Some Applications of Phase–Type Distributions to Insurance and Finance

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ASTIN XXXIV

Berlin, August 26, 2003

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Phase–Type Distributions

(Erlang, 1917, Jensen, 1953, Neuts 1975–)

- Comprises many standard distributions
  - Convolutions of exponentials;
    Gamma's with integer parameter ;
    (Erlang distrn's)
  - Mixtures of exponentials (hyperexponentials)
  - All series/parallel/loop combinations;
    of exponentials
- Makes many calculations explicit;
  (or algorithmically tractable);
  "right generalization of exp distr'n"
- Are dense:

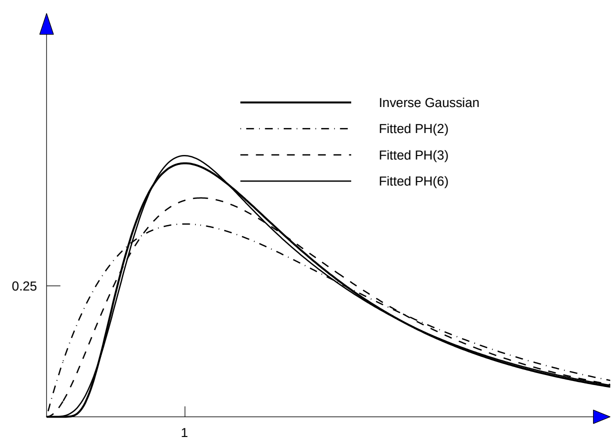
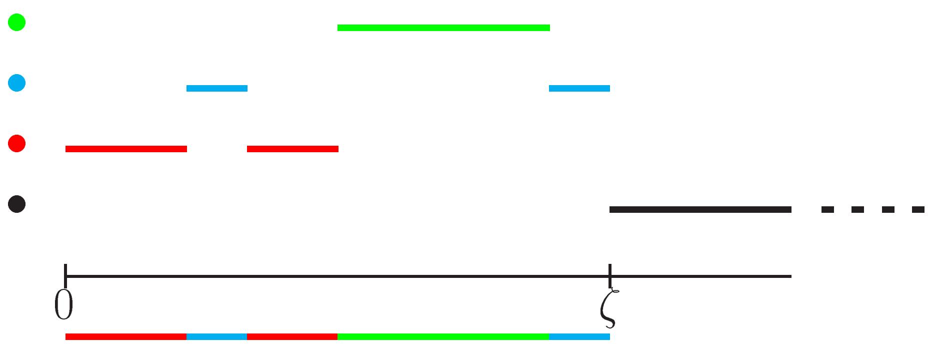


Figure 1: PH approximations to inverse Gaussian distribution

Def'n of PH Distributions

$\{i, j, k, \ldots\} = \{\bullet, \bullet, \bullet, \ldots\}$ Markov states
one $\Delta = \bullet$ absorbing



ζ has PH$(\boldsymbol{\alpha}, \boldsymbol{T})$ distr'n where

$\alpha_i = \alpha_{\bullet} = \mathbb{P}(J_0 = \bullet) = \mathbb{P}(J_0 = i)$ initial prob's

$\boldsymbol{T} = (t_{ij})$ matrix given by

the $t_{ij} = t_{\bullet\bullet} =$ int's jump $\bullet \to \bullet$

the $t_i = t_{\bullet} =$ int's jump $\bullet \to \Delta$ (absorbtion)

$$t_{ii} = -\sum_{j \neq i} t_{ij} - t_i$$

CDF (distr'n fct) $F(x) = \mathbb{P}(\zeta \leq x)$ where

$$1 - F(x) = \mathbb{P}(\zeta > x) = \mathbb{P}(J_x \neq \Delta) = \boldsymbol{\alpha} \mathrm{e}^{\boldsymbol{T}x} \mathbf{1},$$

$$\mathrm{e}^{\boldsymbol{T}x} = \sum_{n=0}^{\infty} \frac{\boldsymbol{T}^n x^n}{n!}.$$

Density $\boldsymbol{\alpha} \mathrm{e}^{\boldsymbol{T}x} \boldsymbol{t}$

Mean $-\boldsymbol{\alpha} \boldsymbol{T}^{-1} \boldsymbol{t}$, 2nd moment $\boldsymbol{\alpha} \boldsymbol{T}^{-2} \boldsymbol{t}$, . . .

Example: Erlang(n)

n exponentials in series (same int'y λ)

= convolution of n exponentials = Gamma(n, λ)

density $\dfrac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}$

$n = 3$

$$\boldsymbol{T} = \begin{pmatrix} \cdot & \lambda & 0 \\ 0 & \cdot & \lambda \\ 0 & 0 & \cdot \end{pmatrix} \qquad \boldsymbol{t} = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

$$\boldsymbol{T} = \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

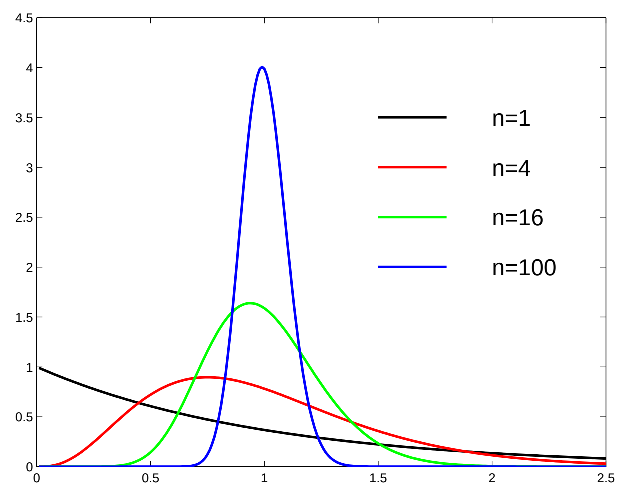
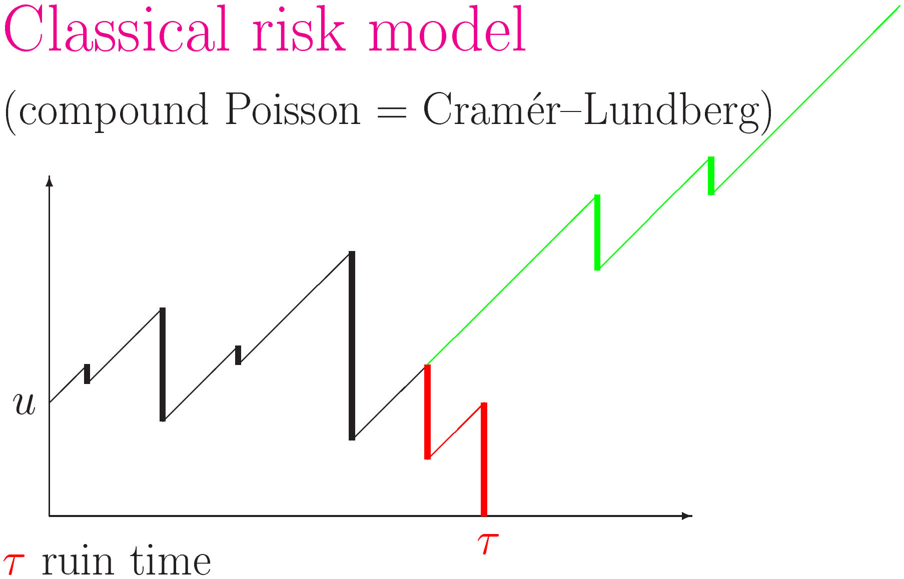


Figure 2: Erlang densities

Classical risk model

(compound Poisson = Cramér–Lundberg)



τ ruin time

$\psi(u) = \mathbb{P}(\tau < \infty)$ ruin prob. (inf. horizon)

Classical expression P–K–B–B

Pollaczek–Khintchine–Beekman–Bowers

β Poisson int'y, unit premium rate

B claim distr'n, $B_I(x) = \dfrac{1}{\mu}\displaystyle\int_0^x \overline{B}(y)\,dy$ integrated tail

$$\psi(u) \;=\; (1-\rho)\sum_{n=0}^{\infty} \rho^n \overline{B}_I^{*n}(u) \qquad \rho = \beta\mu_B < 1$$

Phase–type reduction: B PHT$(\boldsymbol{\alpha}, \boldsymbol{T})$

$\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{1}, \quad \boldsymbol{\alpha}_+ = -\beta\boldsymbol{\alpha}\boldsymbol{T}^{-1}, \boldsymbol{Q} = \boldsymbol{T} + \boldsymbol{t}\boldsymbol{\alpha}_+$

$$\psi(u) \;=\; \boldsymbol{\alpha}_+ \mathrm{e}^{\boldsymbol{Q}u}\boldsymbol{1}$$

Neuts 1981 (M/G/1 queue)

Many special cases in risk literature

SA-Rolski *IME* 1991

Generalizations to many different models

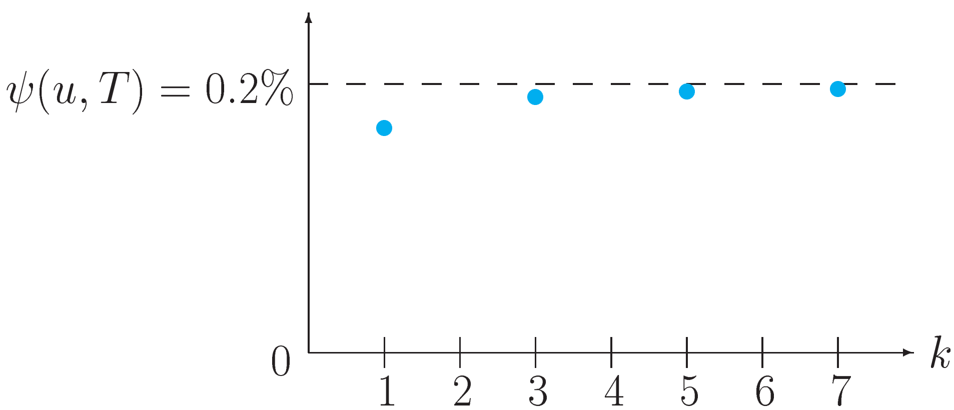
Finite Horizon Problem

SA–Avram–Usabel, *Astin Bull.* 2003

$\psi(u, T) = \mathbb{P}(\tau \leq T)$
No reduction in PHT case
Idea: replace T by PHT r.v. H
Now nice solution
Use denseness: $H = H_k$ where $H_k \to T$, $k \to \infty$

$$\psi(u, T) \;\approx\; \mathbb{E}\psi(u, H_k)$$

Best choice Erlang(k) with mean T
$\dfrac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x}$, $\lambda = \lambda_k = k/T$



Description of Algorithm

1. $n = 1$

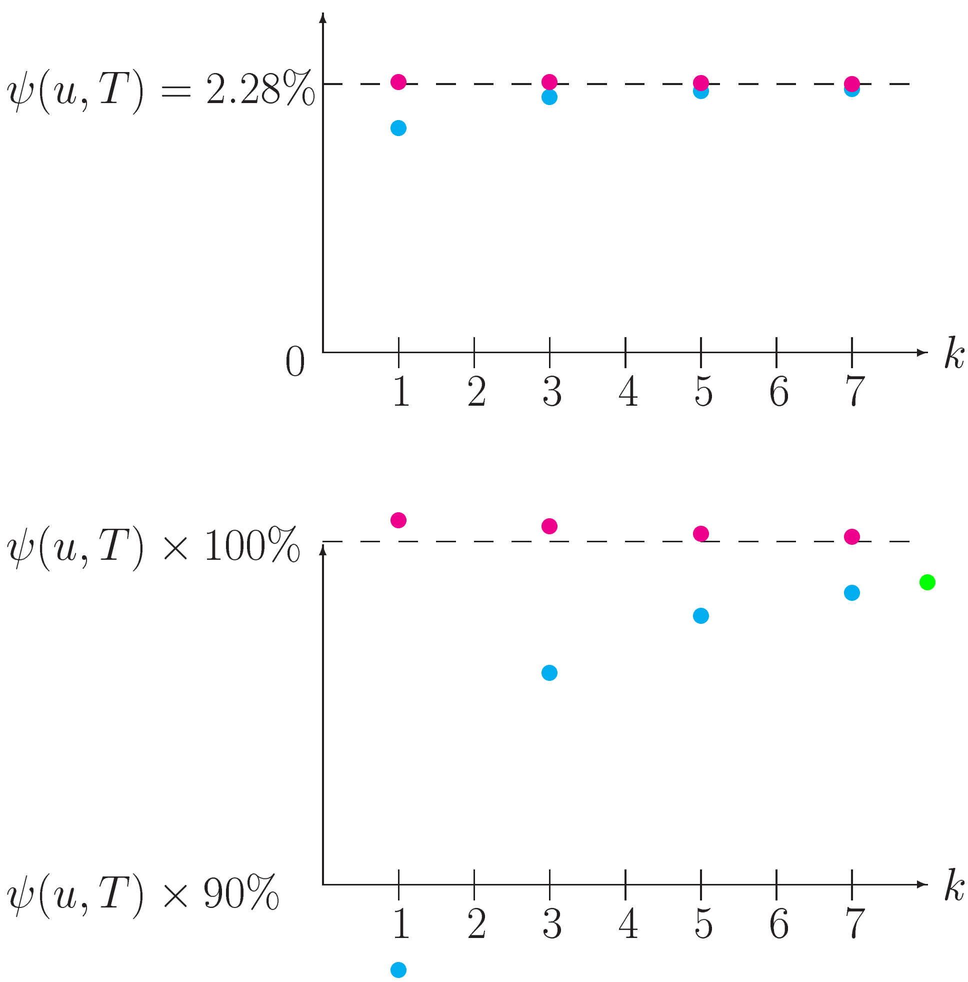
$$\begin{aligned} \mathbb{E}\psi(u,H) &= \int_0^\infty \psi(u,t) f_H(t)\,\mathrm{d}t, \\ E\psi(u,H_1) &= \int_0^\infty \psi(u,t)\lambda \mathrm{e}^{-\lambda t}\,\mathrm{d}t \\ &= \lambda \times \text{Lapl.transf.} \\ &= \lambda \times \boldsymbol{\alpha}_\lambda \mathrm{e}^{\boldsymbol{Q}_\lambda u}\mathbf{1} \end{aligned}$$

$\boldsymbol{\alpha}_\lambda = \beta\boldsymbol{\alpha}(s_\lambda \boldsymbol{I} - \boldsymbol{T}^{-1}, \quad \boldsymbol{Q}_\lambda = \ + \boldsymbol{t}\boldsymbol{\alpha}_\lambda$
s_λ root of $\beta\boldsymbol{\alpha}(-s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t} = \beta + \lambda - s$
(Lundberg equation)

2. $n \longrightarrow n+1$

$$\boldsymbol{\alpha}_\lambda^{(n+1)} = \Big(s_\lambda \boldsymbol{\alpha}_\lambda^{(n)} + \sum_{j=2}^{n} \boldsymbol{\alpha}_\lambda^{n+2-j} \boldsymbol{t}\boldsymbol{\alpha}_\lambda^{(j)}\Big)\big(s_\lambda - \boldsymbol{T} - \boldsymbol{t}\boldsymbol{\alpha}_\lambda^{(1)}\big)^{-1}$$

Improvement by Extrapolation



Richardson extrapolation:

Compute x accurately using sequence $x_k \to x$

Assume convergence rate known: $x - x_k = \frac{c}{k} + \frac{d}{k^{1+\epsilon}} + \cdots$

c is unknown but can be eliminated

Improved approximation $(k+1)x_{k+1} - k x_k$

Implementation:

$$x = \psi(u,T) \;\approx\; \mathbb{E}\psi(u,H_k) = x_k$$

$$\begin{aligned}
&\mathbb{E}\psi(u,H_k) \\
&= \mathbb{E}\Big[\psi(u,T) + \psi_T(u,T)(H_k - T) + \psi_{TT}(u,T)(H_k-T)^2/2 + \cdots\Big] \\
&= \psi(u,T) + 0 + \psi_{TT}(u,T)\mathbb{V}ar(H_k) + \cdots \\
&= \psi(u,T) + \frac{c}{k} + \cdots
\end{aligned}$$

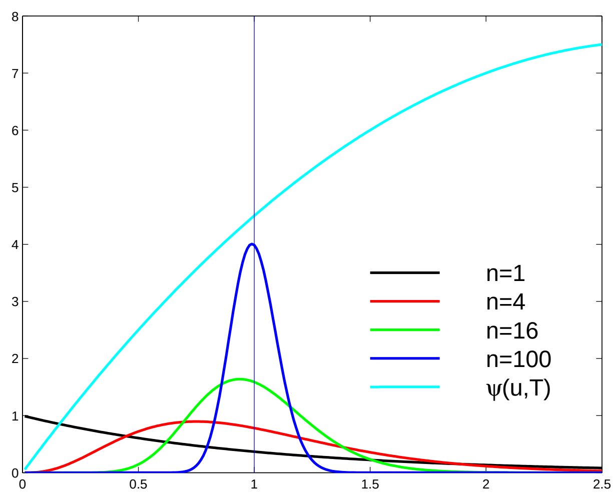
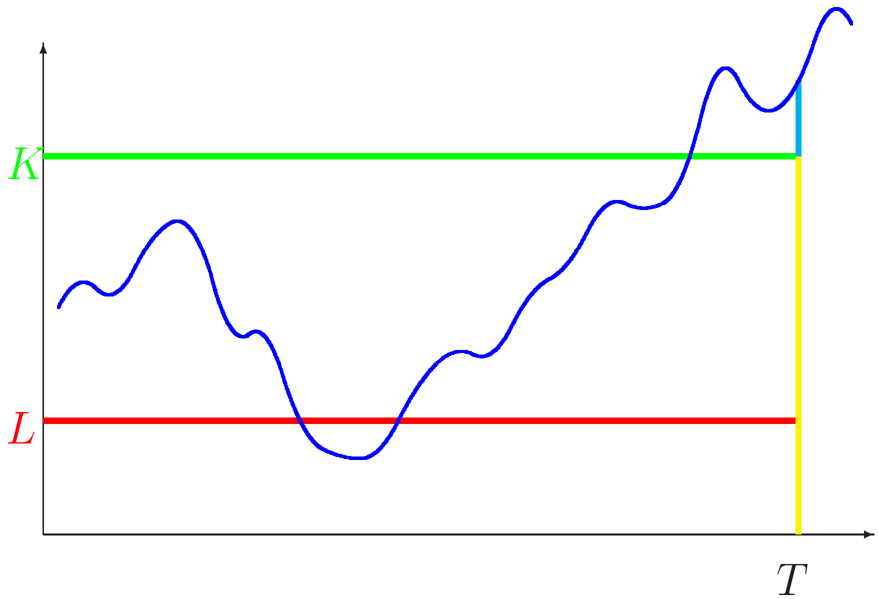


Figure 3: Erlang smoothing

Applications to Option Pricing

Barrier options



Pay–out $\boxed{e^{-aT}\left(S_T - K\right)^+ I\left(\inf_{t\leq T} S_t > L\right)}$

First passage problems occur also for other types of options

Perpetual American Put

Pay–out $\boxed{e^{-a\tau}(K - S_\tau)^+}$

τ stopping time at your disposal

Price $\mathbb{E}^* \inf_{\tau} e^{-a\tau}(K - S_\tau)^+$

$*$: risk–neutral measure

inf attained for $\tau^\# = \inf\{t : S_t \leq k^\#\}$

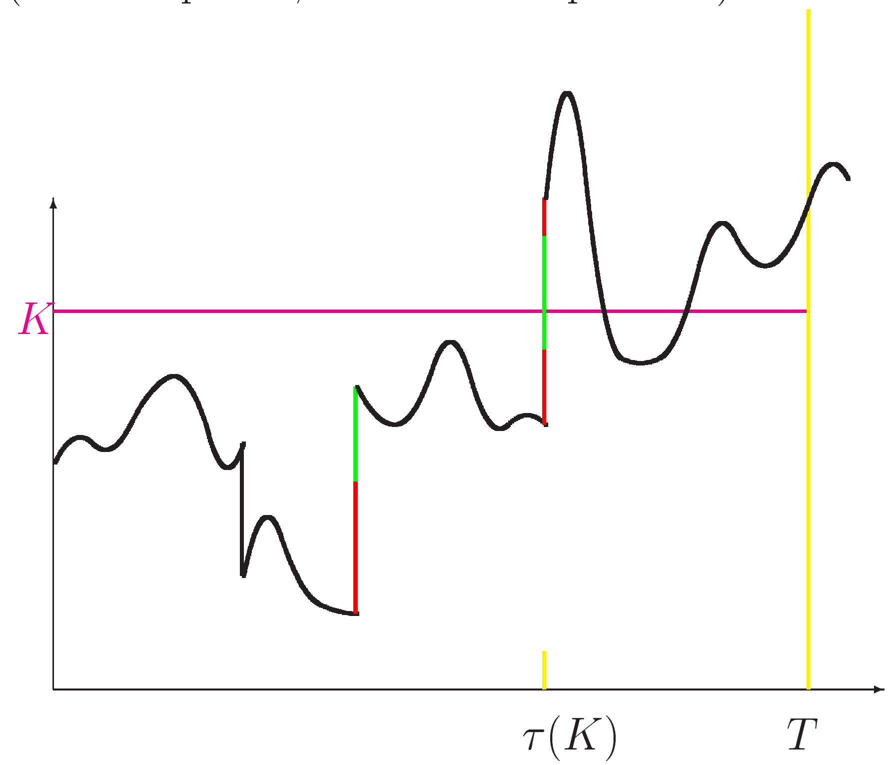
Russian option

Similar but more complicated; details later

Traditional model (Black–Scholes):
$X_t = \log S_t$ Brownian motion
First approx but does not fit data too well
Alternative Lévy models (hyperbolic, NIG, ...)
Price calculations much harder than for BM
For barrier options, feasible with PHT models
No restriction because of denseness

Role of PH Assumptions in Level Crossing Problems

(barrier options; classical ruin problem)



Overshoot at $\tau(K)$? (deficit at ruin)

$S_T \,\big|\, \tau(K) < T$? $\tau(K)$ and overshoot needed

PH assumptions control overshoot

only phase at level crossing needed

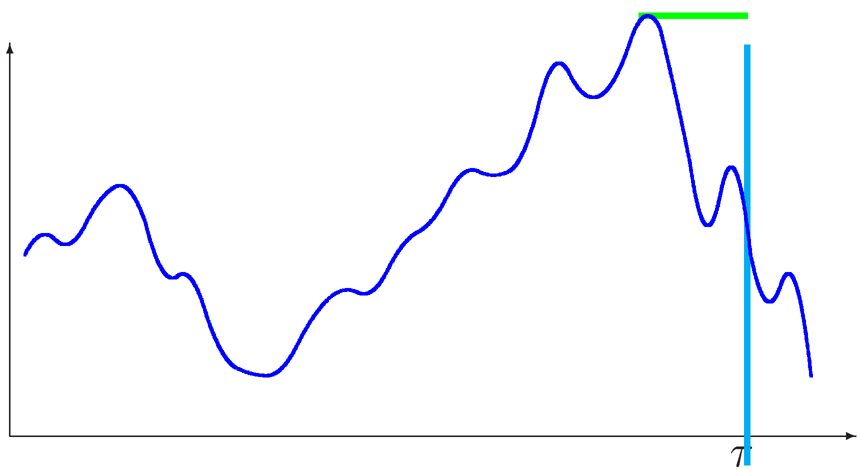
often finite set of linear equations

Implementation for Russian Options

SA-Avram-Pistorius, *Stoch. Proc. Appl.* 2003/04

Introduced by Shepp & Shiryaev 1991

Approximation to perpetual American option



Pay-out at stopping time τ $\quad \mathrm{e}^{-a\tau}\max(m, \max_{t\le\tau} S_t)$

Shepp & Shiryaev 1991: solution for geometric BM

$\tau(k^{\#}) = \inf\{t : \max_{v\le\tau} S_v - S_t \ge k^{\#}\}$ optimal

AAP: solution in dense class of Lévy processes

$$\log S_t \;=\; B_t + \sum_{i=1}^{N_t^+} U_i^+ - \sum_{i=1}^{N_t^-} U_i^-$$

independent compound Poisson, PHT jump $\uparrow$ and $\downarrow$

Solution

1. Risk-neutral measure:

choose as Esscher transform

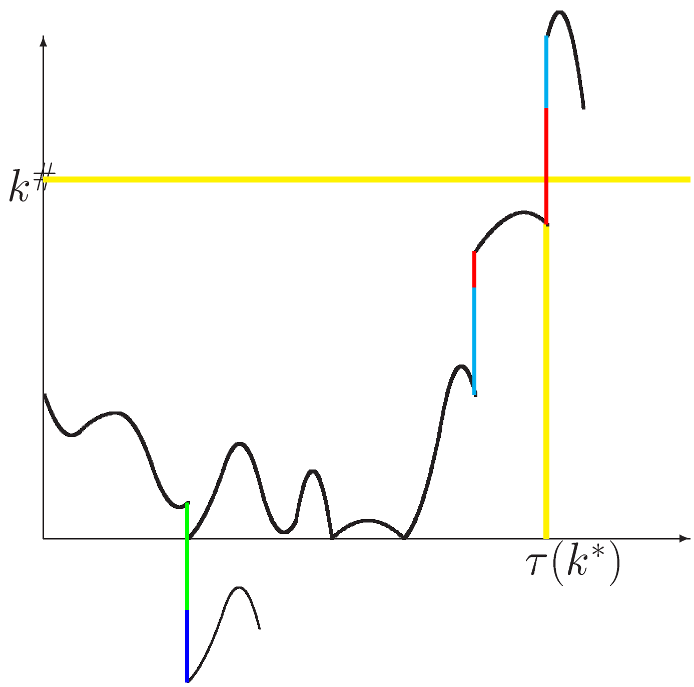
again, $\log S_t \;=\; \mathrm{BM} + \mathrm{CP(PHT)}^+ - \mathrm{CP(PHT)}^-$
(changed parameters)

2. Same optimal stopping time

$\tau(k^{\#}) = \inf\{t : \max_{v \le t} S_v - S_t \ge k^{\#}\}$ as SS

Can be written as $\tau(k^{\#}) = \inf\{t : V_t \ge k^{\#}\}$

$V_t \;=\; \log S_t$ reflected at maximum



3. $2 + p^+ + p^-$ linear equations:

$\mathbb{P}$(●–upcrossing), $\mathbb{P}$(●–upcrossing), ...

$\mathbb{E}$(●–downcrossing), $\mathbb{E}$(●–downcrossing), ...

+2 for BM

obtained by martingale optional stopping