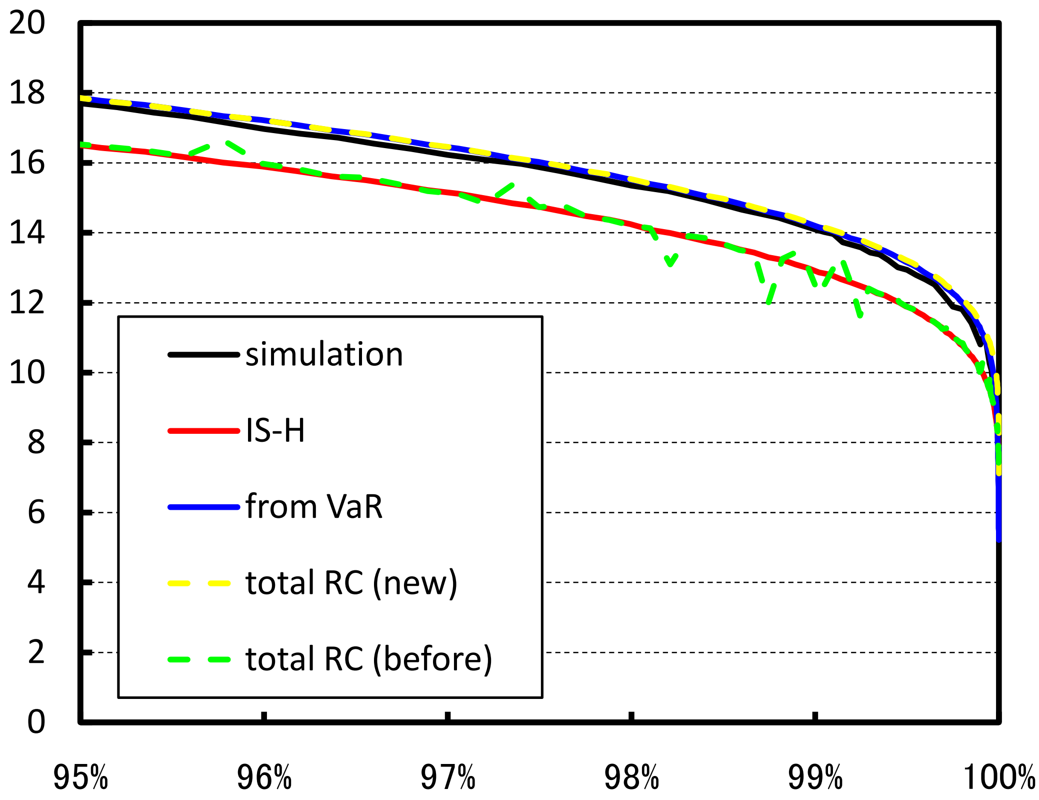


Figure 4. Sum of Risk Contributions

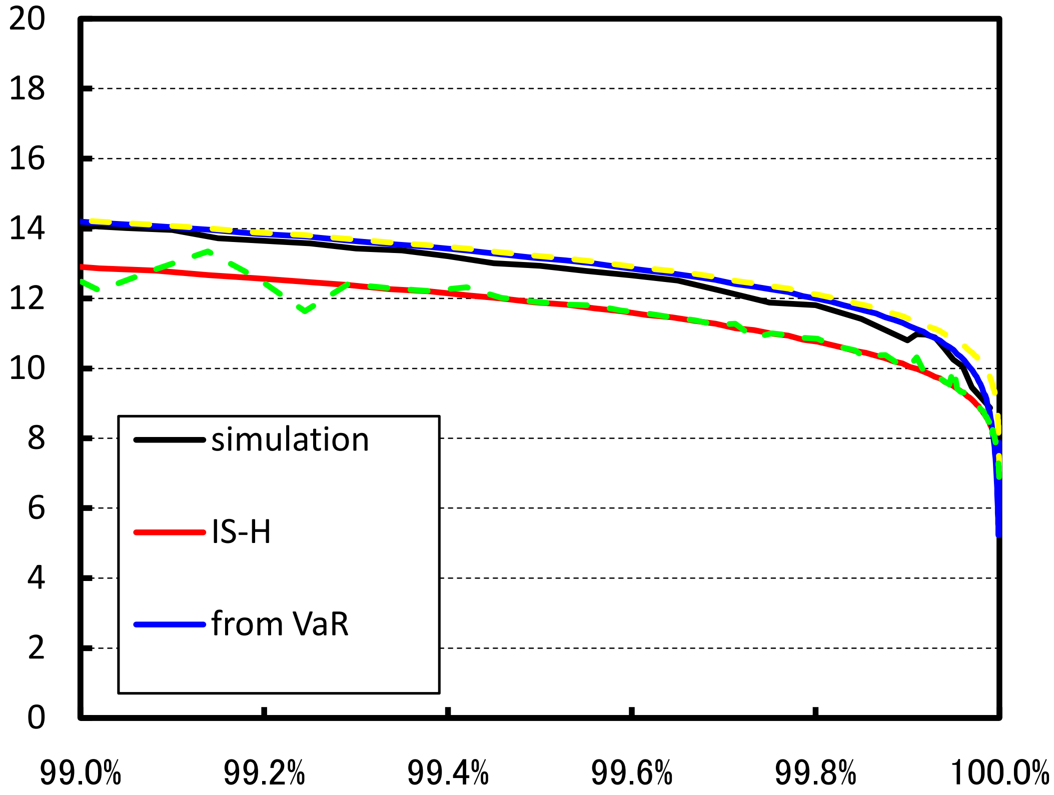


Figure 5.  Sum of Risk Contributions in high confidence levels