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Exposure Rating in Liability Reinsurance

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Abstract:

The well-known inflation-independent exposure rating curves from Property reinsurance (see e.g. [4] or [2]) cannot be deduced in Liability insurance in the same way because here the claims sizes cannot be assumed to be scaled by the sums insured. Instead, German insurance and reinsurance companies apply a specific system of increased limits factors introduced already in 1936 by the pioneer of German non-life insurance mathematics, Paul Riebesell.

In the paper, Riebesell's system is analysed in the light of the Collective Model of Risk Theory. It is shown that Riebesell's system is consistent with the Collective Model only above some threshold $u > 0$ and under the assumption that there the claims sizes have a Pareto distribution $F(x) = 1 - (x/c)^{-\alpha}$ with parameters $c < u$ and $\alpha < 1$. Furthermore, evidence is given that the admissible range of values for α and u is reasonable for practical applications. Thus, Riebesell's system provides an easy and consistent way of exposure rating which does not have to be adjusted for inflation or changes of currency.

1. Introduction

Under an excess-of-loss (XL) reinsurance treaty, the reinsurer pays the amount $\min(\max(X-a, 0), h)$ of each original claim amount X against the direct insurer, i.e. he pays the excess over the retention a, but limited to a maximum amount h. There are two main rating methods for XL treaties, the experience rating method and the exposure rating method. The experience rating method uses the (as if) claims experience of the XL treaty, the exposure rating method uses the risk profile of the portfolio covered, i.e. the sums insured and the original premiums of the individual policies of the direct insurer's portfolio. The essential idea of the latter method is, that a policy with a sum insured $v \leq a$ cannot produce a claim amount $X > a$ and therefore cannot lead to a payment by the XL reinsurer. Therefore, only policies with sums insured > a have to be considered for XL rating purposes. The basic question of the exposure rating approach is: Given a policy with sum insured $v > a$, how should its net premium be allocated to the retained part of the claims and to the reinsured part in excess of a in a fair way?

In this paper, we first shortly review the standard rating method of using exposure curves in Property insurance and show that these curves have the nice property of being invariant against inflation and changes of currency. Unfortunately, it does not seem reasonable to apply these curves in Liability insurance, too. Instead, in Germany a specific system of increased limits factors from direct insurance is often applied in practise. This system consists of applying a constant percentage of premium increase for doubling the limit of indemnity (= sum insured). But it will be shown that this system cannot hold over the full range of all possible sums insured if we accept the Collective Model of Risk Theory as basis. In the central part of the paper it is shown that for sums insured above a threshold $u > 0$ such a system of increased limits factors is in line with the Collective Model and yields exposure curves which are again invariant against inflation and changes of currency. Moreover, it turns out, that the threshold u is below the usual range of XL retentions a, i.e. the system of increased limits factors can be used for XL exposure rating purposes. Above this threshold u, the underlying claims amount distribution is shown to be a Pareto distribution with shape parameter $\alpha < 1$. This seems to contradict to the fact that the observed claims amount distributions in Liability insurance often fit to a Pareto distribution with parameter $\alpha > 1$. But in the final section it is shown that the distribution of the sums insured makes up for the difference.

2. Some Basics

We consider an individual risk i of an insurance portfolio consisting of I risks. Using the Collective Model, risk i has total claims amount (in a fixed period, e.g. one year)

$$S_i = \sum_{n=1}^{N_i} X_{in}$$

with number N_i of claims, independent claims sizes X_{i1}, X_{i2}, … with distribution function

$$F_i(x) = P(X_{in} \le x) .$$

An unlimited excess-of-loss cover with retention a pays

$$R = \sum_{i=1}^{I} \sum_{n=1}^{N_i} \max(X_{in} - a, 0) .$$

Therefore, the price for this cover will be based on (using X_i for any X_{in})

$$\begin{aligned} E(R) &= \sum_{i=1}^{I} E(N_i) E\big(\max(X_i - a, 0)\big) \\ &= \sum_{i=1}^{I} \frac{E(N_i) E\big(\max(X_i - a, 0)\big)}{E(N_i) E(X_i)} E(S_i) \\ &= \sum_{i=1}^{I} \big(1 - r_i(a)\big) E(S_i) \end{aligned}$$

with the so-called loss elimination ratio function $\big($cf. [1], [3], [4], [6], [7]$\big)$

$$r_i(a) := \frac{E\big(\min(X_i, a)\big)}{E(X_i)} = \frac{\int_0^a \big(1 - F_i(x)\big) dx}{\int_0^\infty \big(1 - F_i(x)\big) dx} .$$

The last equality can be shown by partial integration.
Note also that the identity $X_i = \min(X_i, a) + \max(X_i - a, 0)$ has been used.

Usually, the reinsurer estimates $E(S_i)$ from the original premium amount of risk i (as charged by the direct insurer) by deducting the original loading percentage and – if necessary – by correcting for any known bias. Then, the reinsurer needs “only” to know the loss elimination function r_i or the distribution function F_i for every single risk of the portfolio. This seems to be hopeless because risks from different risk classes or with different sums insured will have different distribution functions F_i. But in Property insurance, the problem can be simplified considerably.

3. Exposure Rating in Property Insurance

In Property insurance, a plausible additional assumption makes things much easier. Here it is very common to assume that the risks of the same risk group have the same loss degree variable, i.e. that the distribution G of the loss degree $Y_i = X_i/v_i$, where v_i denotes the sum insured (= total value), does not depend on i for risks of the same risk group. Then we have

$$r_i(a) = \frac{E\left(\min(X_i, a)\right)}{E(X_i)} = \frac{E\left(\min\left(Y_i, \frac{a}{v_i}\right)\right)}{E(Y_i)} = r\left(\frac{a}{v_i}\right)$$

with

$$r(w) := \frac{\int_0^w \left(1-G(y)\right)dy}{\int_0^1 \left(1-G(y)\right)dy} \qquad \text{for } 0 \le w \le 1, \text{ and } r(w) := 1 \text{ for } w \ge 1,$$

where $G(w) = P(Y_i \le w) = P\left(\frac{X_i}{v_i} \le w\right)$ is the distribution of the loss degrees.

On the unit interval, r is strictly increasing and concave. It is important to note that Y_i, G and r are not influenced by currency changes or inflation (as long as the sums insured are adjusted for).

Usually, the reinsurer has several r-curves, each corresponding to a particular risk group. Once the reinsurer has analysed some claims statistics in order to obtain these r-curves, he can use these for many years, see e.g. [2], [4].

4. Increased Limits Factors in Liability Insurance and Riebesell's Rule

In General Liability insurance, the sum insured is not identical to the maximum amount of the possible third party claims (as often such an amount does not exist) but is just a limit of indemnity chosen by the insured up to which he wants to be reimbursed by the insurer in case of any third party claims. Therefore, the assumption that the distribution of X_i/v_i does not depend on i, is not realistic in Liability insurance because risks with identical claims exposure may have greatly varying sums insured. Instead, the assumption "$X_i = \min(X, v_i)$" seems reasonable for the insured claims amounts X_i of risks i = 1, 2, … of the same risk group where X stands for the original third party claim amount. Under this assumption, we have

$$r_i(a) = \begin{cases} \dfrac{E\big(\min(X,a)\big)}{E\big(\min(X,v_i)\big)} = \dfrac{\tilde{r}(a)}{\tilde{r}(v_i)}, & 0 \le a \le v_i \,, \\ 1\,, & a \ge v_i \end{cases}$$

with the loss elimination ratio function $\tilde{r}(a) = E\big(\min(X, a)\big)/E(X)$ of X. But here both X and $\tilde{r}$ are fully affected by inflation and currency changes and therefore nothing is gained by switching from r_i to $\tilde{r}$.

In this situation, German reinsurers use a method stemming from direct insurance where it is used for the pricing of increased sums insured. This method does not have to be changed under inflationary conditions. It has already been described in 1936 in the book "Einführung in die Sachversicherungsmathematik" ("Introduction to General Insurance Mathematics") by the German actuary Paul Riebesell [5]. It works as follows: Let $b_0 = b(v_0)$ be the net premium for the "standard" sum insured v_0 for a risk facing the claims variable X. If this risk wants to double its sum insured, the insurer will increase the net premium to $b(2v_0) = b_0(1+z)$ with e.g. z = 20%. Because here the sum insured acts only as limit of indemnity and not as a measure of the size of the risk like in Property insurance, the premium increases less than the sum insured. If the new sum insured is $4v_0$, the premium will be $b(4v_0) = b_0(1+z)^2$. Thus the general rule is $b(2^k v_0) = b_0(1+z)^k$ or $b(tv_0) = b_0(1+z)^{ld(t)}$ with the binary logarithm ld. Applying this rule for t < 1, too, we finally have Riebesell's formula

$$b(v) = b_0(1+z)^{ld(v/v_0)} = b_0\left(\frac{v}{v_0}\right)^{ld(1+z)} \qquad (1)$$

with $z \in (0; 1)$ for the net premium b(v) at any sum insured v > 0.

If Riebesell's formula (1) constitutes an admissible rule for the net premium of a Liability cover with sum insured v facing claim X, then – according to the Collective Model – the equation

$$b(v) = E(N)E\big(\min(X,v)\big) \qquad (2)$$

must hold for all relevant sums insured v where N denotes the number of claims per period.

(Note that the number N of claims does not depend on the size of the sum insured v.) Then, if risk i with sum insured v_i of the portfolio to be reinsured belongs to the risks facing the claims variable X, its loss elimination ratio function for $a \le v_i$ becomes

$$r_i(a) \;=\; \frac{E\big(\min(X,a)\big)}{E\big(\min(X,v_i)\big)} \;=\; \frac{b(a)}{b(v_i)} \;=\; \left(\frac{a}{v_i}\right)^{ld(1+z)} . \qquad (3)$$

The important aspect of this formula for r_i is the fact that it does not depend on risk i except via the sum insured v_i and it is not affected by currency changes or inflation as was also the case with the corresponding formula $r_i(a) = r(a/v_i)$ in Property insurance, of which (3) would be a special case if there existed an appropriate loss degree distribution function.

5. Can Riebesell's Rule be Explained by the Collective Model?

We are still left with the question whether there are conditions where the Collective Model premium

$$b(v) = E(N)E\big(\min(X, v)\big)$$

leads to Riebesell's rule (1). Denoting with F the distribution function of X, we can deduce the necessary conditions on F from (1) and (2): Under the Collective Model, we have

$$b(v) \;=\; E(N)\int_0^v \big(1 - F(x)\big)dx$$

and therefore

$$b'(v) \;=\; E(N)\big(1 - F(v)\big)$$

or

$$F(v) \;=\; 1 - \frac{b'(v)}{E(N)}$$

for all v > 0. If Riebesell's formula (1) holds for all v > 0, we have

$$b'(v) \;=\; ld(1+z)\,\frac{b_0}{v_0}\cdot\left(\frac{v}{v_0}\right)^{ld(1+z)-1} .$$

Taken together, we have for all v > 0 as the implied claims size distribution function

$$F(v) \;=\; 1 \;-\; \frac{ld(1+z)}{E(N)}\cdot\frac{b_0}{v_0}\left(\frac{v}{v_0}\right)^{ld(1+z)-1}$$

$$\;=\; 1 - \left(\frac{v}{v_1}\right)^{-\alpha}$$

with $\alpha := 1 - \mathrm{ld}(1+z)$,

$$v_1 := v_0 \left(\frac{s_0}{v_0} \cdot (1-\alpha) \right)^{1/\alpha} > 0 ,$$

$$s_0 := b_0/E(N) = E\big(\min(X, v_0)\big) .$$

If this implied function F is a realistic distribution function for third party Liability claims, then (1) constitutes a reasonable premium rule and (3) a reasonable loss elimination function.

First, we remark that $\alpha \in (0; 1)$ because $z \in (0; 1)$. Furthermore, v_1 is much smaller than v_0 because already $s_0 \le v_0$ and the additional influence of α makes v_1 even smaller. Finally, we see that F is precisely a Pareto distribution for $v \ge v_1$, but is negative, i.e. not a distribution (and therefore not admissible) for $v < v_1$. Thus, it seems that we should be able to show that (1) is an admissible premium rule at least for sums insured $v > v_1$. Therefore we consider the claims size variable X* with Pareto distribution

$$F^*(x) = \begin{cases} 1 - \left(\dfrac{x}{v_1} \right)^{-\alpha} & \text{for } x > v_1 , \\ 0 & \text{for } x \le v_1 . \end{cases}$$

But now comes a surprise: For $v > v_1$ we obtain

$$\begin{aligned} E\big(\min(X^*, v)\big) &= \int_0^v \big(1 - F^*(x)\big)dx \\ &= \int_0^{v_1} dx + \int_{v_1}^{v} \left(\frac{x}{v_1} \right)^{-\alpha} dx \\ &= v_1 + \left[\frac{v_1}{1-\alpha} \left(\frac{x}{v_1} \right)^{1-\alpha} \right]_{v_1}^{v} \\ &= v_1 + \frac{v_1}{1-\alpha} \left(\frac{v}{v_1} \right)^{1-\alpha} - \frac{v_1}{1-\alpha} \end{aligned}$$

which – because of $v_1 \neq v_1/(1-\alpha)$ – does not yield the desired form $c \cdot v^{1-\alpha}$ as in (1) for $E(N)E\big(\min(X^*, v)\big)$. Thus, we have to look for a different approach.

6. There are Distribution Functions Leading to Riebesell's Rule.

The decisive idea is to replace the Pareto distribution below some u > v_1 with something more appropriate, i.e. to let

$$\tilde{F}(x) = \begin{cases} 1-\left(\frac{x}{v_1}\right)^{-\alpha} & \text{for } x \geq u > v_1 \, , \\ \text{to be determined} & \text{for } x < u \, . \end{cases}$$

For the corresponding claims amount variable $\tilde{X}$, we have for v > u

$$E\left(\min(\tilde{X}, v)\right) = \int_0^u \left(1-\tilde{F}(x)\right) dx + \int_u^v \left(\frac{x}{v_1}\right)^{-\alpha} dx$$

$$= A(u) + \frac{v_1}{1-\alpha}\left(\frac{v}{v_1}\right)^{1-\alpha} - \frac{v_1}{1-\alpha}\left(\frac{u}{v_1}\right)^{1-\alpha}$$

with $A(u) := \int_0^u \left(1-\tilde{F}(x)\right) dx$.

Therefore, in order to determine $\tilde{F}(x)$ for x < u such that, for v > u, the premium rule (1) holds, we must have

$$A(u) \overset{!}{=} \frac{v_1}{1-\alpha}\left(\frac{u}{v_1}\right)^{1-\alpha} \, . \qquad (4)$$

For x $\in$[0; u], we have $(u/v_1)^{-\alpha} \leq 1-\tilde{F}(x) \leq 1$ and therefore A(u) can take on all values of the interval $[u(u/v_1)^{-\alpha}; u]$ by means of an appropriate choice of $\tilde{F}$. Thus, we can meet condition (4) if and only if

$$u\left(\frac{u}{v_1}\right)^{-\alpha} \leq \frac{v_1}{1-\alpha}\left(\frac{u}{v_1}\right)^{1-\alpha} \leq u \, .$$

Whereas the L.H.S. is always fulfilled because $\alpha \in (0; 1)$, the R.H.S. is equivalent to $u \geq v_1/(1-\alpha)^{1/\alpha}$. Note that the resulting lower bound $v_1/(1-\alpha)^{1/\alpha} = v_0(s_0/v_0)^{1/\alpha}$ for u is not greater than v_0 because $s_0 \leq v_0$. Thus we have shown the following:

Theorem 1: For any parameters v_0, b_0, z of Riebesell's rule (1) we can construct a claims size distribution $\tilde{F}$ and a threshold u, $0 < u \leq v_0$, such that (1) holds under the Collective Model for all sums insured $v \geq u$.

More precisely, we can choose any $u \in [v_0(s_0/v_0)^{1/\alpha}; v_0]$ where $\alpha = 1 - \mathrm{ld}(1+z)$ and $s_0 \le v_0$ is the mean claims amount corresponding to the net premium b_0. (If s_0 is not given, choose any $s_0 \in (0; v_0]$.) Now we define

$$\tilde{F}(x) = \begin{cases} 1 - \frac{1}{1-\alpha}\left(\frac{u}{v_1}\right)^{-\alpha} & \text{for } 0 \le x < u \\ 1 - \left(\frac{x}{v_1}\right)^{-\alpha} & \text{for } x \ge u \end{cases}$$

with $v_1 = v_0\big((1-\alpha)\, s_0/v_0\big)^{1/\alpha}$. For this particular choice of $\tilde{F}$, which is constant below u, we easily see that (4) holds as

$$A(u) = \int_0^u \big(1 - \tilde{F}(x)\big)dx = \frac{u}{1-\alpha}\left(\frac{u}{v_1}\right)^{-\alpha} = \frac{v_1}{1-\alpha}\left(\frac{u}{v_1}\right)^{1-\alpha}.$$

Note that $\tilde{F}$ is well-defined, i.e.

$$1 - \frac{1}{1-\alpha}\left(\frac{u}{v_1}\right)^{-\alpha} \ge 0$$

because this is the same as

$$u \ge v_1/(1-\alpha)^{1/\alpha} = v_0(s_0/v_0)^{1/\alpha}.$$

Finally, we can verify that $\int_0^{v_0} \big(1 - \tilde{F}(x)\big)dx = s_0$ holds.

Of course, such a partially constant claims size distribution is not realistic. We could easily replace $\tilde{F}$ with something which is more plausible below u, but instead we start in the next section with any given claims size distribution and make an appropriate adjustment above u.

7. Realistic Distributions Underlying Riebesell's Rule

Now we start with any realistic claims size distribution F and try to replace F above some point $\big(u; F(u)\big)$ with a Pareto tail such that the formula

$$\int_0^v \big(1 - F(x)\big)dx = c \cdot v^{1-\alpha}$$

holds for all $v \ge u > 0$ with some constant c and for any given $\alpha \in (0; 1)$ stemming from $\alpha = 1 - \mathrm{ld}(1+z)$. This means that we try

$$F_u(x) = \begin{cases} F(x) & \text{for } x \le u\,, \\ F(u) + \big(1 - F(u)\big)\left(1 - \left(\frac{x}{u}\right)^{-\alpha}\right) & \text{for } x \ge u \end{cases}$$

where the conditional distribution of the excess above u is Pareto.

For $v \ge u$ we have

$$\int_0^v \big(1 - F_u(x)\big)\,dx = \int_0^u \big(1 - F(x)\big)\,dx + \int_u^v \big(1 - F(u)\big)\left(\frac{x}{u}\right)^{-\alpha} dx$$

$$= \int_0^u \big(1 - F(x)\big)\,dx + \big(1 - F(u)\big)\frac{u}{1-\alpha}\left(\frac{v}{u}\right)^{1-\alpha} - \big(1 - F(u)\big)\frac{u}{1-\alpha}$$

and we see that we have to look for a threshold u satisfying

$$\int_0^u \big(1 - F(x)\big)\,dx = \big(1 - F(u)\big)\frac{u}{1-\alpha}\ .$$

We thus have to determine u in such a way that for $\alpha \in (0;\ 1)$ given we have

$$\alpha_F(u) = \alpha$$

where

$$\alpha_F(u) := 1 - \frac{u\big(1 - F(u)\big)}{\int_0^u \big(1 - F(x)\big)\,dx}\ .$$

First, we see that $\alpha_F(u) \in [0;\ 1]$ because

$$\int_0^u \big(1 - F(x)\big)\,dx \ \ge\ \int_0^u \big(1 - F(u)\big)\,dx \ =\ u\big(1 - F(u)\big)\ .$$

Second, from (using partial integration)

$$\int_0^u x\,dF(x) + u\big(1 - F(u)\big) \ =\ \int_0^u \big(1 - F(x)\big)\,dx \ \le\ \int_0^\infty \big(1 - F(x)\big)\,dx \ =\ E(X) \ =\ \int_0^\infty x\,dF(x)$$

we infer that $u\big(1-F(u)\big) \to 0$ for $u \to \infty$ if $E(X) < \infty$ which is the case for our F (all claims are limited). Thus we have $\alpha_F(u) \to 1$ for $u \to \infty$.

Third, from

$$\int_0^u \big(1 - F(x)\big)\,dx \ \le\ u$$

we infer the inequality

$$1 - F(u) \le \frac{u(1-F(u))}{\int_0^u (1-F(x))dx} \le 1$$

and therefore $\alpha_F(u) \to 0$ for $u \to 0$.

Taken together, we have shown that $\alpha_F(u)$ takes on any given value $\alpha \in (0; 1)$, if F is continuous with finite expectation. Table 1 gives some examples.

Table 1. Values of $\alpha_F(u)$ for some distributions F.

u/E(X)	Exponential	Gamma α = 0.1	Lognormal σ = 1.8	American Pareto α = 1.5
0.01	0.005	0.092	0.028	0.0147
0.1	0.049	0.151	0.185	0.127
1	0.418	0.294	0.5	0.545
3	0.842	0.442	0.663	0.740
10	0.9996	0.709	0.815	0.867
30		0.952	0.914	0.928
100		0.9999	0.972	0.962

Thus, we have shown the following:

Theorem 2: For any continuous claims amount distribution F with finite expectation and any $\alpha = 1 - ld(1+z) \in (0; 1)$ we can find a threshold u > 0 such that Riebesell's rule (1) holds in $[u; \infty)$ after adjusting F(x) by a Pareto tail $F_u(x)$ for $x \in (u; \infty)$.

For example, let us assume that we use Riebesell's formula with z = 20%, i.e. $\alpha = 1 - ld(1+z) = 1 - \ln(1+z)/\ln(2) = 0.737$, and that the original claims amount distribution is lognormal with $\sigma = 1.8$. Then, according to the Table 1, this means that we replace the lognormal above $u \approx 5 \cdot E(X)$ with a Pareto distribution. In Liability insurance, E(X) is usually below € 50,000, and the sums insured start at about € 500,000. This shows that a condition like $u \approx 5 \cdot E(X)$ gives enough room for the application of Riebesell's rule to the full range of existing sums insured.

For such a realistic claims amount distribution F_u, the loss elimination function formula (3) $r_i(a) = (a/v_i)^{1-\alpha}$ only holds for the values $v_i \ge a \ge u$. But this suffices for the purpose of exposure rating because the retention a is usually high enough to be larger than u. Moreover, we see from equation (2) that the behaviour of F_u above the area where we want to apply Riebesell's law does not matter at all, i.e. we can replace F_u above some very high amount $w >> u$ with something else, e.g. in order to ascertain a finite expectation.

8. Is the Size of α Realistic?

Whereas in view of the results of the Extreme Value Theory it seems very realistic to assume a Pareto-like tail for the claims amount distribution in Liability insurance above a certain threshold, it still seems questionable if the resulting interval (0; 1) for the Pareto parameter α is realistic. This would imply that the corresponding third party claims amount variable does not have a finite expected value. But this is not a real problem because the insurance cover is always finite. Moreover, as was remarked above, the distribution F_u can, above the maximum sum insured of the portfolio, be adjusted such that it has a finite expectation. On the other hand, the claims size distributions of real Liability portfolios often indicate an α-parameter in the neighbourhood of $\alpha = 2$ for the large claims area. This seems to contradict the α-values found above or could indicate that Riebesell's rule already contains a certain risk load.

But the claims size distribution G of a Liability portfolio is a mixture

$$G(x) = \int F(x \mid v)\, dH(v)$$

of the claims size distributions (as obtained in the previous section but now censored by the sum insured v)

$$F(x \mid v) = \begin{cases} \text{almost arbitrary}, & x < u, \\ 1 - \left(\dfrac{x}{v_1}\right)^{-\alpha}, & u \le x < v, \\ 1, & x \ge v \end{cases}$$

of the individual risks, mixed with the distribution H of the various sums insured where each sum insured is weighted with the expected value E(N) of the corresponding claims number. In practise, this mixing distribution can often be approximated by a truncated Pareto distribution

$$H(v) \;=\; \begin{cases} 0\ , & v < c\ , \\ \dfrac{c^{-\beta} - v^{-\beta}}{c^{-\beta} - d^{-\beta}}, & v \in [c;d)\ , \\ 1\ , & v \ge d \end{cases}$$

where c and d are the lowest and highest sum insured, respectively. Under the realistic assumption c > u, we obtain for the mixed distribution for x > c

$$\begin{aligned} G(x) \;&=\; \int_0^\infty F(x\,|\,v)\,dH(v) \\ &=\; \int_0^\infty \left\{ 1_{\{x \ge v\}} + 1_{\{x<v\}} \cdot \left(1 - \left(\frac{x}{v_1}\right)^{-\alpha} \right) \right\} dH(v) \\ &=\; \int_0^x dH(v) + \int_x^\infty \left(1 - \left(\frac{x}{v_1}\right)^{-\alpha} \right) dH(v) \\ &=\; H(x) + \left(1 - \left(\frac{x}{v_1}\right)^{-\alpha} \right) (1 - H(x)) \\ &=\; 1 \;-\; \left(\frac{x}{v_1}\right)^{-\alpha} (1 - H(x))\ . \end{aligned}$$

Now, if d >> x, then $H(x) \approx 1 - \left(\frac{x}{c}\right)^{-\beta}$ and

$$G(x) \;\approx\; 1 \;-\; \left(\frac{x}{v_1}\right)^{-\alpha} \left(\frac{x}{c}\right)^{-\beta} \;=\; 1 - \left(\frac{x}{v_2}\right)^{-(\alpha+\beta)}\ .$$

Thus, if e.g. z = 20% ⇒ $\alpha = 1 - \mathrm{ld}(1.20) = 0.737$, and if the distribution H of the sums insured has $\beta = 1.3$, the mixed distribution G is approximately Pareto with parameter $\alpha+\beta = 2.037$ in the large claims area.

Similarly, in Property insurance, the analysis of empirical loss degrees for Industrial Fire risks in [4] has suggested a Pareto parameter $\alpha = 0.65$ whereas the loss amounts of Fire portfolios often have a Pareto parameter of about 1.5.

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