Stochastic orders in dynamic reinsurance markets

Thomas Møller
PFA Pension
Sundkrogsgade 4
DK-2100 Copenhagen Ø
Denmark

and

Laboratory of Actuarial Mathematics
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark

Phone: +45 39 17 60 66
Fax: +45 39 17 59 52
e-mail: thm@pfa.dk

Abstract. We consider a dynamic reinsurance market, where the traded risk process is driven by a jump-diffusion and where claim amounts are unbounded. These markets are known to be incomplete, and there are typically infinitely many martingale measures. In this case, no-arbitrage pricing theory can typically only provide wide bounds on prices of reinsurance claims. Optimal martingale measures such as the minimal martingale measure and the minimal entropy martingale measure are determined, and some comparison results for prices under different martingale measures are provided. This leads to a simple stochastic ordering result for the optimal martingale measures. Moreover, these optimal martingale measures are compared with other martingale measures that have been suggested in the literature on dynamic reinsurance markets.

Key words: Compound Poisson process, change of measure, minimal martingale measure, minimal entropy martingale measure, convex order, stop-loss contract.

JEL Classification: G10

Mathematics Subject Classification (2000): 62P05, 60J75, 60G44

(This version: April 11, 2003)

1 Introduction

Dynamic reinsurance markets have been studied in a continuous time framework using no-arbitrage conditions by Sondermann (1991), Delbaen and Haezendonck (1989) and de Waegenaere and Delbaen (1992), among others. The main idea underlying these papers is to allow for dynamic rebalancing of proportional reinsurance covers. Typically it is assumed that some process related to an *insurance risk process* (accumulated premiums minus claims) of some insurance business is tradeable and that positions can be rebalanced continuously. For example, this could mean that reinsurers can change at any time (continuously) the amount of insurance business that they have accepted. Thus, the insurance risk process can essentially be viewed as a traded security, and this already imposes no-arbitrage bounds on premiums (prices) for other (traditional) reinsurance contracts such as stop-loss contracts.

This paper studies the situation where the traded index X is defined as claims less premiums on some insurance portfolio. This process is driven by a marked point process, where claims occur according to an inhomogeneous Poisson process and claim amounts are distributed according to some time-dependent distribution. In contrast to the classical papers on dynamic reinsurance markets, we moreover allow for a diffusion component in the traded process. This leads to an incomplete market with two traded assets, a savings account and the price process related to the insurance risk process. On the other hand, if we allow claims to be described by an absolutely continuous distribution, there are essentially infinitely many sources of risk. As a consequence of this incompleteness, contingent claims (reinsurance contracts) cannot be priced uniquely by using no-arbitrage theory alone. In particular, there will be infinitely many martingale measures. For most contracts, different measures will lead to different prices, and it is not clear which measure one should apply. In this setting, we determine optimal martingale measures known from the literature on incomplete financial markets, such as the minimal martingale measure and the minimal entropy martingale measure. These are candidate measures that can be used for pricing or valuation. For a treatment of their properties, see e.g. Schweizer (2001) and Grandits and Rheinländer (2002). The main result of this paper is a criterion for the ordering of prices under two given martingale measures, which is sufficiently general to show that these two optimal martingale measures are ordered in the so-called convex order. More precisely, we show that for any convex function Φ, the price of the contract with payoff $\Phi(X_T)$ is smaller under the minimal martingale measure than under the minimal entropy martingale measure. For example, this is relevant for a stop-loss reinsurance contract $\Phi(X_T) = (X_T - K)^+$, which covers claims above some level (the sum of the premiums and the level K). The results show that the minimal entropy martingale measure in our model can be viewed as a more conservative attitude to risk than the minimal martingale measure.

Existing literature
In order to illustrate the relation of our results to the ones obtained by Sondermann (1991) and Delbaen and Haezendonck (1989), we introduce some notation and give a short review of their main results; a similar review can be found in Møller (2002). Let U_t be the accumulated claims during $[0, t]$ from some insurance business and let $\widetilde{p} = (\widetilde{p}_t)_{0 \le t \le T}$ be a predictable process related to the premiums on this business; we assume that all amounts have already been discounted with the interest rate on the market. Define the process X by $X_t = U_t + \widetilde{p}_t$. Sondermann (1991) takes $-\widetilde{p}_t$ to be the premiums paid during $[0, t]$, so that $-X_t$ is the insurance risk process. Thus, X_t can be viewed as the value at time t of an account where claims are added and premiums subtracted as they incur. In particular, in the special case where premiums are paid continuously at a fixed rate κ, $\widetilde{p}_t = -\kappa t$. Reinsurers can now

participate in the risk by trading the asset X. Sondermann (1991) demonstrated that in this setting of a dynamic market for proportional reinsurance contracts, traditional reinsurance contracts such as stop-loss contracts can be viewed as contingent claims and that these claims should be priced so that no arbitrage possibilities arise. Delbaen and Haezendonck (1989) interpreted $\widetilde{p}_t$ as the premium at which the direct insurer can sell the remaining risk $U_T - U_t$ on the reinsurance market. In their framework, X_t represents the insurer's liabilities at time t. In the special case where the direct insurer receives continuously paid premiums at rate κ and provided that this premium is identical to the one charged by the reinsurers, we obtain that $\widetilde{p}_t = \kappa(T-t)$, so that $\widetilde{p}_t$ in this situation differs from Sondermann's choice only by the constant κT. Delbaen and Haezendonck (1989) assumed that U is a compound Poisson process, i.e. $U_t = \sum_{i=1}^{N_t} Y_i$, where N is a Poisson process and $Y_1, Y_2, \dots$ is a sequence of i.i.d. non-negative random variables which are independent of N. They investigated the set of equivalent measures Q which are such that U is also a Q-compound Poisson process. For each such measure Q, a predictable premium process $\widetilde{p}$ was obtained by requiring that X be a Q-martingale. This procedure guaranteed that no arbitrage possibilities could arise from trading in X. In this way, Delbaen and Haezendonck (1989) recovered several traditional actuarial valuation principles on a certain subspace of claims from no-arbitrage considerations, namely the expected value principle, the variance principle and the Esscher principle. A more detailed account of the results of Delbaen and Haezendonck (1989) is also given by Embrechts (2000).

Outline

The present paper is organized as follows. Section 2 introduces the basic model for the traded price process and decomposes the process into the driving martingale parts. In Section 3, we study various martingale measures. Section 3.1 recalls Girsanov's Theorem for jump-diffusions, and Section 3.2 considers some simple examples of martingale measures suggested by Delbaen and Haezendonck (1989). Sections 3.3 and 3.4 are devoted to studies of the minimal martingale measure and the minimal entropy martingale measure. In Section 4 we give some results that are related to a comparison result for martingale measures obtained by Henderson and Hobson (2002) within exponential jump-diffusion models. Section 5 contains the main results. Here, we present our general result on ordering of martingale measures, and it is showed that the minimal entropy martingale measure in our model leads to higher prices than the minimal martingale measure. Finally, we compare these two optimal martingale measures with the measures suggested by Delbaen and Haezendonck (1989). In the special case with exponentially distributed claims, where all measures can be characterized explicitly, we present some numerical results.

2 Model and notation

We consider the discounted traded price process

$$dX_t = \int_{(0,\infty)} y \, dN_t^*(dy) + \sigma_t \, dW_t - p_t \, dt, \tag{2.1}$$

where $X_0 = x_0$, W is a standard Brownian motion, and where N^* is a marked point process with intensity

$$\nu(dt, dy) = G_t(dy) \, \lambda_t \, dt. \tag{2.2}$$

All processes involved are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and we consider a fixed finite time horizon T. We assume that $\mathbb{F}$ is the P-augmentation of the natural filtration of X; furthermore, we take $\mathcal{F}_T = \mathcal{F}$ and assume that $\mathcal{F}_0$ is trivial. Moreover, we restrict to the case where the parameters λ_t, σ_t and p_t are deterministic functions of time t and assume that for each t, $G_t(\cdot)$ is a distribution function on $(0,\infty)$, which only depends on t. This means that X has independent increments. In comparison to the classical papers on dynamic reinsurance markets mentioned above, we include a term driven by a Brownian motion. This term may be interpreted as a description of small claims, expenses or adjustments of claim amounts.

Safety-loading
We introduce the notation

$$\mu_{k,t} = \int_{(0,\infty)} y^k \, G_t(dy),$$

which is the k'th moment (under the measure P) of claims occurring at t. We require that $\mu_{k,t} < \infty$ for all $k > 0$ and that

$$p_t = (1 + \vartheta_t)\, \lambda_t \, \mu_{1,t} > \lambda_t \, \mu_{1,t}, \tag{2.3}$$

where $\vartheta_t > 0$ is the deterministic safety-loading parameter. This condition means that we assume that there is a non-negative safety-loading on the risk process. Finally, we assume that the distributions G_t have finite exponential moments, i.e. that

$$\int_{(0,\infty)} e^{\eta_t y} G_t(dy) < \infty,$$

for η_t in some open interval containing 0.

Compound Poisson process with noise
The marked point process N^* models the occurrence of claims and their sizes. The parameter λ_t may be interpreted as the intensity for arrival of claims, and G_t describes the claim size distribution. If we assume that G and λ are independent of t (time-homogeneous jump part), we may write X as

$$X_t = X_0 + \sum_{i=1}^{N_t} Y_i + \int_0^t \sigma_s \, dW_s - \int_0^t p_s \, ds, \tag{2.4}$$

where N is a homogeneous Poisson process with intensity λ, and where N is independent of the sequence $Y_1, Y_2, \dots$ of i.i.d. random variables. Thus, $-X$ is similar to a classical insurance risk process added noise from a Brownian motion.

Change of measure for compound Poisson processes
In the subsequent sections, we study equivalent martingale measures for the process X via Girsanov's Theorem. Here, we specialize to the time-homogeneous pure jump case, where X is given by (2.4) with $\sigma = 0$. For simplicity, we take $T = 1$. Below, we consider measures Q defined by $\frac{dQ}{dP} = Z_1$, with density process $Z_t = \mathrm{E}[Z_1 | \mathcal{F}_t]$ given by

$$Z_t = e^{-\lambda t \mathrm{E}[\phi(Y_1)]} e^{\sum_{j=1}^{N_t} \log(1+\phi(Y_j))},$$

where ϕ is some function with $\phi(y) > -1$ for all $y > 0$. Delbaen and Haezendonck (1989) showed that the process $U_t = \sum_{j=1}^{N_t} Y_j$ is a compound Poisson process under Q with parameters

$$\lambda^Q = \lambda(1 + \mathrm{E}[\phi(Y_1)]), \tag{2.5}$$

$$G^Q(dy) = \frac{1+\phi(y)}{1+\mathrm{E}[\phi(Y_1)]} G(dy). \tag{2.6}$$

Here, we show that U_1 is a compound Poisson variable under Q with parameters (λ^Q, G^Q), given by (2.5) and (2.6), by simply calculating the characteristic function of U_1. Recall that the characteristic function of the compound Poisson variable $U_1 = \sum_{j=1}^{N_1} Y_j$ with parameters (λ, G) is

$$\mathrm{E}\left[e^{isU_1}\right] = e^{\lambda\left(\mathrm{E}[e^{isY_1}]-1\right)},$$

which involves λ and the characteristic function of Y_1 with distribution G. Under Q, the characteristic function is

$$\begin{aligned}
\mathrm{E}_Q[e^{isU_1}] &= \mathrm{E}\left[e^{-\lambda \mathrm{E}[\phi(Y_1)]} e^{\sum_{j=1}^{N_1}\log(1+\phi(Y_j))} e^{is\sum_{j=1}^{N_1} Y_j}\right] \\
&= e^{-\lambda \mathrm{E}[\phi(Y_1)]}\mathrm{E}\left[\prod_{j=1}^{N_1}\left(\frac{e^{isY_j}(1+\phi(Y_j))}{1+\mathrm{E}[\phi(Y_1)]}(1+\mathrm{E}[\phi(Y_1)])\right)\right] \\
&= e^{\lambda(1+\mathrm{E}[\phi(Y_1)])(\mathrm{E}_\phi[e^{isY_1}]-1)}. \qquad (2.7)
\end{aligned}$$

Here, the first equality follows by using the definition of Q, and the second equality follows by straightforward calculations; in the last equality, which follows by using the properties of N and $Y_1, Y_2, \ldots$ under P, we have moreover introduced the notation

$$\mathrm{E}_\phi[e^{isY_1}] = \int_{(0,\infty)} e^{isy}\frac{1+\phi(y)}{1+\mathrm{E}[\phi(Y_1)]}G(dy) = \int_{(0,\infty)} e^{isy}G^Q(dy),$$

where G^Q is defined by (2.6). This shows that (2.7) is indeed the characteristic function of a compound Poisson variable with parameters (2.5) and (2.6).

2.1 Random measures and the canonical decomposition

Sometimes it is convenient to model the jumps in X via an integer valued random Poisson measure $\gamma(dt, dy)$ (i.e. a marked point process) with compensator $\nu(dt, dy)$ given by (2.2). Now rewrite (2.1) as

$$dX_t = \int_{(0,\infty)} y(\gamma(dt, dy) - \lambda_t\, G_t(dy)\, dt) + \sigma_t\, dW_t + (\lambda_t\, \mu_{1,t} - p_t)\, dt. \qquad (2.8)$$

This shows that the process X can be written as $X = X_0 + M + A$, where M is a martingale, and where A is predictable and of finite variation with

$$\begin{aligned}
dM_t &= \int_{(0,\infty)} y(\gamma(dt, dy) - \lambda_t\, G_t(dy)\, dt) + \sigma_t\, dW_t, \\
dA_t &= (\lambda_t\, \mu_{1,t} - p_t)\, dt = -\vartheta_t\, \lambda_t\, \mu_{1,t}\, dt.
\end{aligned}$$

Under the above assumptions, A is actually deterministic. For later use, we give here the so-called predictable quadratic variation process $\langle M\rangle$ of the martingale M; for its precise definition and properties, see Jacod and Shiryaev (1987). This process is determined by

$$d\langle M\rangle_t = \int_{(0,\infty)} y^2 G_t(dy)\, \lambda_t dt + \sigma_t^2 dt = (\mu_{2,t}\lambda_t + \sigma_t^2)dt.$$

In addition, we note that $dA_t = \alpha_t\, d\langle M\rangle_t$, where

$$\alpha_t = -\frac{\vartheta_t\, \lambda_t\, \mu_{1,t}}{\lambda_t\, \mu_{2,t} + \sigma_t^2}. \qquad (2.9)$$

In particular, A is absolutely continuous with respect to $\langle M\rangle$, which is known to be a necessary condition for the existence of an equivalent martingale measure for X, see Ansel and Stricker (1992, Théorème 4).

3 Martingale measures

It is well-known that there can be many martingale measures for X even in the situation where $\sigma = 0$, i.e. when the Brownian motion W does not affect the traded risk process, see e.g. Chan (1999) and references therein. In this case, a martingale measure can, loosely speaking, be defined by changing the intensity λ_t for the occurrence of claims, by changing the distribution $G_t(\cdot)$ for the claim sizes, or by combinations of these two methods.

In the literature, one can find notions such as the minimal martingale measure and the minimal entropy martingale measure. In this section, we list these martingale measures for the process X and give a discussion of its dynamics under these new measures. The results in this section are well-known and can essentially be compiled from various other papers on pricing of options on assets driven by jump-diffusions; for related results on log-Lévy processes, see e.g. Chan (1999), Henderson and Hobson (2002) and references therein.

3.1 Girsanov's Theorem

This section recalls Girsanov's Theorem for our situation, which is covered by Theorem III.3.24 of Jacod and Shiryaev (1987); see also Chan (1999). Consider a probability measure Q, whose density process with respect to P is given by

$$Z_t = \mathrm{E}\left[\frac{dQ}{dP} \middle| \mathcal{F}_t \right],$$

where Z is a P-martingale, $Z_0 = 1$, and where

$$dZ_t = Z_{t-} \left(\psi_t dW_t + \int_{(0,\infty)} \phi_t(y)(\gamma(dt,dy) - \nu(dt,dy)) \right),$$

with $1 + \phi_t(y) > 0$ for predictable processes ψ and ϕ. Girsanov's Theorem states that W^Q defined by $dW_t^Q = dW_t - \psi_t dt$ is a Q-standard Brownian motion and that $\gamma(dt,dy) - \nu^Q(dt,dy)$ is a Q-martingale increment, where $\nu^Q(dt,dy) = (1+\phi_t(y))\nu(dt,dy)$. This means that ν^Q is the Q-compensator of γ. If we insert this in (2.1), we get that

$$\begin{aligned} dX_t &= \sigma_t dW_t^Q + \psi_t \sigma_t dt + \int_{(0,\infty)} y(\gamma(dt,dy) - \nu^Q(dt,dy)) \\ &\quad + \int_{(0,\infty)} y \nu^Q(dt,dy) - (1+\vartheta_t)\mu_{1,t}\lambda_t dt. \end{aligned}$$

Thus, X is a (local) Q-martingale (so that Q is a martingale measure) provided that

$$\psi_t \sigma_t + \lambda_t \int_{(0,\infty)} y(1+\phi_t(y))G_t(dy) - (1+\vartheta_t)\mu_{1,t}\lambda_t = 0, \tag{3.1}$$

for all t. We shall also refer to (3.1) as the martingale equation. If $\phi_t(y)$ is deterministic, i.e. a function of (t,y) only, then it is natural to rewrite the Q-compensator as $\nu^Q(dt,dy) = \lambda_t^Q G_t^Q(dy)$, where $\lambda_t^Q = (1+\widetilde{\phi}_t)\lambda_t$, with

$$\widetilde{\phi}_t = \int_{(0,\infty)} \phi_t(y) G_t(dy),$$

and where

$$G_t^Q(dy) = \frac{1+\phi_t(y)}{1+\widetilde{\phi}_t} G_t(dy) = \frac{\lambda_t}{\lambda_t^Q}(1+\phi_t(y))G_t(dy). \tag{3.2}$$

This means that claims occur under Q according to an inhomogeneous Poisson process with intensity $\lambda_t^Q = \lambda_t(1+\widetilde{\phi}_t)$ and with time-dependent claim size distribution given by (3.2).

We finally note that $\frac{dQ}{dP} = Z_T$ and that the density process Z may be written as

$$\begin{aligned} Z_t &= \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) \exp\left(\int_0^t \int_{(0,\infty)} \log(1+\phi_s(y))\gamma(dt,dy)\right) \\ & \exp\left(-\int_0^t \int_{(0,\infty)} \phi_s(y)\nu(ds,dy)\right). \end{aligned}$$

3.2 Martingale measures in the time-homogeneous case: Examples

We consider three examples of martingale measures proposed by Delbaen and Haezendonck (1989). They considered the time-homogeneous case and did not include a Brownian motion in the traded price process. They therefore focused on measures Q^β with density

$$\frac{dQ^\beta}{dP} = C\exp\left(\int_{(0,T)}\int_{(0,\infty)} \beta(y)\gamma(dt,dy)\right),$$

which means that $\beta(y) = \log(1+\phi(y))$ in the notation used above.

One example is to take β constant, for example $\beta(y) = \log(1+\zeta)$, $\zeta > -1$. In this case, the martingale equation (3.1) leads to

$$\lambda \int y(1+\zeta)G(dy) = \lambda(1+\vartheta)\mu_1,$$

which implies that $\zeta = \vartheta$. Thus, under this martingale measure, the Poisson intensity is changed to $(1+\vartheta)\lambda$, while the distribution (3.2) of the claims remains unchanged. We refer to this measure as $Q^{(1)}$ and the parameters are called $(\lambda^{(1)}, G^{(1)})$.

Another example is $\beta(y) = \log(1+b(y-\mu_1))$, for some $b > 0$. In this case, the martingale equation becomes

$$\lambda \int y(1+b(y-\mu_1))G(dy) = \lambda(1+\vartheta)\mu_1,$$

which shows that this martingale measure $Q^{(2)}$ is determined by $b = \vartheta\mu_1/(\mu_2-(\mu_1)^2)$. Note that the condition $\phi(y) > -1$ for all y is only satisfied if $\frac{\vartheta(\mu_1)^2}{\mu_2-(\mu_1)^2} < 1$, which means that the safety-loading ϑ should not be too big. Under this martingale measure, the Poisson parameter is unchanged, i.e. $\lambda^{(2)} = \lambda$, whereas the distribution for the claim amounts is given by

$$G^{(2)}(dy) = \left(1 + \frac{\vartheta\mu_1}{\mu_2-(\mu_1)^2}(y-\mu_1)\right) G(dy).$$

This defines the martingale measure $Q^{(2)}$ with parameters $(\lambda^{(2)}, G^{(2)})$.

A third example is $\beta(y) = \rho y - \log(\int e^{\rho y} G(dy))$, for some $\rho > 0$. In this case, $1+\phi(y) = e^{\rho y}/\int e^{\rho y}G(dy)$, so that the martingale equation for this martingale measures $Q^{(3)}$ becomes

$$\frac{\lambda}{\int e^{\rho y'}G(dy')}\int y e^{\rho y} G(dy) = \lambda(1+\vartheta)\mu_1.$$

Under this measure $Q^{(3)}$, the Poisson parameter $\lambda^{(3)}$ is again unaffected by the change of measure and equal to λ, whereas the distribution of the claim amounts changes to $G^{(3)}(dy) = (1+\phi(y))G(dy)$. We analyse these measures in Section 5.3 and compare them with the minimal martingale measure and the minimal entropy martingale measure.

3.3 The minimal martingale measure

Define a local martingale $\widehat{Z}$ via

$$\widehat{Z}_t = \mathcal{E}\left(-\int \alpha\, dM\right)_t ,$$

where α is defined by (2.9). This means that $\widehat{Z}$ is the solution of the stochastic differential equation $d\widehat{Z} = -\widehat{Z}\alpha dM$, with $\widehat{Z}_0 = 1$. It is well known and follows by Girsanov's Theorem that $\widehat{Z}X$ is a local martingale under P, so that we can define an equivalent martingale measure $\widehat{P}$ via $\frac{d\widehat{P}}{dP} = \widehat{Z}_T$, provided that $\widehat{Z}_T$ is strictly positive and $\mathrm{E}[\widehat{Z}_T] = 1$. The measure $\widehat{P}$ is known as the *minimal martingale measure*, see Föllmer and Schweizer (1990), and it is well-known that this measure is in general only a *signed measure* for discontinuous processes; see also Schweizer (1995). This phenomenon can occur since $\widehat{Z}_T$ may attain negative values. Thus, in general, additional assumptions are needed in order to guarantee that $\widehat{P}$ is a true probability measure.

Introducing $\Delta M_t = M_t - M_{t-} = \int y\, \gamma(\{t\}, dy)$ and $\tilde{M}_t = M_t - \sum_{s\le t} \Delta M_s$, we can rewrite the process $\widehat{Z}_t$ as

$$\widehat{Z}_t = \exp\left(-\int_0^t \alpha_s\, d\tilde{M}_s - \frac{1}{2}\int_0^t \alpha_s^2\, \sigma_s^2\, ds\right) \prod_{s\le t} \left(1 - \alpha_s \Delta M_s\right), \tag{3.3}$$

see e.g. Jacod and Shiryaev (1987, Theorem I.4.67). In order for the measure $\widehat{P}$ to be a probability measure, we need that $\widehat{Z}_T$ is strictly positive. This is indeed the case provided that all factors $(1 - \alpha_s \Delta M_s)$ appearing in (3.3) are strictly positive. However, the assumption (2.3) of a positive safety-loading already guarantees that α defined by (2.9) is negative, so that $(1 - \alpha_s \Delta M_s) \geq 1$. This ensures that $\widehat{Z}_T$ is strictly positive.

The dynamics of X under the minimal martingale measure can now be found in the following way: Rewrite the density process $\widehat{Z}$ as

$$\begin{aligned} \widehat{Z}_t &= \exp\left(-\int_0^t \alpha_s\, \sigma_s dW_s - \frac{1}{2}\int_0^t \alpha_s^2\, \sigma_s^2\, ds\right) \\ &\quad \exp\left(\int_0^t \alpha_s\, \lambda_s\, \mu_{1,s}\, ds\right) \exp\left(\int_0^t \int_{(0,\infty)} \log(1 - \alpha_s\, y)\gamma(ds, dy)\right). \end{aligned} \tag{3.4}$$

By Girsanov's Theorem, $\widehat{W}_t = W_t + \int_0^t \alpha_s \sigma_s\, ds$ is a standard Brownian motion under $\widehat{P}$, and the $\widehat{P}$-compensator of $\gamma(ds, dy)$ is

$$\widehat{\nu}(dt, dy) = (1 - \alpha_t\, y)\nu(dt, dy) = \frac{1 - \alpha_t\, y}{1 - \alpha_t\, \mu_{1,t}} G_t(dy)(1 - \alpha_t\, \mu_{1,t})\lambda_t\, dt. \tag{3.5}$$

In this case, the martingale equation becomes

$$-\alpha_t \sigma_t^2 + \lambda_t \int_{(0,\infty)} y(1 - \alpha_t y) G_t(dy) - (1 + \vartheta_t)\lambda_t \mu_{1,t} = 0. \tag{3.6}$$

This equation could also simply be obtained from (2.9). Under the minimal martingale measure $\widehat{P}$, claims occur according to an inhomogeneous Poisson process with intensity

$$\widehat{\lambda}_t := (1 - \alpha_t\, \mu_{1,t})\lambda_t, \tag{3.7}$$

which exceeds the intensity λ_t under P, whereas a claim occurring at time t has distribution

$$\widehat{G}_t(dy) := \frac{1-\alpha_t\, y}{1-\alpha_t\, \mu_{1,t}} G_t(dy).$$

The distribution $\widehat{G}_t$ differs from G_t in that it increases the probability for claims above the mean $\mu_{1,t}$ and reduces the probability for claims below the mean. Moreover, it follows that

$$\int_{(0,\infty)} y G_t(dy) \leq \int_{(0,\infty)} y \widehat{G}_t(dy).$$

Thus, the change of measure from P to the minimal martingale measure $\widehat{P}$ increases the intensity for occurrence of claims as well as expected claim sizes.

3.4 The minimal entropy martingale measure

An equivalent martingale measure is called the *minimal entropy martingale measure* if it minimizes the so-called relative entropy with respect to the measure P, i.e. if it minimizes

$$\mathrm{E}\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right]$$

over all equivalent martingale measures Q. It follows e.g. from Grandits and Rheinländer (2002) that the density of the minimal entropy martingale measure $\overline{P}$ is of the form

$$\frac{d\overline{P}}{dP} = C\exp\left(\int_0^T \eta_s\, dX_s\right), \tag{3.8}$$

for some constant C and some integrable process η. More precisely, it is shown by Grandits and Rheinländer (2002) that an equivalent martingale measure on the form (3.8), where η satisfies some additional integrability conditions, is in fact the minimal entropy martingale measure; see also Chan (1999).

In our model, the structure of the minimal entropy measure can be exploited to determine the process η in the following way: Consider a measure $\overline{P}^\eta$ on the form (3.8) where η is some *deterministic* process. (Of course, for most choices of η, $\overline{P}^\eta$ will not be a martingale measure.) Using the fact that $\nu(dt, dy)$ and the process A appearing in the canonical decomposition for X are also deterministic, (3.8) may be rewritten as (here C is a constant which may change from line to line):

$$\begin{aligned} \frac{d\overline{P}^\eta}{dP} &= C\exp\left(\int_0^T \eta_s\, d(M_s + A_s)\right) = C\exp\left(\int_0^T \eta_s\, dM_s\right) \\ &= C\mathcal{E}\left(\int \eta\sigma dW\right)_T \exp\left(\int_0^T \eta_t \int_{(0,\infty)} y(\gamma(dt,dy) - \lambda_t G_t(dy)dt)\right) \\ &= C\mathcal{E}\left(\int \eta\sigma dW\right)_T \exp\left(\int_0^T \int_{(0,\infty)} \log e^{\eta_t y}\gamma(dt,dy)\right). \end{aligned} \tag{3.9}$$

Thus, for each deterministic function η, we have an equivalent measure $\overline{P}^\eta$, which has density (3.9) with respect to P. The question is now whether η can be chosen such that $\overline{P}^\eta$ becomes a martingale measure, and hence the minimal entropy martingale measure. To answer this question, note that the structure of (3.9) is similar to (3.4), so that we can immediately derive the properties of the processes W and $\gamma(dt, dy)$ which drive X.

We see that $d\overline{W}_t = dW_t - \eta_t\sigma_t dt$ is a standard Brownian motion under $\overline{P}^\eta$ and that the $\overline{P}^\eta$-compensator of $\gamma(dt, dy)$ is

$$\overline{\nu}(dt, dy) = e^{\eta_t y}\nu(dt, dy) = \overline{G}_t(dy)\overline{\lambda}_t dt,$$

where

$$\overline{G}_t(dy) = \frac{e^{\eta_t y}}{\psi_t(\eta_t)}G_t(dy),$$

and $\overline{\lambda}_t = \psi_t(\eta_t)\lambda_t$. Here, we have introduced the notation

$$\psi_t(\eta_t) = \int_{(0,\infty)} e^{\eta_t y}G_t(dy). \tag{3.10}$$

We can now express dX in terms of the $\overline{P}^\eta$-martingale increments $d\overline{W}$ and $\gamma(dt, dy) - \overline{\nu}(dt, dy)$, which leads to the martingale equation

$$\eta_t\sigma_t^2 + \lambda_t\int_{(0,\infty)} ye^{\eta_t y}G_t(dy) - (1 + \vartheta_t)\lambda_t\mu_{1,t} = 0. \tag{3.11}$$

A similar result was obtained by Chan (1999), who showed that this equation will have a unique solution provided that $\psi_t(\eta_t)$ is finite for η_t in some open interval containing 0. Thus, the minimal entropy martingale measure is uniquely determined via the solutions η_t to the equations (3.11). By rewriting the above equation for η_t as

$$\eta_t\sigma_t^2 + \lambda_t\int_{(0,\infty)} y(e^{\eta_t y} - 1)G_t(dy) - \vartheta_t\lambda_t\mu_{1,t} = 0, \tag{3.12}$$

we can realize that $\eta_t > 0$. To see this, recall that by assumption $\vartheta_t > 0$. Thus, for $\eta_t < 0$ all terms on the left side of the equation are negative, which leads to a contradiction.

The case of gamma distributed claims
Assume as an example that G is the gamma distribution with parameters (β, δ), i.e. that $G(dy) = g_{(\beta,\delta)}(y)dy$, where

$$g_{(\beta,\delta)}(dy) = \frac{\delta^\beta}{\Gamma(\beta)}y^{\beta-1}e^{-\delta y}. \tag{3.13}$$

Then, it follows that

$$\int_{(0,\infty)} ye^{\eta y}G(dy) = \frac{\beta\delta^\beta}{(\delta - \eta)^{\beta+1}},$$

for $\eta < \delta$. In particular, for α and β integer valued, solving (3.11) involves finding a real root in a polynomial of order $\beta + 2$. If $\beta = 1$ (exponentially distributed claims), this amounts to solving a cubic equation (quadratic, if $\sigma = 0$). We return to this example in Section 5.3.

4 Comparisons of optimal martingale measures

A natural question is whether there is a systematic ordering of prices computed under the two optimal martingale measures introduced above. This question has recently been studied in a different context within an exponential jump-diffusion model by Henderson and Hobson (2002), among others, who gave a simple criterion for ordering of option prices under various martingale measures. We first recall their result in our setting. We then show that their criterion does not allow for a comparison of prices under the minimal entropy martingale measure and the minimal martingale measure within our model.

Let Q be some martingale measure for X and let

$$X_T = X_0 + \int_0^T \int_{(0,\infty)} y(\gamma(dt,dy) - (1+\phi_t^Q(y))\nu(dt,dy)) + \int_0^T \sigma_t \, dW_t^Q, \tag{4.1}$$

where W^Q is a standard Brownian motion under Q, and where $(1+\phi_t^Q(y))\nu(dt,dy)$ is the Q-compensator of $\gamma(dt,dy)$. According to (3.5), $\phi_t^{\widehat{P}}(y) = -\alpha_t y$ for the minimal martingale measure. For the minimal entropy martingale measure, Section 3.4 shows that $\phi_t^{\overline{P}}(y) = \exp(\eta_t y) - 1$, where η_t solves the equation

$$\eta_t \sigma_t^2 + \lambda_t \int_{(0,\infty)} y(e^{\eta_t y} - 1) G_t(dy) - \vartheta_t \lambda_t \mu_{1,t} = 0. \tag{4.2}$$

Consider now another martingale measure $\tilde{Q}$ and the corresponding decomposition

$$X_T = X_0 + \int_0^T \int_{(0,\infty)} y(\gamma(dt,dy) - (1+\phi_t^{\tilde{Q}}(y))\nu(dt,dy)) + \int_0^T \sigma_t \, dW_t^{\tilde{Q}}. \tag{4.3}$$

A first ordering result is now obtained via the following:

Proposition 4.1 *(Henderson and Hobson 2002)*
Let $H = \Phi(X_T)$ for some convex function Φ and consider two martingale measures Q and $\tilde{Q}$ with ϕ^Q and $\phi^{\tilde{Q}}$ deterministic. If $\phi_t^Q(y) \geq \phi_t^{\tilde{Q}}(t,y)$ for all (t,y), then $\mathrm{E}_Q[\Phi(X_T)] \geq \mathrm{E}_{\tilde{Q}}[\Phi(X_T)]$.

This result is taken from Theorem 6.1 of Henderson and Hobson (2002), who prove this via coupling and Jensen's inequality for conditional expectations. Since our situation essentially only differs from their setup in that they consider an *exponential* jump-diffusion model, we omit the proof of this result here; the result can also be proved with the techniques introduced in Section 5 below.

The following result gives a strict lower bound for the value of the parameter η_t. However, this bound does not allow for a uniform comparison of the kernels $\phi^{\overline{P}}(t,y)$ and $\phi^{\widehat{P}}(t,y)$ for the minimal entropy and minimal martingale measures. To see this, we first show the following:

Lemma 4.2 *The parameter η_t appearing in the kernel $\phi_t^{\overline{P}}(y) = e^{\eta_t y} - 1$ is strictly positive, and $\eta_t < -\alpha_t$.*

Proof: We can realize this by a straightforward examination of (4.2). By adding and subtracting the term $\eta_t y$, the integrand can be rewritten as $e^{\eta_t y} - 1 = \eta_t y + f(\eta_t, y)$, where f is defined by $f(\eta_t, y) = e^{\eta_t y} - (1 + \eta_t y)$, which is strictly positive for $\eta_t > 0$ and $y > 0$ (Taylor expansion of e^x at 0). Thus, the integral appearing in the equation (4.2) for η_t becomes

$$\begin{aligned} \lambda_t \int_{(0,\infty)} y(e^{\eta_t y} - 1) G_t(dy) &= \eta_t \lambda_t \int_{(0,\infty)} y^2 G_t(dy) + \lambda_t \int_{(0,\infty)} y f(\eta_t, y) G_t(dy) \\ &= \eta_t \lambda_t \mu_{2,t} + \lambda_t F(\eta_t), \end{aligned}$$

where $F(\eta_t) > 0$. By inserting this in (4.2), we get that $\eta_t\left(\sigma_t^2 + \lambda_t\mu_{2,t}\right) + \lambda_t F(\eta_t) = \vartheta_t\lambda_t\mu_{1,t}$, which shows that

$$\eta_t < \eta_t + \frac{\lambda_t F(\eta_t)}{\sigma_t^2 + \lambda_t\mu_{2,t}} = \frac{\vartheta_t\lambda_t\mu_{1,t}}{\sigma_t^2 + \lambda_t\mu_{2,t}} = -\alpha_t.$$

This completes the proof. □

The lemma can be used to see that the minimal martingale measure and the minimal entropy martingale measures in our case do not admit the uniform ordering needed in Proposition 4.1. This follows immediately by comparing $e^{\eta_t y} - 1$ and $-\alpha_t y$. First recall that $0 < \eta_t < -\alpha_t$. Then, define the function

$$v(y) = e^{\eta_t y} - 1 + \alpha_t y = \phi_t^{\overline{P}}(y) - \phi_t^{\widehat{P}}(y),$$

which is the difference between the kernels under the minimal entropy martingale measure and the minimal martingale measure. It follows that $v(y)$ is strictly positive for y sufficiently big, whereas $v(0) = 0$ and $v'(0) = \eta_t + \alpha_t < 0$, so that $v(y)$ is negative for small y. This shows that in the case where claims take values in any interval of $(0, \infty)$ with positive probability, a uniform ordering of $\phi_t^{\overline{P}}(y)$ and $\phi_t^{\widehat{P}}(y)$ does not exist. More precisely, we see that $\phi_t^{\overline{P}}(y) < \phi_t^{\widehat{P}}(y)$ for y small and $\phi_t^{\overline{P}}(y) > \phi_t^{\widehat{P}}(y)$ for y sufficiently big.

Remark 4.3 Equations (3.6) and (3.11) show that

$$-\alpha_t\sigma_t^2 + \lambda_t \int_{(0,\infty)} y(1 - \alpha_t y)G_t(dy) = \eta_t\sigma_t^2 + \lambda_t \int_{(0,\infty)} y e^{\eta_t y} G_t(dy),$$

which implies that

$$\begin{aligned}\widehat{\lambda}_t\widehat{\mu}_{1,t} = \lambda_t \int_{(0,\infty)} y(1 - \alpha_t y)G_t(dy) &= (\eta_t + \alpha_t)\sigma_t^2 + \lambda_t \int_{(0,\infty)} y e^{\eta_t y} G_t(dy) \\ &\leq \lambda_t \int_{(0,\infty)} y e^{\eta_t y} G_t(dy) = \overline{\lambda}_t\overline{\mu}_{1,t}.\end{aligned}$$

This shows that the expected value under the minimal entropy martingale measure of the random Poisson part is bigger than the expected value under the minimal martingale measure. The last calculation combined with the above lemma shows that we have "=" if and only if $\sigma = 0$. □

4.1 Simple comparisons

Ordering of Poisson intensities in the pure jump case
Assume now that $\sigma = 0$. Under this simplifying assumption, we show that $\widehat{\lambda}_t \geq \overline{\lambda}_t$, i.e., the Poisson arrival intensity under the minimal martingale measure exceeds the one for the minimal entropy martingale measure. Using the definition of $\overline{\lambda}_t$, Taylor expansion for the exponential function and Tonelli's Theorem, we first rewrite the Poisson intensity under the minimal entropy martingale measure as

$$\begin{aligned}\overline{\lambda}_t &= \lambda_t \int_{(0,\infty)} e^{\eta_t y} G_t(dy) \\ &= \lambda_t \int_{(0,\infty)} \left(\sum_{m=0}^{\infty} \frac{(\eta_t y)^m}{m!}\right) G_t(dy) \\ &= \lambda_t \left(1 + \sum_{m=1}^{\infty} \eta_t^m \frac{\mu_{m,t}}{m!}\right). \qquad (4.4)\end{aligned}$$

Here, η_t is found from the equation (4.2), which via similar calculations simplifies to

$$\vartheta_t \mu_{1,t} = \int_{(0,\infty)} y(e^{\eta_t y} - 1) G_t(dy) = \sum_{m=1}^{\infty} \eta_t^m \frac{\mu_{m+1,t}}{m!}. \tag{4.5}$$

Under the minimal martingale measure, the Poisson intensity (3.7) is

$$\widehat{\lambda}_t = \lambda_t (1 - \alpha_t \mu_{1,t}) = \lambda_t \left(1 + \vartheta_t \frac{\mu_{1,t}^2}{\mu_{2,t}}\right). \tag{4.6}$$

By inserting (4.5) in (4.6), we get that

$$\widehat{\lambda}_t = \lambda_t \left(1 + \frac{\mu_{1,t}}{\mu_{2,t}} \sum_{m=1}^{\infty} \eta_t^m \frac{\mu_{m+1,t}}{m!}\right),$$

which can now be compared directly with (4.4). To see that $\widehat{\lambda}_t \geq \overline{\lambda}_t$ is only remains to verify the following inequality for the moments of the distribution G_t:

$$\mu_{1,t}\, \mu_{m+1,t} \geq \mu_{m,t}\, \mu_{2,t}, \text{ for } m = 1, 2, \ldots. \tag{4.7}$$

The inequalities (4.7) are in fact a direct consequence of a generalized version of Chebychev's other inequality, see e.g. Fink and Jodeit (1984) and Mitrinović and Vasić (1974). For reasons of completeness we recall that result here:

Lemma 4.4 *(Chebychev's other inequality, generalized)*
Consider a probability measure G^* and two functions f and g. Then it holds for any interval $(a, b]$ that

$$\int_{(a,b]} G^*(dy) \int_{(a,b]} f(y')g(y')G^*(dy') \geq \int_{(a,b]} f(y)G^*(dy) \int_{(a,b]} g(y')G^*(dy'),$$

if

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0, \text{ for all } (x, y) \in (a, b]^2.$$

Note that the necessary condition is indeed satisfied if for example the functions f and g are both increasing.

Applying Lemma 4.4 with the probability measure $G_t^*(dy) = \frac{y}{\mu_{1,t}} G_t(dy)$, the functions $f(y) = y^{m-1}$, $g(y) = y$, which clearly satisfy the condition, and by considering intervals $(0, b]$ as b goes to infinity, we get that

$$\int_{(0,\infty)} yG_t(dy) \int_{(0,\infty)} y^{m+1}G_t(dy) \geq \int_{(0,\infty)} y^m G_t(dy) \int_{(0,\infty)} y^2 G_t(dy),$$

which shows (4.7).

Ordering of expected values in claim size distributions
In the general case with a diffusion term, where $\sigma \neq 0$, we cannot conclude that $\widehat{\lambda}_t \geq \overline{\lambda}_t$ via similar calculations. However, similar arguments show that

$$\widehat{\mu}_{1,t} = \frac{\int y(1 - \alpha_t y)G_t(dy)}{\int (1 - \alpha_t y)G_t(dy)} \leq \frac{\int y e^{\eta_t y} G_t(dy)}{\int e^{\eta_t y} G_t(dy)} = \overline{\mu}_{1,t}, \tag{4.8}$$

that is, the expected values in the claim size distributions are ordered. This result is essential for the proofs of our main results in the next section. The inequality in (4.8) can be established by using Taylor expansions of the exponential functions appearing on the right of the inequality. Thus, it is sufficient to show that

$$(\mu_{1,t} - \alpha_t \mu_{2,t}) \sum_{m=0}^{\infty} \mu_{m,t} \frac{\eta_t^m}{m} \leq (1 - \alpha_t \mu_{1,t}) \sum_{m=0}^{\infty} \mu_{m+1,t} \frac{\eta_t^m}{m},$$

and this follows via Lemma 4.4.

5 Stochastic orders and optimal martingale measures

5.1 Some results from the theory on stochastic orders

This section reviews some standard results from the theory on stochastic orders that will prove useful below; references are Müller and Stoyan (2002) and Shaked and Shanthikumar (1994). For applications of stochastic orders in actuarial science, see e.g. Goovaerts, De Vylder and Haezendonck (1984), Kaas, van Heerwanden and Goovaerts (1994) and references therein.

Consider two random variables Y and Z, with distribution functions F_Y and F_Z, respectively. The random variable Y is said to be *stochastically smaller* than Z if $F_Y(y) \geq F_Z(y)$ for all y. In this case we write $Y \preceq_d Z$ or $F_Y \preceq_d F_Z$; note than this is a condition on the distribution functions for Y and Z and that Y and Z need not be defined on the same probability space. It is not difficult to see that if Y and Z are non-negative and if $Y \preceq_d Z$, then $\mathrm{E}[Y^r] \leq \mathrm{E}[Z^r]$ for all $r \geq 0$. The random variable Y is said to be smaller than Z in the *increasing convex order* (written $Y \preceq_c Z$ or $F_Y \preceq_c F_Z$) if

$$\mathrm{E}[(Y-x)^+] = \int_x^{\infty} (y-x) F_Y(dy) \leq \int_x^{\infty} (y-x) F_Z(dy) = \mathrm{E}[(Z-x)^+]$$

for all x. It follows by Jensen's inequality that $\mathrm{E}[Y] \preceq_c Y$, i.e. Y is larger in the increasing convex order than its expected value. It can be shown that $Y \preceq_c Z$ if and only if $\mathrm{E}[\Phi(Y)] \leq \mathrm{E}[\Phi(Z)]$ for all increasing convex functions Φ. If, moreover, $\mathrm{E}[Y] = \mathrm{E}[Z]$, then this inequality holds for all (not necessarily increasing) convex functions. In this situation, we simply say that Y is smaller than Z in the convex order.

Let Θ be another random variable and denote by $F_{Y,\theta}$ and $F_{Z,\theta}$ the conditional distributions of Y and Z given $\Theta = \theta$. If $F_{Y,\theta} \preceq_c F_{Z,\theta}$ for all ϑ, then $F_Y \preceq_c F_Z$. Thus, the (increasing) convex order is *closed under mixture*. If $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ are sequences of independent random variables such that $Y_i \preceq_c Z_i$ for $i = 1, 2, \ldots$ then $g(Y_1, \ldots, Y_m) \preceq_c g(Z_1, \ldots, Z_m)$ for any increasing, component-wise convex function $g : \mathbb{R}^m \mapsto \mathbb{R}$. In particular, this shows that $Y_1 + \ldots + Y_m \preceq_c Z_1 + \ldots + Z_m$, so that the (increasing) convex order is *closed under convolution*. If moreover each of the sequences $(Y_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ are i.i.d. and if N and M are two integer-valued non-negative random variables with $M \preceq_c N$ which are independent of the sequences of Y_i's and Z_i's, then

$$\sum_{i=1}^{M} Y_i \preceq_c \sum_{i=1}^{N} Z_i. \tag{5.1}$$

This shows that the (increasing) convex order is *closed under random summation*. For example, if M and N are Poisson variables with parameters λ_M and λ_N, then $M \preceq_c N$ if

$\lambda_M \leq \lambda_N$. A useful sufficient criterion for the (increasing) convex ordering of two random variables Y and Z with distribution functions F_Y and F_Z is the so-called *cut criterion*: If $\mathrm{E}[Y] \leq \mathrm{E}[Z]$ and if there exists some finite ξ such that

$$F_Y(y) \leq F_Z(y) \text{ for all } y < \xi, \text{ and } F_Y(y) \geq F_Z(y) \text{ for all } y > \xi,$$

then $Y \preceq_c Z$. If the distribution functions F_Y and F_Z admit densities f_Z and f_Y with respect to some measure and have equal means, then the cut criterion is satisfied if for example the function $(f_Z - f_Y)$ has exactly two sign changes, with sign sequence $+$, $-$, $+$.

5.2 Main results: Convex ordering for martingale measures

In this section, we formulate our main result, which gives a sufficient criterion for the convex ordering of the distribution of X_T for two equivalent martingale measures Q and $\widetilde{Q}$ with *deterministic* kernels (ψ^Q, ϕ^Q) and $(\psi^{\widetilde{Q}}, \phi^{\widetilde{Q}})$, respectively; see Section 3.1 for the definitions of Q and $\widetilde{Q}$ in terms of their kernels. In this situation, the jump parts of X_T are essentially generated by compound Poisson processes, such that we can use the results on convex orders reviewed in the previous section. We refer to the pairs (λ_t^Q, G_t^Q) and $(\lambda_t^{\widetilde{Q}}, G_t^{\widetilde{Q}})$, consisting of the Poisson intensities and the claim size distributions (both possibly time dependent), as the parameters. Moreover, we denote by $\mu_{1,t}^Q$ and $\mu_{1,t}^{\widetilde{Q}}$ the means in the claim size distributions. Finally, F_Q and $F_{\widetilde{Q}}$ are the distribution functions of X_T under Q and $\widetilde{Q}$. Note that for any martingale measures Q and $\widetilde{Q}$, we have that $\mathrm{E}_Q[X_T] = \mathrm{E}_{\widetilde{Q}}[X_T] = X_0$, so that we don't need to distinguish between the increasing convex order and the convex order when comparing the distribution of X_T under different martingale measures.

Here is our main result:

Theorem 5.1 *Consider two equivalent martingale measures Q and $\widetilde{Q}$ with deterministic kernels (ψ_t^Q, ϕ_t^Q) and $(\psi_t^{\widetilde{Q}}, \phi_t^{\widetilde{Q}})$ and parameters (λ_t^Q, G_t^Q) and $(\lambda_t^{\widetilde{Q}}, G_t^{\widetilde{Q}})$. Let*

$$f_t(y) = \lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}} (1 + \phi_t^Q(y)) - \lambda_t^Q \mu_{1,t}^Q (1 + \phi_t^{\widetilde{Q}}(y)). \tag{5.2}$$

Assume that for all $t \in [0, T]$:

1. $\lambda_t^Q \mu_{1,t}^Q \geq \lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}}$ *and* $\mu_{1,t}^Q \geq \mu_{1,t}^{\widetilde{Q}}$.
2. *There exist constants $0 \leq y_t^1 \leq y_t^2 < \infty$, such that $f_t(y) \geq 0$ for $y \in (0, y_t^1) \cup (y_t^2, \infty)$ and $f_t(y) \leq 0$ for $y \in (y_t^1, y_t^2)$.*

Then $F_{\widetilde{Q}} \preceq_c F_Q$, i.e. for any convex function Φ, $\mathrm{E}_{\widetilde{Q}}[\Phi(X_T)] \leq \mathrm{E}_Q[\Phi(X_T)]$.

Remark 5.2 We comment a bit on the conditions needed in the theorem. The first condition guarantees that the means of the claim size distributions and the means of the compound Poisson parts in X_T under Q and $\widetilde{Q}$ are ordered. The second condition is needed in order to ensure that we can apply the cut criterion on certain transformed densities related to the two measures. Note that if $\sigma = 0$, we have that $\lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}} = \lambda_t^Q \mu_{1,t}^Q = (1+\vartheta_t)\mu_{1,t}\lambda_t = p_t$, see (3.1) and (2.3), which implies that (5.2) reduces to $p_t(\phi_t^Q(y) - \phi_t^{\widetilde{Q}}(y))$. $\square$

We postpone the proof of Theorem 5.1 to Section 5.4 below. Here, we formulate and prove instead Proposition 5.3 by using this theorem. Denote by $\widehat{F}$ and $\overline{F}$ the distribution functions of X_T under the minimal martingale measure $\widehat{P}$ and the minimal entropy martingale

Measure	Poisson intensity	Claim size distribution	$1+\phi(y)$
$Q^{(1)}$	$(1+\vartheta)\lambda$	$G(dy)$	$1+\vartheta$
$Q^{(2)}$	λ	$(1+\frac{\vartheta\mu_1}{\mu_2-(\mu_1)^2}(y-\mu_1))G(dy)$	$1+\frac{\vartheta\mu_1}{\mu_2-(\mu_1)^2}(y-\mu_1)$
$Q^{(3)}$	λ	$e^{\rho y}/(\int e^{\rho y'}G(dy'))G(dy)$	$e^{\rho y}/(\int e^{\rho y'}G(dy'))$
$\widehat{P}$	$(1+\vartheta\frac{\mu_1^2}{\mu_2})\lambda$	$(1+\frac{\vartheta\mu_1}{\mu_2+\vartheta(\mu_1)^2}(y-\mu_1))G(dy)$	$1+\frac{\vartheta\mu_1}{\mu_2+\vartheta(\mu_1)^2}(y-\mu_1)$
$\overline{P}$	$\lambda\int e^{\eta y'}G(dy')$	$e^{\eta y}/(\int e^{\eta y'}G(dy'))G(dy)$	$e^{\eta y}$

Table 1: *Martingale measures and their parameters in the pure jump case.*

measure $\overline{P}$, respectively. In addition, we use the notation $\widehat{\mathrm{E}}$ and $\overline{\mathrm{E}}$ for the expectations $\mathrm{E}_{\widehat{P}}$ and $\mathrm{E}_{\overline{P}}$, respectively. We show that $\widehat{F} \preceq_c \overline{F}$; according to the results listed in the previous section, this can be formulated in the following way:

Proposition 5.3 *For any convex function Φ, the price of $\Phi(X_T)$ under the minimal martingale measure is smaller than the price of $\Phi(X_T)$ under the minimal entropy martingale measure, i.e.* $\widehat{\mathrm{E}}[\Phi(X_T)] \le \overline{\mathrm{E}}[\Phi(X_T)]$.

The result can also be interpreted by saying that the minimal entropy martingale measure represents a more conservative attitude to risk than the minimal martingale measure.

Proof of Proposition 5.3: First recall that the kernels for the minimal martingale measure $\widehat{P}$ and the minimal entropy martingale measure $\overline{P}$ are indeed deterministic. Secondly, we know from Remark 4.3 that $\widehat{\lambda}_t\widehat{\mu}_{1,t} \le \overline{\lambda}_t\overline{\mu}_{1,t}$, and (4.8) shows that $\widehat{\mu}_{1,t} \le \overline{\mu}_{1,t}$. This establishes the first condition of the theorem. For the measures $(\widehat{P}, \overline{P})$, the function (5.2) is given by

$$f_t(y) = \widehat{\lambda}_t\widehat{\mu}_{1,t}e^{\eta_t y} - \overline{\lambda}_t\overline{\mu}_{1,t}(1-\alpha_t y).$$

In this case, Condition 2 is clearly satisfied with $y_t^1 = 0$ and y_t^2 given as the unique stricly positive solution to the equation $f_t(y) = 0$. This completes the proof. □

5.3 Optimal measures versus ad-hoc choices in the pure jump case

In this section, we compare the measures determined by ad-hoc considerations in Section 3.2 with the two optimal martingale measures in the time-homogenous pure jump case. The various measures can be found in Table 1 with their parameters, the Poisson intensity and the claim size distribution. We end this section by considering an example with exponentially distributed claims and characterize the various measures that have been introduced so far within this example.

Proposition 5.4 *In the time-homogeneous pure jump case where $\sigma = 0$, the martingale measures are ordered in the following way:*

$$F_{Q^{(1)}} \preceq_c F_{\widehat{P}} \preceq_c F_{\overline{P}} \preceq_c F_{Q^{(3)}}. \tag{5.3}$$

If moreover $\vartheta(\mu_1)^2/(\mu_2-(\mu_1)^2) < 1$, then $Q^{(2)}$ is well-defined and we have that

$$F_{Q^{(1)}} \preceq_c F_{\widehat{P}} \preceq_c F_{Q^{(2)}} \preceq_c F_{Q^{(3)}}.$$

Measure	Poisson intensity	Claim size distr. (density)
P	λ	$g_{(1,\delta)}$
$Q^{(1)}$	$(1+\vartheta)\lambda$	$g_{(1,\delta)}$
$Q^{(2)}$	λ	$(1-\vartheta)g_{(1,\delta)}+\vartheta g_{(2,\delta)}$
$Q^{(3)}$	λ	$g_{(1,\delta/(1+\vartheta))}$
$\widehat{P}$	$(1+\vartheta/2)\lambda$	$\frac{1}{1+\vartheta/2}g_{(1,\delta)}+\frac{\vartheta/2}{1+\vartheta/2}g_{(2,\delta)}$
$\overline{P}$	$\lambda\sqrt{1+\vartheta}$	$g_{(1,\frac{\delta}{\sqrt{1+\vartheta}})}$

Table 2: *Probability measures and their parameters in the pure jump case under exponentially distributed claims.*

Proof: As mentioned in Remark 5.2, the means in the compound Poisson parts under the various martingale measures coincide in the pure jump case, such that the first part of Condition 1 in Theorem 5.1 will automatically be satisfied for any pair of martingale measures. We first verify the three ordering relations in (5.3) by showing in each case that the conditions of the theorem are satisfied:

$F_{Q^{(1)}} \preceq_c F_{\widehat{P}}$: Compare first the Poisson intensities under the two measures: Since $\lambda^{Q^{(1)}} = (1+\vartheta)\lambda \geq (1+\vartheta\frac{(\mu_1)^2}{\mu_2}) = \widehat{\lambda}$, we see that $\mu_1^{Q^{(1)}} \leq \widehat{\mu}_1$. This shows the second part of Condition 1. To see the second condition, note that $f_t(y) = p\left(|\alpha|y-\vartheta\right)$, which clearly satisfies Condition 2. (Here, α and p are defined by (2.9) and (2.3).)

$F_{\widehat{P}} \preceq_c F_{\widehat{P}}$: This already follows from Proposition 5.3.

$F_{\overline{P}} \preceq_c F_{Q^{(3)}}$: The first condition of Theorem 5.1 is verified by noting that $\lambda^{Q^{(3)}} = \lambda \leq \overline{\lambda}$, and the second condition follows by noting that $f_t(y) = p\left(e^{\rho y}/(\int e^{\rho y'}G(dy')) - e^{\eta y}\right)$, where $\rho > \eta$. Thus, $f_t(y)$ satisfies the second condition of the theorem. This completes the proof of (5.3). The condition on ϑ for the existence of $Q^{(3)}$ follows from Section 3.2, and the ordering relations for the measures $Q^{(i)}$, $i=1,2,3$, and $\widehat{P}$ follows from calculations similar to the ones used for the proof of (5.3). □

The case of exponentially distributed claims
Consider the situation, where the claim size distribution G under P is exponential with parameter δ. Recall that $g_{(\beta,\delta)}$ is the density for the gamma distribution with parameters (β,δ), see (3.13). Thus, in the example $G(dy) = g_{(1,\delta)}(y)dy$, so that $\mu_1 = 1/\delta$ and $\mu_2 = 2/\delta^2$. Using the various defining equations for the parameters appearing in the martingale measures presented in Table 1, we can characterize the Poisson intensities and the claim size distributions under the martingale measures in this example, see Table 2. This table shows that the claims are also exponentially distributed under the measures $Q^{(1)}$, $Q^{(3)}$ and $\overline{P}$, whereas the claim size distributions under $Q^{(2)}$ and $\widehat{P}$ are mixtures of certain exponential and gamma distributions with shape parameter 2. (Note in addition that the measure $Q^{(2)}$ is only defined if $\vartheta < 1$.)

We consider a numerical example, where we take $\lambda = \delta = T = 1$ and $\vartheta = 0.5$. Stop-loss premiums for retentions between 0.5 (which corresponds to 1/3 of the premium, since $(1+\vartheta)\mu_1\lambda = 1.5$) and 6 (4 times the premium) can be found in Figure 1. (All numbers have been computed via simulation; an alternative idea would be to apply the so-called Panjer recursion, see Panjer (1981).) The figure illustrates the results in Proposition 5.4, in that the premiums are ordered for any retention levels such that indeed $F_{Q^{(1)}} \preceq_c F_{\widehat{P}} \preceq_c F_{\overline{P}} \preceq_c F_{Q^{(3)}}$.

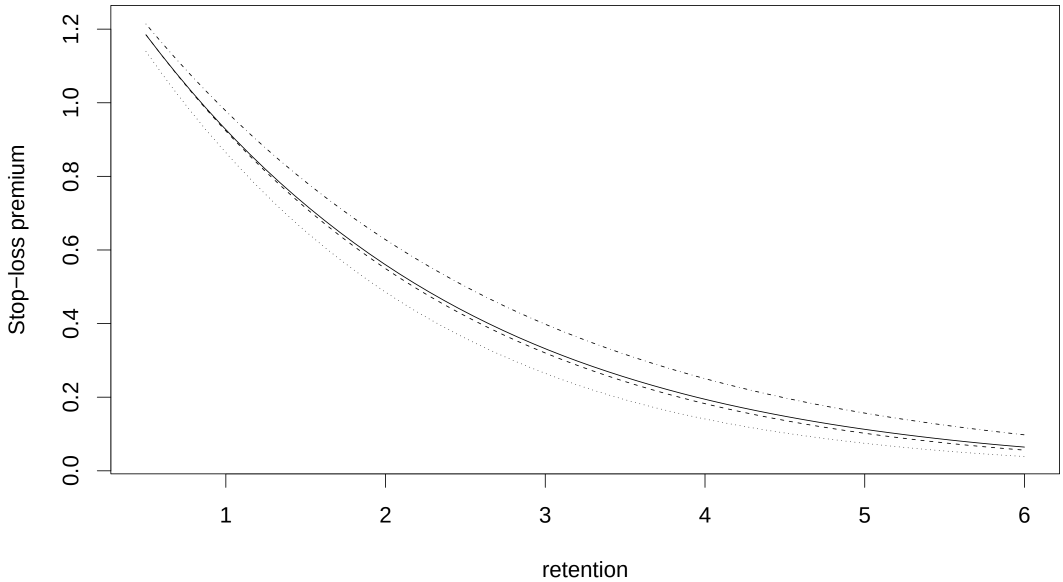

Figure 1: *Stop-loss premiums under exponential claims. The dot-dashed line (at the top) corresponds to the measure $Q^{(3)}$, the solid line is the minimal entropy measure, the dashed line is the minimal martingale measure, and the dotted line (at the bottom) corresponds to $Q^{(1)}$.*

In particular, the figure confirms that premiums computed under the minimal martingale measure are smaller than the ones computed under the minimal entropy martingale measure. However, the difference between the premiums under these two measures is relatively small. For comparison, we have listed some of the premiums in Table 3. This table could seem to indicate that there is also an ordering between $\overline{P}$ and $Q^{(2)}$, i.e. that $F_{\overline{P}} \preceq_c F_{Q^{(2)}}$ in our example. However, we point out that our theorem cannot be used to establish such

Measure/retention	0	1	2	3	4	5	6
$Q^{(1)}$	1.50	0.86	0.49	0.26	0.14	0.08	0.04
$\widehat{P}$	1.50	0.92	0.55	0.32	0.18	0.10	0.06
$\overline{P}$	1.50	0.93	0.56	0.33	0.19	0.11	0.06
$Q^{(2)}$	1.50	0.97	0.60	0.37	0.22	0.13	0.07
$Q^{(3)}$	1.50	0.98	0.63	0.40	0.25	0.16	0.10

Table 3: *Stop-loss premiums under the various martingale measures in the case of exponentially distributed claims.*

an ordering of the two measures; the problem is that the mean in the $Q^{(3)}$-claim size distribution exceeds the mean in the $\overline{P}$-distribution, but the corresponding kernels do not allow for application of the cut criterion.

5.4 Proof of Theorem 5.1

We first present a proof for the time-homogeneous case, where $\sigma = 0$. Then, we give the proof for the general case.

Time-homogeneous pure jump case
Assume for simplicity that $T = 1$. Consider two measures Q and $\widetilde{Q}$ with decompositions (4.1) and (4.3) and take $\sigma = 0$. The terms $\int_0^1 \int y\gamma(dt, dy)$ are distributed as standard compound Poisson variables both under Q and $\widetilde{Q}$. Since X is a martingale under both Q and $\widetilde{Q}$, and since $\sigma = 0$, the expected values of the jump parts under the two optimal martingale measures coincide, see Remark 5.2. Thus, it is sufficient to check that the two compound Poisson distributions are ordered under the convex order. However, according to Condition 1, $\lambda^{\widetilde{Q}} \geq \lambda^Q$, so that we cannot use the results reviewed in Section 5.1 directly; the problem is that the claim arrival Poisson intensity under $\widetilde{Q}$ exceeds the Q-intensity, and we want to show that $F_{\widetilde{Q}} \preceq_c F_Q$.

In order to prepare for an application of the results on convex ordering, we first apply standard results for compound Poisson variables. The distribution of the compound Poisson variable under Q is identical to the distribution of another compound Poisson variable $\sum_{i=1}^{N_1'} Y_i'$, where $N_1' \sim$Poisson$(\lambda^{\widetilde{Q}})$, and where $Y_1', Y_2', \ldots$ are i.i.d., independent of N_1', with distribution G' on $[0, \infty) = \{0\} \cup (0, \infty)$ given by

$$\begin{aligned} G'(dy) &= \frac{\lambda^Q}{\lambda^{\widetilde{Q}}} G^Q(dy) + \left(1 - \frac{\lambda^Q}{\lambda^{\widetilde{Q}}}\right) \varepsilon_0(dy) \\ &= \frac{\lambda^Q}{\lambda^{\widetilde{Q}}} \frac{\lambda}{\lambda^Q}(1 + \phi^Q(y))G(dy) + \left(1 - \frac{\lambda^Q}{\lambda^{\widetilde{Q}}}\right) \varepsilon_0(dy), \end{aligned} \tag{5.4}$$

where $\varepsilon_0(y)$ is the Dirac-measure at 0. For simplicity, one can take (here and in the following) all random variables equipped with a $'$ to be defined on a separate probability space $(\Omega', \mathcal{F}', P')$. Thus, we have increased the Q-Poisson intensity from λ^Q to $\lambda^{\widetilde{Q}}$ by the factor $\frac{\lambda^{\widetilde{Q}}}{\lambda^Q} \geq 1$ and replaced the claim size distribution G^Q by G', which is a mixture of the original Q-claim size distribution and the Dirac measure at 0, without affecting the distribution of the compound variable. (To see this, simply compute the characteristic functions of the two compound Poisson variables, and check that they are identical.) Thus, if we can show that $G^{\widetilde{Q}} \preceq_c G'$, then the assertion follows by using a result similar to (5.1). To see that $G^{\widetilde{Q}} \preceq_c G'$, we now apply the cut criterion to G', defined by (5.4), and to $G^{\widetilde{Q}}$ defined by

$$G^{\widetilde{Q}}(dy) = \frac{\lambda}{\lambda^{\widetilde{Q}}}(1 + \phi^{\widetilde{Q}}(y))G(dy).$$

First note that the mean μ_1' in the distribution G' is identical to the mean $\mu_1^{\widetilde{Q}}$ in the distribution $G^{\widetilde{Q}}$, since the mean of the two compound Poisson variables coincide in the pure jump case.

Denote by g' and $g^{\widetilde{Q}}$ the densities for the distributions G' and $G^{\widetilde{Q}}$ with respect to the convolution of G and the Dirac measure at 0. Then the difference between the two densities is

$$g'(y) - g^{\widetilde{Q}}(y) = \left(1 - \frac{\lambda^Q}{\lambda^{\widetilde{Q}}}\right) 1_{\{y=0\}} + \frac{\lambda}{\lambda^{\widetilde{Q}}} \left((1 + \phi^Q(y)) - (1 + \phi^{\widetilde{Q}}(y))\right) 1_{\{y \neq 0\}}.$$

Since $\lambda^{\widetilde{Q}}\mu^{\widetilde{Q}} = \lambda^Q\mu^Q$, Condition 2 of the above theorem guarantees the existence of $0 \leq y^1 \leq y^2 < \infty$ such that

$$\mathrm{sign}(g'(y) - \widehat{g}(y)) = \begin{cases} +, & y \in [0, y^1), \\ -, & y \in (y^1, y^2), \\ +, & y \in (y^2, \infty). \end{cases} \tag{5.5}$$

Thus, according to the sufficient condition for the cut criterion, $G^{\widetilde{Q}} \preceq_c G'$, so that (5.1) gives that $F_{\widetilde{Q}} \preceq_c F_Q$ as claimed. In the case $\lambda^Q = \lambda^{\widetilde{Q}}$, no adjustment of the Poisson intensities is needed and the cut criterion follows by examining the original distribution functions G^Q and $G^{\widetilde{Q}}$ directly. This shows the result in the time-homogeneous pure jump case. □

Proof in the general case:
Consider now the time-inhomogeneous case, with σ not necessarily equal to 0. As in the homogeneous case, the result will be proved by examining the representation formulas for X_T under the two measures Q and $\widetilde{Q}$. However, in the present situation, where parameters are time-dependent, we cannot immediately apply the results on the closedness of the convex order under random summation. In this situation, we apply a result of Norberg (1993) which allows us to represent the jump part $\int_0^T \int y\gamma(dt, dy)$ as a standard compound Poisson variable.

Consider first the jump part under $\widetilde{Q}$. By Theorem 1 and Corollary 1 of Norberg (1993), we get that

$$\int_0^T \int_{(0,\infty)} y\gamma(dt, dy) \overset{\mathcal{D}}{=} \sum_{i=1}^{\widetilde{N}'_T} \widetilde{Y}'_i, \tag{5.6}$$

where the standard compound Poisson variable has parameters $\widetilde{\Lambda}_T = \int_0^T \lambda_t^{\widetilde{Q}} dt$ and

$$\widetilde{G}^*(dy) = \frac{\int_0^T \lambda_t^{\widetilde{Q}} G_t^{\widetilde{Q}}(dy)dt}{\int_0^T \lambda_t^{\widetilde{Q}} dt}. \tag{5.7}$$

Here, it is crucial that we have fixed the time T. The result says that, when considering the total claim amount during $[0, T]$, we can equivalently view claims as taken from the same distribution $\widetilde{G}^*$. More precisely, $\widetilde{G}^*$ is a mixture of the distributions $\{G_\theta^{\widetilde{Q}} | \theta \in [0, T]\}$ with a mixing distribution, which has density $h(\theta) = \frac{\lambda_\theta^{\widetilde{Q}}}{\widetilde{\Lambda}_T}$ on $[0, T]$. The proof consists in deriving a similar characterization of the distribution of the jump part under Q and then using the closedness of the convex order under mixtures.

We now turn to the distribution of the jump part under Q. Since the expected values of the jump parts under the two measures might differ, see Condition 1, we first decompose the Q-jump part into two (inhomogeneous) Poisson random measures. Let

$$\lambda'_t = \frac{\lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}}}{\mu_{1,t}^Q} \text{ and } \lambda''_t = \lambda_t^Q - \lambda'_t.$$

First part of Condition 1 of the theorem ensures that $\lambda'_t \leq \lambda_t^Q$, such that $\lambda''_t \geq 0$, and second part shows that $\lambda'_t \leq \lambda_t^{\widetilde{Q}}$. Let γ' and γ'' be independent random Poisson measures with compensators $\nu'(dt,dy) = \lambda'_t G_t^Q(dy)dt$ and $\nu''(dt,dy) = \lambda''_t G_t^Q(dy)dt$, respectively. Then it follows that under Q:

$$\int_0^T \int_{(0,\infty)} y(\gamma - \nu^Q) \stackrel{\mathcal{D}}{=} \int_0^T \int_{(0,\infty)} y(\gamma' - \nu') + \int_0^T \int_{(0,\infty)} y(\gamma'' - \nu'').$$

Here, the compensated jump parts have mean zero. In particular, Jensen's inequality implies that the last term is larger in the convex order than 0 (its mean); consequently, this term can be ignored. We will derive an ordering result for the first jump part and (5.6). Since by construction,

$$\int_0^T \int_{(0,\infty)} y\nu'(dt,dy) = \lambda'_t \mu_{1,t}^Q = \lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}} = \int_0^T \int_{(0,\infty)} y\nu^{\widetilde{Q}}(dt,dy),$$

for all t, we can focus on the jumps only.

Next, we identify γ' with another random Poisson measure γ''' with compensator

$$\nu'''(dt,dy) = \lambda_t^{\widetilde{Q}} dt \left(\frac{\lambda'_t}{\lambda_t^{\widetilde{Q}}} G_t^Q(dy) + \left(1 - \frac{\lambda'_t}{\lambda_t^{\widetilde{Q}}}\right) \varepsilon_0(dy) \right) =: \lambda_t^{\widetilde{Q}} dt G_t'''(dy).$$

Thus, we can increase the intensity of claims from λ'_t to $\lambda_t^{\widetilde{Q}}$ and replace the distribution G_t^Q by a mixture of this distribution and the Dirac measure at 0. This change does not affect the distribution of the total claim amount, so that we get

$$\int_0^T \int_{(0,\infty)} y\gamma'(dt,dy) \stackrel{\mathcal{D}}{=} \int_0^T \int_{(0,\infty)} y\gamma'''(dt,dy) \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N'_T} Y'_i, \tag{5.8}$$

where the last term is a standard compound Poisson variable with parameters $\widetilde{\Lambda}_T$ and

$$G^*(dy) = \frac{\int_0^T \lambda_t^{\widetilde{Q}} G_t'''(dy)dt}{\int_0^T \lambda_t^{\widetilde{Q}} dt}.$$

The second equality in distribution in (5.8) is again a consequence of Norberg (1993, Corollary 1). Condition 2 of the theorem again ensures that we can apply the sufficient condition for the cut criterion on the distribution functions $G_t^{\widetilde{Q}}$ and G_t''' for each t. To see this, note that the difference between the densities for G_t''' and $G_t^{\widetilde{Q}}$ with respect to the convolution of G and the Dirac measure at 0 is

$$\begin{aligned} g_t'''(y) - g_t^{\widetilde{Q}}(y) &= \frac{\lambda'_t}{\lambda^{\widetilde{Q}_t}} \frac{\lambda_t}{\lambda_t^Q}(1 + \phi_t^Q(y)) - \frac{\lambda_t}{\lambda_t^{\widetilde{Q}}}(1 + \phi_t^{\widetilde{Q}}(y)) \\ &= \frac{\lambda_t}{\lambda_t^{\widetilde{Q}} \lambda_t^Q \mu_t^Q} \left(\lambda_t^{\widetilde{Q}} \mu_{1,t}^{\widetilde{Q}} (1 + \phi_t^Q(y)) - \lambda_t^Q \mu_{1,t}^Q (1 + \phi_t^{\widetilde{Q}}(y)) \right), \end{aligned}$$

for $y \in (0,\infty)$, where we have used the definition of λ'_t in the second equality. Similarly,

$$g_t'''(0) - g_t^{\widetilde{Q}}(0) = 1 - \frac{\lambda'_t}{\lambda_t^{\widetilde{Q}}} = 1 - \frac{\mu_{1,t}^{\widetilde{Q}}}{\mu_{1,t}^Q},$$

The second condition in the theorem now implies that there exist $0 \le y_t^1 \le y_t^2 < \infty$, such that

$$\mathrm{sign}(g_t'''(y) - g_t^{\widetilde{Q}}(y)) = \begin{cases} +, & y \in [0, y_t^1), \\ -, & y \in (y_t^1, y_t^2), \\ +, & y \in (y_t^2, \infty). \end{cases}$$

The cut criterion now gives that $G_t^{\widetilde{Q}} \preceq_c G_t'''$ for each t, and the closedness of the convex order under mixtures implies that $\widetilde{G}^* \preceq_c G^*$. Finally, the result in the theorem follows by using that the convex order is closed under convolution and random summation. □

Note

An updated version of this paper will be available from my homepage at the University of Copenhagen, www.math.ku.dk/˜tmoller.

References

Ansel, J. P. and Stricker, C. (1992). Lois de martingale, densités et décomposition de Föllmer-Schweizer, *Annales de l'Institut Henri Poincaré* **28**, 375–392.

Chan, T. (1999). Pricing Contingent Claims on Stocks Driven by Lévy Processes, *Annals of Applied Probability* **9**, 504–528.

de Waegenaere, A. and Delbaen, F. (1992). A Dynamic Reinsurance Theory, *Insurance: Mathematics and Economics* **14**, 205–218.

Delbaen, F. and Haezendonck, J. (1989). A Martingale Approach to Premium Calculation Principles in an Arbitrage Free Market, *Insurance: Mathematics and Economics* **8**, 269–277.

Embrechts, P. (2000). Actuarial versus Financial Pricing of Insurance, *Journal of Risk Finance* **1**(4), 17–26.

Fink, A. M. and Jodeit, Jr., M. (1984). On Chebyshev's other inequality, *Inequalities in Statistics and Probability (Lincoln, Neb., 1982)*, Vol. 5 of *IMS Lecture Notes Monogr. Ser.*, Inst. Math. Statist., Hayward, CA, , 115–120.

Föllmer, H. and Schweizer, M. (1990). Hedging of Contingent Claims under Incomplete Information, *in* M. H. A. Davis and R. J. Elliott (eds), *Applied Stochastic Analysis*, Stochastic Monographs, Vol. 5, Gordon and Breach, 389–414.

Goovaerts, M. J., De Vylder, F. and Haezendonck, J. (1984). *Insurance Premiums: Theory and Applications*, North-Holland.

Grandits, P. and Rheinländer, T. (2002). On the Minimal Entropy Martingale Measure, *Annals of Probability* **30**, 1003–1038.

Henderson, V. and Hobson, D. (2002). Coupling and Option Price Comparisons in a Jump-Diffusion Model, *To appear in Stochastics and Stochastics Reports.*

Jacod, J. and Shiryaev, A. N. (1987). *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin.

Kaas, R., van Heerwanden, A. E. and Goovaerts, M. J. (1994). *Ordering of Actuarial Risks*, CAIRE Education Series 1, Brussels.

Mitrinović, D. S. and Vasić, P. M. (1974). History, variations and generalisations of the Čebyšev inequality and the question of some priorities, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461–497*, 1–30.

Møller, T. (2002). On Valuation and Risk Management at the Interface of Insurance and Finance, *British Actuarial Journal* **8**(4), 787–828.

Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*, Wiley, Chichester.

Norberg, R. (1993). Prediction of Outstanding Liabilities in Non-Life Insurance, *ASTIN Bulletin* **23**(1), 95–115.

Panjer, H. (1981). Recursive Evaluation of a Family of Compound Distributions, *ASTIN Bulletin* **11**, 22–26.

Schweizer, M. (1995). On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition, *Stochastic Analysis and Applications* **13**, 573–599.

Schweizer, M. (2001). A Guided Tour through Quadratic Hedging Approaches, *in* E. Jouini, J. Cvitanić and M. Musiela (eds), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, 538–574.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and their Applications*, Probability and Mathematical Statistics, Academic Press Inc., Boston, MA.

Sondermann, D. (1991). Reinsurance in Arbitrage-free Markets, *Insurance: Mathematics and Economics* **10**, 191–202.