Cash Management with Futures in Passive Investment

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Abstract

Despite of the importance of effective cash management in passive investment, few analytical results could be found in this area except Connor & Leland(1995). They analyzed the cash management as a stochastic control problem. But futures are not taken into account in their model. Though the use of futures in hedging has disadvantages of possible large tracking error and extra trading cost due to rollovers, it has advantages such as low trading cost. This paper tries to prove the effectiveness of the use of futures through the stochastic analysis and numerical experiments.

First, we formulate "spot model"(without futures) and "spot and future model"(with futures) respectively as a discrete Markov chain. We then calculate the expected tracking errors under several cash management policies of threshold type. The comparison of calculation indicates that there is not large difference of the expected tracking errors, but spot and future model performs well for all threshold values while performance of spot model can degrade considerably depending on the threshold levels. We also compare the average interval of trading index portfolio, and find that spot and future model has longer interval, which enables us to reduce office cost. Finally, we research robustness against fluctuation of parameters with sensitivity analysis and illustrate that spot and future model is robust enough for practical use.

Keywords

futures, passive investment, cash management, Markov chain, discrete

1 Preface

Commingled funds of trust banks are widely used for Japanese pension funds. Use of commingled funds has the advantage of low trading cost, whereas it has the disadvantage of the occurrence of cash flow due to the trading by other pension funds which also have some portion in the commingled fund. The existence of cash inflow and outflow makes the commingled fund to have some cash position. The problem is what is the optimal level of cash portion. If the cash portion is small, tracking errors is small, but trading cost is large due to the frequent trading of index-tracking portfolio. If the cash portion is large, tracking error is large, but trading cost is small thanks to the less frequent trading of portfolio.

Connor & Leland[1] derived a closed form solution for such a tradeoff problem of cash position in a fund by the use of the continuous-time stochastic control theory with constant parameters. Let ω_t denote the chosen cash weight at time t, and let de denote the cash inflow(a negative value of d_t denotes a cash outflow). The optimal cash management policy is controlled by a threshold value denoted by L ,which is the upper limit of the cash position. If $\omega_{t-1} + d_t > L$, then the manager decreases the cash position to L. The manager increases the cash weight only if $\omega_{t-1} + d_t<0$, in which case the manager increases the cash weight to 0. If $0 \leqq \omega_{t-1} + d_t \leqq L$, the manager does not rebalance. Connor & Leland[1] obtained the way to calculate L.

Their model deals with spot (of index-tracking portfolio) and cash model but does not deal with futures. From the practical point of view, fund managers hesitate to use futures to hedge cash with index futures, because the price fluctuation of the index futures is not the same as that of index itself , and because futures is accompanied by rollover risk.

The purpose of this paper is the proposal of effective cash management with futures in passive index-tracking investment. First we formulate " spot model " for the cash management technique currently used in the actual investment. Then we define "spot and future model" as the cash management technique with futures. We develop a model without rollovers first, then we proceed to a model with rollovers. We illustrate the superiority of spot and future model to spot model.

This paper comprises four sections. The second section deals with spot model, following this first section. After formulating the model as a Markov chain, we develop the numerical calculation method of the tracking error. We then explain the calculation for a passive fund. In the third section we formulate a spot and future model without rollovers using the similar concept to the spot model .Next we develop a spot and future model by considering the Markov chain of which the time interval is the interval between rollovers. We then explain the result of the use of futures by numerical calculation in the case of a passive fund tracking the domestic equity index.

In the fourth section we compare the cash model with the cash and spot model by investigating the expected tracking errors and the average trading interval of spot portfolio. We summarize the above result in the fifth section, and we propose the spot and future model as a superior strategy to spot model. Further research subject is also described.

2 Spot Model

2.1 Formulation of spot model

The structure of our cash management problem can be shown in the following Figure:

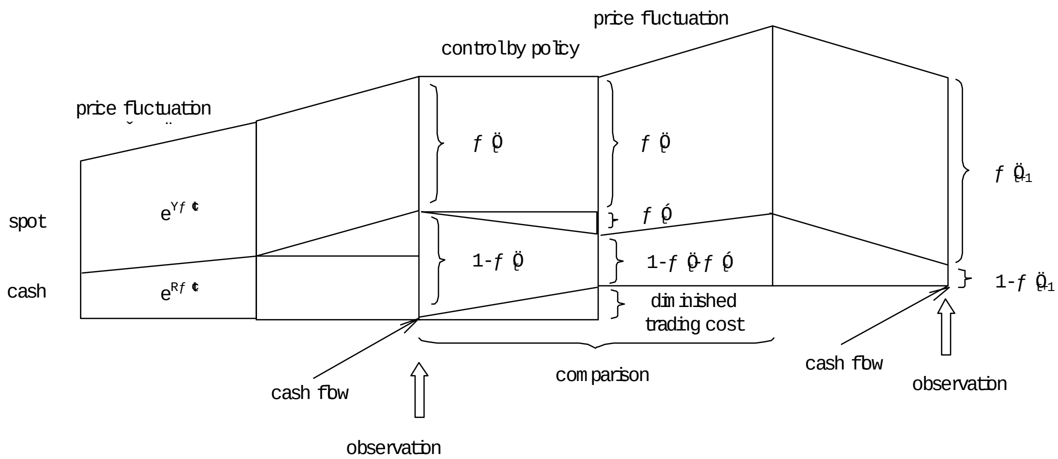


Figure 1 : Structure of the spot model

First we observe the spot portion ω_t in the fund and cash portion 1-ω_t by measuring the market value of the fund and the spot index portfolio at the closing time of the date when a cash flow occurred. This time is called "observation time". The cash flow is supposed to occur just before the observation time. The control of cash portion is realized by selling or buying the spot portfolio according to the trading portion ϕ_t determined by the observed the cash portion. Spot trading reduces cash position due to the trading cost. Until the next observation time of cash flow, the cash portion changes by the fluctuation of spot price and the interest arising from cash. At the next observation time we again observe cash portion, and we adjust the cash portion according to the policy. We repeat these process hereafter.

The practitioners' need can be expressed by the following minimizing problem:

[Objective function]

$$\min \qquad \frac{1}{\Lambda+1} E\left[\sqrt{\sum_{\lambda=0}^{\Lambda}(r_\lambda - \bar{r}_\lambda)^2}\right] \qquad (1)$$

[Time-weighted average rate of return]

$$r_\lambda = \prod_{t=1}^{T_\lambda} v_t - 1, \quad v_t = \frac{V_t}{V_{t-1} + d_{t-1}}, \quad \bar{r}_\lambda = \prod_{t=1}^{T_\lambda} b_t - 1, \quad b_t = \frac{B_t}{B_{t-1} + d_{t-1}} \quad (2)$$

where the notations are as follows:

Λ : number of observations for tracking errors
T_λ : The final time of an interval of observation period λ to measure the time-weighted rate of return
r_λ :the time-weighted annual rate of return of a passive fund in the λ th observation period
$\overline{r_\lambda}$:the time-weighted annual rate of return of a benchmark in the λ th observation period
b_t :annual rate of return of a benchmark from time t-1 to t
B_t: market value of a benchmark at time t (t=1,...,T_λ)
v_t: annual rate of return of a passive fund from time t-1 to t
V_t: market value of a passive fund at time t (t=1,...,T_λ)
d_t: the amount of the cash flow occurring at time t (random variable)

Although Connor & Leland[1] formulate their model as a continuous-time type, we formulate our model in terms of discrete type, because tracking errors are defined within discrete concept in practice. We assume that the distribution of the rates of return of cash flow , benchmark, and spot are i.i.d. This assumption is called hypothesis 1 .

For example, T_λ is 31 in case of measuring the time-weighted monthly return using daily data, Λ is 24 in the case of measuring 2 year tracking error based on monthly the time-weighted monthly return .

Though the number of measurement interval of tracking errors is finite, we assume the number as infinite for the easier calculation. We calculate time-weighted rate of return at the interval of cash flow which occurs periodically. The length of each interval is assumed to be the same, and the assumption is called Hypothesis 2.

Because the time horizon of pension investment is usually over several decades, if cash flows occur every several years, chance to control cash position would be several thousands times. This justifies our assumption , infinite number of measurement intervals. Calculation of time-weighted rate of return at every cash flow interval is deemed as conservative handling, because in practice time-weighted rate of return is measured at longer period ,for example monthly. We deal with expected tracking errors, instead of historical tracking errors as in the practical investment.

Under the above assumptions, we formulate our problem as follows:

[Problem 1] $$\min \quad \limsup_{T\to\infty} \frac{1}{T+1} E\left[\sum_{t=0}^{T}(v_t - b_t)^2\right] \quad (3)$$

We can assume that the return of cash flow, the return of benchmark, the return of spot are independent one another. We only deal with "stationary policy", which means not to admit short selling. As for a "stationary policy", we aim at minimizing the expected tracking errors in only one cash flow period.

We first describe the rate of return of passive fund in terms of spot price and cash price. We derive the following formula considering that index-tracking portfolio does not completely trace the benchmark, and that trading cost exists, and that cash return is different from benchmark return.

[Return of passive fund]

$$v_t = \frac{(G_{t-1} + g_{t-1})e^{Y\Delta t} + (H_{t-1} - g_{t-1} - |g_{t-1}| \cdot p(g_{t-1}) + d_{t-1})e^{R\Delta t}}{(G_{t-1} + H_{t-1} + d_{t-1})} \quad (4)$$

At the same time market value of benchmark and the market value of passive fund can be written as follows:

$$B_t = (B_{t-1} + d_{t-1})e^{X\Delta t} \quad (5)$$

$$V_t = (G_{t-1} + g_{t-1})e^{Y\Delta t} + (H_{t-1} - g_{t-1} - |g_{t-1}| \cdot p(g_{t-1}) + d_{t-1})e^{R\Delta t} \quad (6)$$

where the notation is as follows:
Δt : time interval of the occurrence of cash flow.
R: annual rate of cash return
X: annual rate of benchmark return (random variable)
Y: rate of return of spot index-tracking portfolio (random variable)
G_t: market value of spot index-tracking portfolio at time t
H_t: the amount of cash position at time t
g_t: trading amount of spot at time t
p(・): trading cost of spot trading per trading amount

Assuming that p(・) is constant k (Hypothesis 3) , the excess rate of return of passive fund over benchmark can be expressed as follows:

$$v_t - b_t = \frac{V_t}{V_{t-1} + d_{t-1}} - e^{X\Delta t}$$
$$= \frac{G_{t-1}(e^{Y\Delta t} - e^{X\Delta t}) + (H_{t-1} + d_{t-1})(e^{R\Delta t} - e^{X\Delta t}) - g_{t-1}(e^{R\Delta t} - e^{Y\Delta t}) - |g_{t-1}| \cdot ke^{R\Delta t}}{G_{t-1} + H_{t-1} + d_{t-1}} \quad (7)$$

Assuming further that cash flow at time t is the product of random variable θ_t and spot market value at time t-1, namely $d_t = \theta_t G_{t-1}$(Hypothesis 4), we can derive the following formula by the use of spot portion rate ω_t and trading rate ϕ_t from the above formula.

$$v_t - b_t = \omega_{t-1}(e^{Y\Delta t} - e^{X\Delta t}) + (1-\omega_{t-1})(e^{R\Delta t} - e^{X\Delta t}) - \phi_{t-1}(e^{R\Delta t} - e^{Y\Delta t}) - |\phi_{t-1}| \cdot ke^{R\Delta t} \quad (8)$$

We can now express the expected value of squares of excess rate of return of passive fund in one cash flow period as follows:

$$\begin{aligned} E[(v_t - b_t)^2] =& E[(a_{22} - 2a_{02} + a_{00})(\omega_{t-1}^2 + 2\omega_{t-1}\phi_{t-1} + \phi_{t-1}^2) + k^2 a_{00}\phi_{t-1}^2 + (a_{00} - 2a_{01} + a_{11}) \\ &+ 2(-k(a_{02} - a_{00})(\omega_{t-1} + \phi_{t-1})|\phi_{t-1}| + (a_{02} - a_{12} - a_{00} + a_{01})(\omega_{t-1} + \phi_{t-1}) \\ &- k(a_{00} - a_{01})|\phi_{t-1}|)] \end{aligned} \quad (9)$$

where

$$a_{00} = e^{2R\Delta t},\ a_{11} = e^{2X\Delta t},\ a_{22} = e^{2Y\Delta t},$$
$$a_{01} = e^{R\Delta t}e^{X\Delta t},\ a_{12} = e^{X\Delta t}e^{Y\Delta t},\ a_{02} = e^{R\Delta t}e^{Y\Delta t}$$

Under the above preparation, we can calculate the objective function.

2.2 The solution of spot model

We analyze the problem in the threshold policy framework as in Connor & Leland[1]. Although threshold policy is not proved optimal for the discrete model, this policy might be at the neighborhood of optimal policy, because Connor & Leland[1] proved that the threshold policy is optimal for the continuous-time model. At least, the confinement to threshold model is not the obstacle for the comparison of two models, “spot model” and “spot and future model”.

If we confine our analysis within the threshold policy, we can treat the problem as a Markov chain problem, allocating every spot portion s_i to a state i under the given boundaries. A state is determined by giving boundaries. Let s_i be the spot portion of state i such that $a_{i-1} < s_{i-1} < a_i$ for $i = 1,...,m$, where i is the number of states, a_i is the boundary of each state, $a_0 = -\infty$ and $a_m = \infty$. We represent the spot portion for each state i , as the value of the middle point between boundaries. The difference between spot portion s_i and s_{i-1} is set to be equal for all i.

We thus create the Markov chain as shown in Figure 2

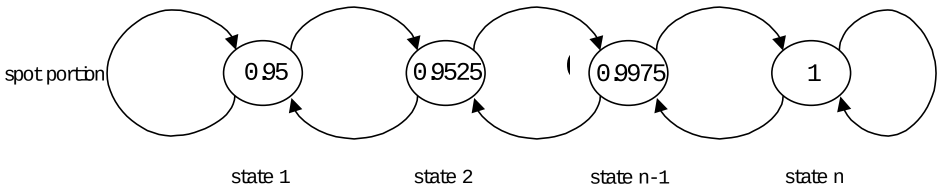


Figure 2: Markov chain

Figure 2 illustrates that the state moves from one state to another state through time. We calculate the transition probability p(j|i,n) which is the probability from state i to state n after the application of the threshold policy, and to state j just before the occurrence of the next cash flow. We calculate the probability with simulation using the following relationship.

$$\omega_t = \frac{(\omega_{t-1} + \phi_{t-1})e_2}{(e_2 - e_0)(\omega_{t-1} + \phi_{t-1}) + e_0 - ke_0|\phi_{t-1}| + \theta_t \omega_{t-1}} \quad (10)$$

We define the transition probability matrix P as follows.

$$\mathbf{P} = \begin{pmatrix} p(1|1,n) & \dots & p(m|1,n) \\ \vdots & \ddots & \vdots \\ p(1|m,n) & \dots & p(m|m,n) \end{pmatrix} \quad \textbf{(11)}$$

We define the expected tracking error vector when the state i moves to state n by a policy as follows:

$$\mathbf{C} = \begin{pmatrix} C(1,n) \\ \vdots \\ C(m,n) \end{pmatrix} \quad \textbf{(12)}$$

We then define the average cost, which is the expected tracking errors in a cash flow interval. The average cost is calculated as follows:

1. Determine transition probability matrix P and expected tracking error vector C according to the adopted policy.
2. Calculate initial probability distribution π from the equation $\pi \bullet P = \pi$ and $\pi \bullet 1 = 1$
3. Calculate average cost c_{mean} from the equation $c_{mean} = \pi \bullet C$

We consider negligence policy and constant policy as well as threshold policy for the purpose of comparison. Negligence policy is the policy which do no rebalance regardless of the change of cash portion, expect the cash portion goes to negative, when cash portion is adjusted to 0 by selling the spot index-tracking portfolio. Constant policy is the policy which fix the cash portion regardless of any change of cash portion.

2.3 Numerical example of spot model calculation

We experiment numerically to confirm the superiority of threshold policy over negligence policy and constant policy. Spot portion is used as the representation of its state and we consider 42 kinds of spot portion : 0.94875, 0.95, 0.9525, ..., 1.045, 1.0475, 1.04875. We show the parameter of numerical example in the case of the domestic passive equity fund in table 1. We assume the normal distribution for the distribution of cash flow, rate of return of benchmark, rate of return of spot, rate of return of futures.

Table 1 : parameters

parameter	value
average annual return of benchmark	10%
standard derivation of the annual return of benchmark	20%
average annual excess return of spot	0%
average annual excess return of futures	-0.5%
standard derivation of annual excess return of spot	0.15%
standard derivation of annual excess return of futures	0.35%
risk free annual rate of return	0.5%
margin rate of futures	3%
annual return of margin for futures	0%
average of θ_t	0
standard derivation of θ_t	20/6000
rate of trading cost of spot	0.1%
rate of trading cost of futures	0.05%
annual occurrence of cash flow	150
correlation between spot excess return and benchmark return	0
correlation between futures excess return and benchmark return	0
correlation between spot excess return and futures excess return	0

Under the above assumptions we calculate the average cost for three policies. The result is illustrated in Figure 3, where the average cost is the square root of expected tracking error multiplied by 150, annual occurrence of cash flow .

Negligence policy is represented at the single point on the vertical line at the spot portion 0.9475. The horizontal line represent spot portion in case of constant policy, threshold spot portion for threshold policy. The Figure 3 shows that the average cost is minimum at the spot portion 1 in case of threshold policy. This shows that cash minimizing policy is most cost-saving for the spot model.

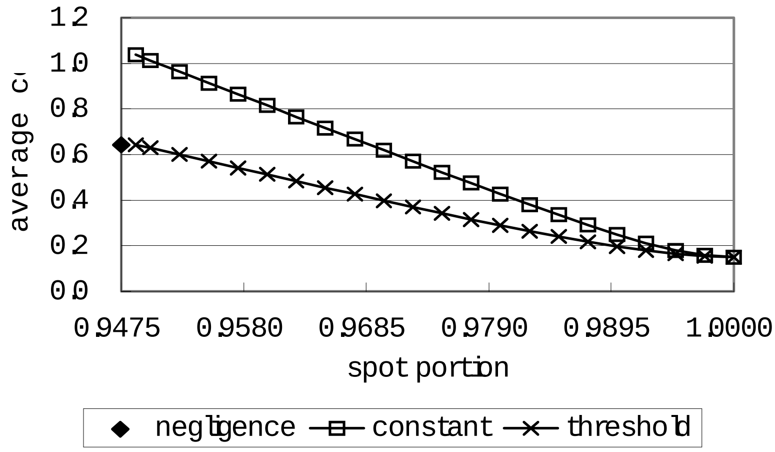


Figure 3:Comparison among negligence policy, constant policy, and threshold policy (vertical: %)

For constant policy, cash minimizing policy is most cost-saving as in the case of threshold policy. And if the cash portion is 0, the average cost is equal for the constant policy and the threshold policy. If the cash portion is less than 1, threshold policy realizes lower average cost than constant policy. Figure 3 shows that the average cost of threshold policy is lower than that of negligence policy.

3. **Spot and Future model**

We explain first the spot and future model without the consideration of rollovers. This model can be formulated almost in the same way as the spot model. We explain the model with rollovers which contains the model without rollovers.

3.1 Spot and Future model without the consideration of rollovers

3.1.1 **Formulation of Spot and Future model without the consideration of rollovers**

Figure 4 shows the structure of the model:

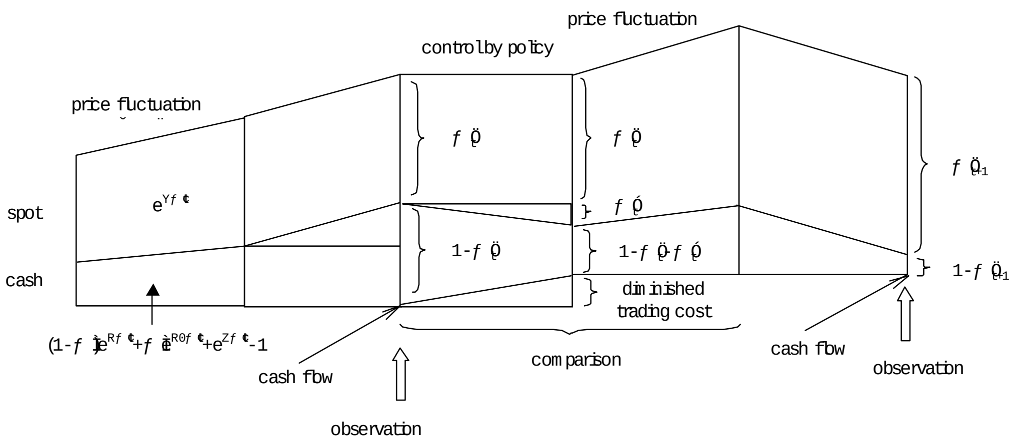


Figure 4: Structure of spot and future model without rollovers

The model structure is almost same as the spot model, but the cash return is different because of future hedging. When we rebalance according to a policy, future trading cost is deducted from cash as well as the consumption of spot trading cost. We assume that gains and losses of futures are realized at every occurrence of cash flow, though they are realized in the settlement of futures in practice.

The objective function for spot and future model is just the same as the formula (1) for spot model. The problem to be solved can be expressed also as the formula (3) same as in the case of spot model.

We now prepare to express the objective function in terms of spot portion ω_t and spot trading rate ϕ_t. The return of passive fund can be formulated in (14) shown below, which is different from (8), because index-tracking portfolio does not track completely down the benchmark, trading cost for futures is necessary, and the return of cash and margin is different from benchmark return.

[Rate of Return of Passive Fund]

$$\begin{aligned} v_t =& ((G_{t-1}+g_{t-1})e^{Y\Delta t} + (H_{t-1}-g_{t-1}+d_{t-1} \\ & - |g_{t-1}|\cdot p(g_{t-1}) - |f_{t-1}|\cdot q(f_{t-1}))((1-\xi)e^{R\Delta t}+\xi e^{R_0\Delta t}) + (H_{t-1}-g_{t-1}+d_{t-1})(e^{Z\Delta t}-1)) \\ & /(G_{t-1}+H_{t-1}+d_{t-1}) \end{aligned} \tag{14}$$

Now we can express the market value of passive fund as follows. The market value of benchmark is the same as that of spot model.

[The Market Value of Passive Fund]

$$\begin{aligned} V_t =& (G_{t-1}+g_{t-1})e^{Y\Delta t} + (H_{t-1}-g_{t-1}+d_{t-1}-|g_{t-1}|\cdot p(g_{t-1}) - |f_{t-1}|\cdot q(f_{t-1})) \\ & ((1-\xi)e^{R\Delta t}+\xi e^{R_0\Delta t}) + (H_{t-1}-g_{t-1}+d_{t-1})(e^{Z\Delta t}-1) \end{aligned} \tag{15}$$

where the notation is as follows:
Z: annual rate of return of future (random variable)
R_0: annual rate of return of margin
ξ : margin rate of future
F_t: market price of future at time t (yen)
f_t: trading amount of money of futures at time t (yen)
q(・) : trading cost rate for future trading in terms of future amount

We calculate excess return of portfolio over benchmark, as in the case of spot model.

[excess return of portfolio over benchmark]

$$\begin{aligned} v_t - b_t &= \frac{V_t}{V_{t-1}+d_{t-1}} - e_1 \\ &= \frac{G_{t-1}(e_2-e_1)+(H_{t-1}+d_{t-1})(e_4-e_1+e_3)-g_{t-1}(e_4-e_2+e_3)-(|g_{t-1}|\cdot p(g_{t-1})+|f_{t-1}|\cdot q(f_{t-1}))e_4}{G_{t-1}+H_{t-1}+d_{t-1}} \end{aligned} \tag{16}$$

where the notation is as follows:

$$e_1 = e^{X\Delta t},\ \ e_2 = e^{Y\Delta t},$$
$$e_3 = e^{Z\Delta t},\ \ e_4 = (1-\xi)e^{R\Delta t} + \xi e^{R_0\Delta t}$$

Assuming p(･)and q(･) to be constant, namely p(･)= κ and q(･)= μ , where κ and μ are cost rates, we can rewrite the formula of excess return as follows in terms of spot portion ω_{t-1} , spot trading rate ϕ_{t-1} , and cash flow rate η_{t-1} just after the occurrence of cash flow.

$$\begin{aligned} v_t - b_t =& \omega_{t-1}(e_2 - e_1) + (1-\omega_{t-1})(e_4 - e_1 + e_3) - \phi_{t-1}(e_4 - e_2 + e_3) \\ & - (|\phi_{t-1}|k + |\eta_{t-1} - \phi_{t-1}|\mu)e_4 \end{aligned}$$

we can transform the formula into the following simpler one.

$$v_t - b_t = (\omega_{t-1} + \phi_{t-1})(e_2 - e_4 - e_3) + (e_4 - e_1 + e_3) - (|\phi_{t-1}|k + |\eta_{t-1} - \phi_{t-1}|\mu)e_4$$

We now derive the formula of the expected value of square of excess return of passive fund.

[Expansion of formula for one interval]

$$\begin{aligned} & E[(v_t - b_t)^2] \\ =& E[(\omega_{t-1} + \phi_{t-1})^2(a_{44} + a_{22} + a_{33} - 2a_{42} + 2a_{43} - 2a_{23}) + (a_{44} + a_{11} + a_{33} - 2a_{41} - 2a_{13} + 2a_{43}) \\ & + (|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu)^2 a_{44} \\ & + 2((\omega_{t-1} + \phi_{t-1})(a_{42} - a_{12} + a_{23} - a_{44} + a_{41} - 2a_{43} + a_{13} - a_{33}) \\ & - (a_{44} - a_{41} + a_{43})(|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu) \\ & - (\omega_{t-1} + \phi_{t-1})(|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu)(a_{42} - a_{44} - a_{43}))] \end{aligned} \tag{17}$$

where we assume $\eta_{t-1} = \theta_{t-1}\ \omega_{t-1}$.

We also assume $d_{t-1} = \theta_{t-1} G_{t-2}$ and $G_{t-1} \fallingdotseq G_{t-2}$, and we call this assumption as hypothesis 5. The definition of notation is as follows:

$$e_1^2 = a_{11},\ \ e_2^2 = a_{22},\ \ e_3^2 = a_{33},\ \ e_4^2 = a_{44},$$
$$e_1e_2 = a_{12},\ \ e_1e_3 = a_{13},\ \ e_1e_4 = a_{14},\ \ e_2e_3 = a_{23},\ \ e_2e_4 = a_{24},\ \ e_3e_4 = a_{34}$$

We calculate the value of objective function using formula (17) similar to that of the spot model, and we will determine the policy which minimizes the tracking errors in cash flow period.

3.1.2 Solution of Spot and Future model without the consideration of rollovers

We can solve the problem by treating the model as Markov chain if we confine the scope of analysis within the threshold policy as in our study of spot model. We can calculate the average cost almost same as in the case of spot model. But the formula used in the calculation of transition probability is slightly different, shown as follows:

$$\omega_t = \frac{(\omega_{t-1}+\phi_{t-1})e_2}{(e_2-e_4-e_3)(\omega_{t-1}+\phi_{t-1})+(e_4+e_3)-(|\phi_{t-1}|k+|\eta_{t-1}-\phi_{t-1}|\mu)e_4+\theta_t\omega_{t-1}}$$
$$= \frac{(\omega_{t-1}+\phi_{t-1})e_2}{(e_2-e_4-e_3)(\omega_{t-1}+\phi_{t-1})+(e_4+e_3)-(|\phi_{t-1}|k+|\theta_{t-1}\omega_{t-1}-\phi_{t-1}|\mu)e_4+\theta_t\omega_{t-1}} \quad \textbf{(18)}$$

where $d_t = \theta_t G_{t-1}$.

Using the above formula(18), we can calculate transition probability matrix for the spot and future model without rollovers. Thus we can calculate the average cost in the model.

3.2 Spot and future model with consideration of rollovers

We now formulate the spot and future model with rollovers.

3.2.1 Formulation an solution of spot and future model with rollovers

The structure of the model can be shown in Figure 5.

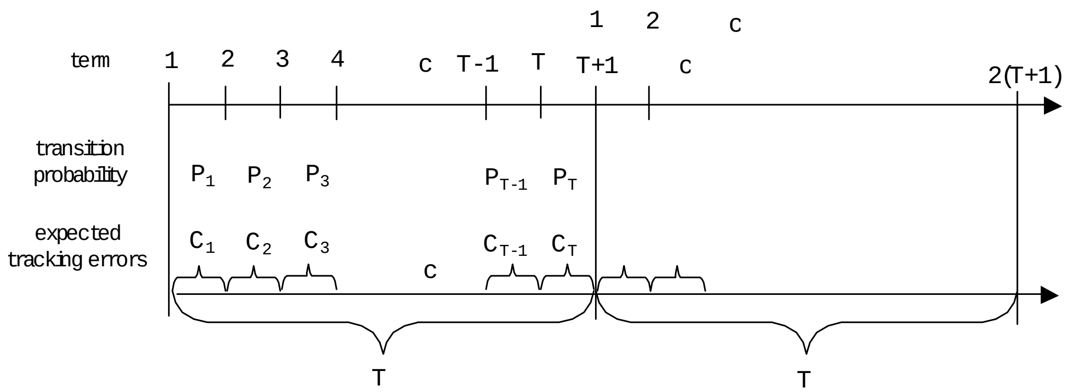


Figure 5: Structure of the spot and future model with rollovers

Figure 5 shows the rollovers of futures for every T times of cash flow. Like in the case of spot model, we can confine our scope of policy selection within ordinary policy, which regards T intervals of cash flow as one interval. If we treat the ordinary policy such as threshold policy, we can think the model as Markov chain. The spot portion for each cash flow interval is allowed to be different from one another. For example, we can adopt a policy where the threshold cash portion is gradually increased as the approach of time of rollover.

In this model, we use transition probability and expected tracking error for spot and future model without rollovers about cash flow from the first time to T-1 th cash flow. We use transition probability and expected tracking error for spot and future model with rollovers for the T th cash flow.

We can calculate initial distribution from second cash flow to T th cash flow by multiplication, if we know the probability distribution of the previous interval, on condition that the initial probability distribution of the first interval is determined. We calculate the first initial probability distribution by regarding the problem as a Markov chain of which state changes once for T times of cash flow. The transition matrix P can be expressed as follows:

$$\mathcal{P} = \prod_{j=1}^{T} \mathbf{P}_j \tag{19}$$

where P_j is the transition probability for each cash flow.

We calculate the initial probability distribution π for every T cash flows by solving the equation $\pi P = \pi$, $\pi \cdot 1 = 1$. We can calculate average cost C by the use of the following formula (20) after the calculation of initial probability distribution.

$$\begin{aligned} \mathcal{C} &= \frac{1}{T} E[C_1 + C_2 + \cdots + C_{T-1} + C_T] \\ &= \frac{1}{T}(E[C_1] + E[C_2] + \cdots + E[C_{T-1}] + E[C_T]) \end{aligned} \tag{20}$$

where C_k(k=1,...,T) is the average cost for each cash flow, and its expected value is expressed as follows:

$$E[C_k] = \sum_i E[\mathbf{C}_k(i)]\mathbf{P}_k(i) \tag{21}$$

where
$C_k(i)$: expected tracking error vector at the k-th initial state i.
$P_k(i)$:probability matrix for k-the initial state.

We then present the formula to calculate the expected tracking errors with rollovers.

$$\begin{aligned} & E[(v_t - b_t)^2] \\ =& E[(\omega_{t-1} + \phi_{t-1})^2(a_{44} + a_{22} + a_{33} - 2a_{42} + 2a_{43} - 2a_{23}) + (a_{44} + a_{11} + a_{33} - 2a_{41} - 2a_{13} + 2a_{43}) \\ & + (|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu + 2\mu|1 - \omega_{t-1}|)^2 a_{44} \\ & + 2((\omega_{t-1} + \phi_{t-1})(a_{42} - a_{12} + a_{23} - a_{44} + a_{41} - 2a_{43} + a_{13} - a_{33}) \\ & - (a_{44} - a_{41} + a_{43})(|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu + 2\mu|1 - \omega_{t-1}|) \\ & - (\omega_{t-1} + \phi_{t-1})(|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu + 2\mu|1 - \omega_{t-1}|)(a_{42} - a_{44} - a_{43}))] \end{aligned} \tag{22}$$

Next we show the formula to calculate the transition probability matrix.

$$\begin{aligned} \omega_t =& \frac{(\omega_{t-1} + \phi_{t-})e_2}{(e_2 - e_4 - e_3)(\omega_{t-1} + \phi_{t-1}) + (e_4 + e_3) - (|\phi_{t-1}|k + |\eta_{t-1} - \phi_{t-1}|\mu + 2\mu|1 - \omega_{t-1}|)e_4 + \theta_t\omega_{t-1}} \\ =& \frac{(\omega_{t-1} + \phi_{t-1})e_2}{(e_2 - e_4 - e_3)(\omega_{t-1} + \phi_{t-1}) + (e_4 + e_3) - (|\phi_{t-1}|k + |\theta_{t-1}\omega_{t-1} - \phi_{t-1}|\mu + 2\mu|1 - \omega_{t-1}|)e_4 + \theta_t\omega_{t-1}} \end{aligned} \tag{23}$$

We can now calculate the average cost by the use of above formulas.

3.2.2 **Numerical example of spot and future model with rollovers and the comparison of policies**

We will experiment numerically to confirm the superiority of threshold policy over negligence policy and constant policy like in the case of spot model. Figure 6 shows the comparison of average costs among 3 policies where the threshold cash portions are constant for every cash flow within the interval between rollovers.

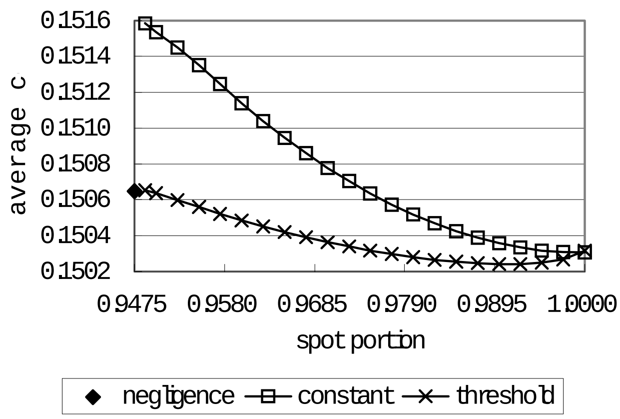


Figure 6 Comparison among negligence policy, constant policy, and threshold policy (vertical: %)

The average cost is minimum if the threshold spot portion is 0.99, and the average cost is lower in the threshold policy than in the negligence policy or in the constant policy for all spot portions.

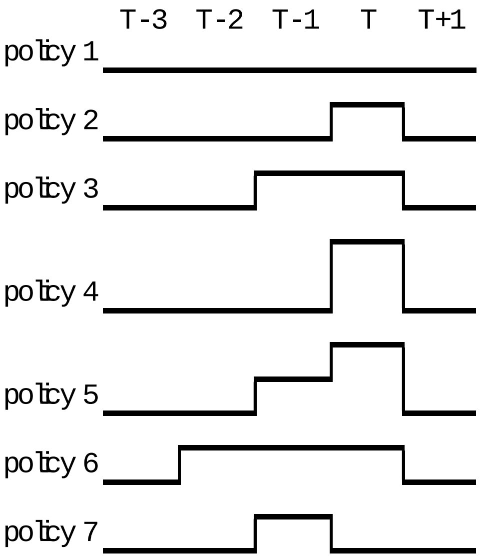
3.2.3 Comparison of various threshold policies for spot and futures model with rollovers

We can consider to vary the threshold spot portion for each cash flow within the rollover interval instead of the constant threshold as described in 3.2.2. For example there is some possibility to reduce the average cost if the threshold spot portion is increased as the time of rollover approaches, because the trading cost of rollover is smaller owing to the small cash portion. To investigate this possibilities, we will survey the following 7 threshold policies.

Policy 1 : Threshold value is constant and has the value of state n.
Policy 2 :Threshold value corresponds to state n+1 for time T, and n otherwise.
Policy 3 :Threshold value corresponds to state n+1 for time T and T-1, and n otherwise.
Policy 4 :Threshold value corresponds to state n+2 for time T, and n otherwise.
Policy 5 :Threshold value corresponds to state n+2 for time T, and n+1 for time T-1, and n otherwise.
Policy 6 :Threshold value corresponds to state n+1 for time T, T-1,T-2, and n otherwise.

Policy 7 :Threshold value corresponds to state n+1 for time T-2, and n otherwise.

Where T is the time of the occurrence of cash flow and is also the time of rollover,
T-1 is the cash flow interval just before T, T-2 is the cash flow interval just before T-1.



example of threshold value 0.99

	T-3	T-2	T-1	T	T+1
policy 1	0.99	0.99	0.99	0.99	0.99
policy 2	0.99	0.99	0.99	0.9925	0.99
policy 3	0.99	0.99	0.9925	0.9925	0.99
policy 4	0.99	0.99	0.99	0.995	0.99
policy 5	0.99	0.99	0.9925	0.995	0.99
policy 6	0.99	0.9925	0.9925	0.9925	0.99
policy 7	0.99	0.99	0.9925	0.99	0.99

Figure 7: Image of policies and the example of threshold value 0.99

Figure 7 shows the image of policy 1 through policy 7, and the transition of threshold under the assumption that the standard threshold for n is 0.99. Other assumptions of parameters are the same for the spot model shown in table 1.

Figure 8 shows the comparison of 7 policies in the average cost for the spot and future model with rollovers.

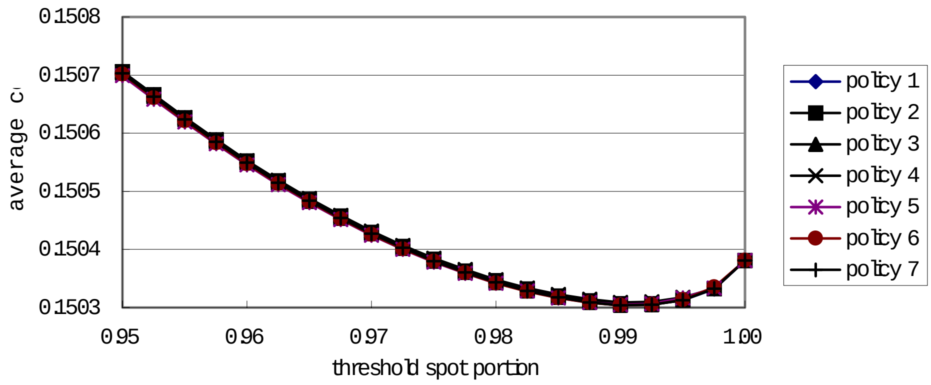


Figure 8(a): Comparison of the average cost among various policies:(vertical: %)

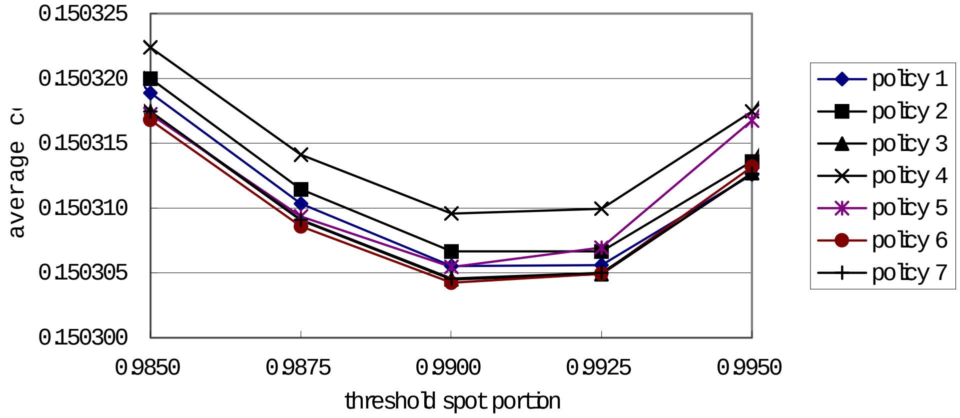


Figure 8(b): Detailed exhibit of the average cost among various policies:(vertical :%)

Table 2 shows the average cost for each policy.

Table 2: Comparison of the average cost among various policies(%)

spot portion	policy 1	policy 2	policy 3	policy 4	policy 5	policy 6	policy 7
0.94875	0.1507216	0.1507217	0.1507210	0.1507216	0.1507190	0.1507205	0.1507210
0.95000	0.1507056	0.1507057	0.1507027	0.1507056	0.1506989	0.1507012	0.1507031
0.95250	0.1506656	0.1506659	0.1506623	0.1506660	0.1506585	0.1506608	0.1506628
0.95500	0.1506262	0.1506266	0.1506228	0.1506269	0.1506192	0.1506213	0.1506234
0.95750	0.1505880	0.1505885	0.1505848	0.1505890	0.1505813	0.1505833	0.1505853
0.96000	0.1505517	0.1505523	0.1505486	0.1505529	0.1505455	0.1505472	0.1505492
0.96250	0.1505177	0.1505183	0.1505148	0.1505192	0.1505119	0.1505134	0.1505152
0.96500	0.1504859	0.1504867	0.1504831	0.1504876	0.1504805	0.1504819	0.1504836
0.96750	0.1504565	0.1504573	0.1504539	0.1504584	0.1504515	0.1504527	0.1504542
0.97000	0.1504294	0.1504303	0.1504269	0.1504316	0.1504248	0.1504258	0.1504273
0.97250	0.1504047	0.1504056	0.1504024	0.1504071	0.1504005	0.1504013	0.1504026
0.97500	0.1503823	0.1503833	0.1503802	0.1503850	0.1503786	0.1503792	0.1503804
0.97750	0.1503625	0.1503635	0.1503605	0.1503654	0.1503593	0.1503596	0.1503607
0.98000	0.1503452	0.1503463	0.1503434	0.1503483	0.1503425	0.1503426	0.1503435
0.98250	0.1503306	0.1503317	0.1503289	0.1503339	0.1503284	0.1503282	0.1503291
0.98500	0.1503189	0.1503200	0.1503174	0.1503224	0.1503172	0.1503168	0.1503175
0.98750	0.1503103	0.1503115	0.1503090	0.1503141	0.1503094	0.1503086	0.1503091
0.99000	0.1503056	0.1503066	0.1503045	0.1503096	0.1503055	0.1503042	0.1503046
0.99250	0.1503056	0.1503067	0.1503049	0.1503100	0.1503070	0.1503050	0.1503050
0.99500	0.1503126	0.1503136	0.1503127	0.1503174	0.1503168	0.1503132	0.1503127
0.99750	0.1503319	0.1503329	0.1503341	0.1503329	0.1503341	0.1503351	0.1503330
1.00000	0.1503806	0.1503806	0.1503806	0.1503806	0.1503806	0.1503806	0.1503806

This result tells us that the average cost of policy 3 and policy 6 is smaller than the cost of policy 1. This means that the average cost will be reduced if the spot portion is slightly raised as the time to rollover approaches. But the difference of the result for each policies is very small, so we can neglect it. Therefore the average cost does not vary much regardless of the change of threshold.

3.2.4 Effect of rollover to the average cost

Figure 9 shows the comparison between the average cost of spot and future model with rollovers, and that of spot and future model without rollovers. The parameters are the same as shown in the Table 1 for spot model.

The Figure tells us that the average cost of spot and future model with rollovers is higher than that of the model without rollovers. This would be the result of the additional trading cost of futures at the time of rollovers. The Figure shows also that the difference of the average cost is small under our parameters

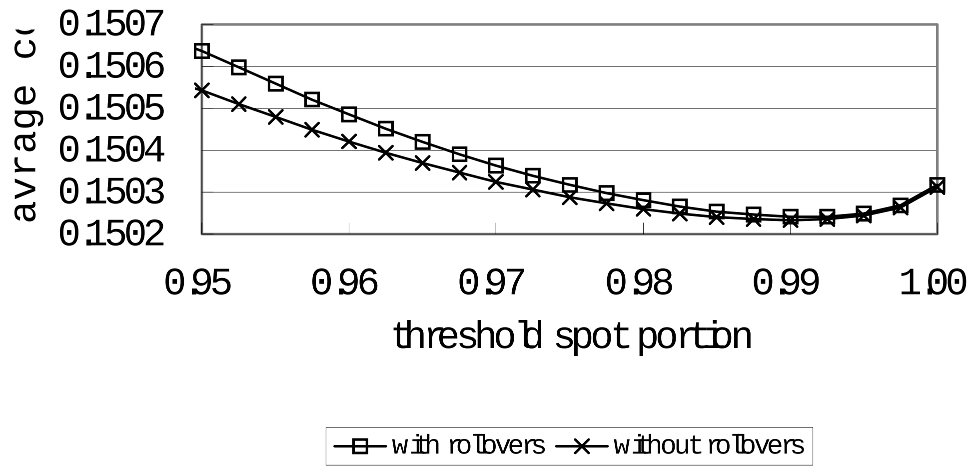


Figure 9: Comparison of the average cost between with rollovers and without rollovers (vertical: %)

4. Spot model vs. Spot and future model

Based on the above discussion, we compare the spot model with spot and future model. Hereafter the spot and future model includes rollovers.

4.1 The average cost

Figure 10 and Table 3 shows the result of calculation of the average cost for spot model , and spot and future model , under the threshold policy.

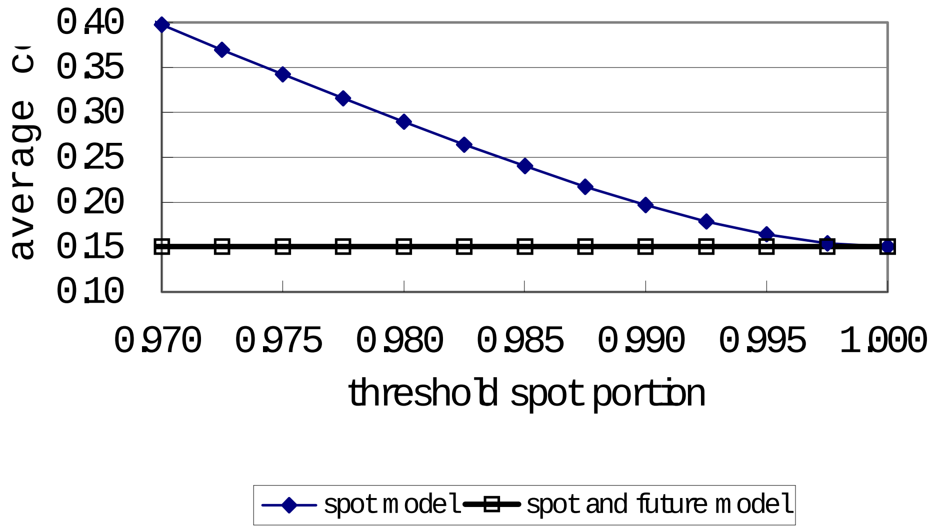


Figure 10: Comparison of the average cost

Table 3: Comparison of the average cost (%)

threshold spot portion	spot model	spot and future model	threshold spot portion	spot model	spot and future model
0.94875	0.641319	0.150653	0.97500	0.342406	0.150317
0.95000	0.629634	0.150637	0.97750	0.315537	0.150297
0.95250	0.600266	0.150597	0.98000	0.289370	0.150280
0.95500	0.570896	0.150559	0.98250	0.264100	0.150266
0.95750	0.541603	0.150521	0.98500	0.239952	0.150254
0.96000	0.512448	0.150485	0.98750	0.217286	0.150246
0.96250	0.483488	0.150451	0.99000	0.196644	0.150241
0.96500	0.454700	0.150420	0.99250	0.178660	0.150242
0.96750	0.426113	0.150391	0.99500	0.164186	0.150249
0.97000	0.397837	0.150364	0.99750	0.154194	0.150269
0.97250	0.369865	0.150339	1.00000	0.150202	0.150318

Table 3 shows that the minimum average cost of spot model is 0.15202 for threshold value 1, whereas the minimum average cost of spot and future model is 0.15241 for threshold value 0.99, which informs that the average cost of spot model is smaller. But the difference is very small, so we can conclude that there is little difference between the both models if the optimal policy is adopted.

4.2 Robustness

In spite of the little difference of average cost as shown in Figure 10, the robustness for the difference of threshold value is completely different. As the estimation error is inevitable for parameter evaluation, the threshold value calculated from a model is not necessarily the optimal value that minimize the average cost in the practical cash management. The average cost of spot model varies considerably with the threshold value ,on the other hand the cost of spot and future model does not vary widely. This robustness of spot and future model means that the average cost does not change considerably even if the estimated threshold value is different from the optimal value.

4.3 Average interval of spot trading

Next we confirm that the average interval of spot trading in a spot and future model is longer than that in a spot model, or we verify that the frequency of spot trading in a spot and future model is smaller than that in a spot model. For modeling, considerable office work is necessary for the trading of spot index-tracking portfolio because of the purchase and sale of several tens or hundreds of stocks. The less frequent occurs spot trading, the less necessary is the office work.

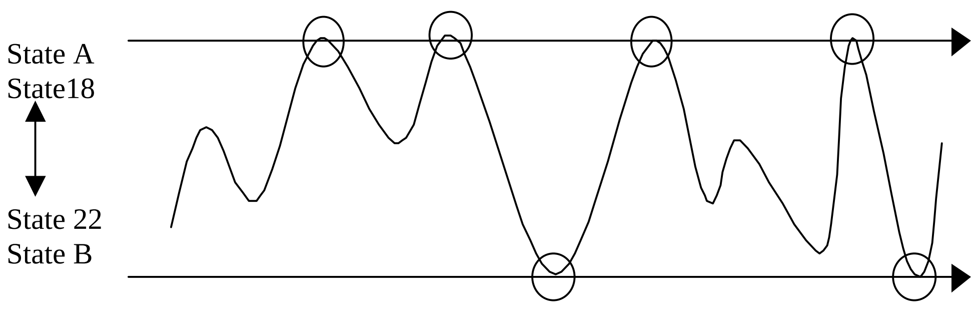


Figure 11 Model for the calculation of the average interval of spot trading

State A and state B are both absorbing states. The state 18 corresponds to the threshold spot portion 0.99, and the state 22 corresponds to the threshold spot portion 1. We assume that the current threshold value is at the state 18. If the spot portion is absorbed in the state A, spot purchase will take place, on the other hand if the spot portion is absorbed in the state B, spot sale will happen. The frequency of absorption into state A or B is measured and the average interval from one absorption to another one is calculated with the absorbing Markov chain model. Let Q be the square matrix representing the temporary transition of states. We define the basic matrix M as follows:

$$\mathbf{M} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{pmatrix} m_{18,18} & \cdots & m_{18,22} \\ \vdots & \ddots & \vdots \\ m_{22,18} & \cdots & m_{22,22} \end{pmatrix} \quad (24)$$

where I is a unit matrix.

Next we calculate the average number of steps m_{18} from state 18 to absorbing state A and m_{22} from state 22 to absorbing state B by the use of the following formula:

$$m_{18} = \sum_{j=18}^{22} m_{18,j}$$
$$m_{22} = \sum_{j=18}^{22} m_{22,j}$$

After reaching the absorbing state, the state is transferred to threshold state(state 18 or state 22) by spot trading, and state is transferred from one state to another until it reaches one of the absorbing states. We can regard these movements as the Markov chain comprising 2 states, A and B, as shown in Figure 12.

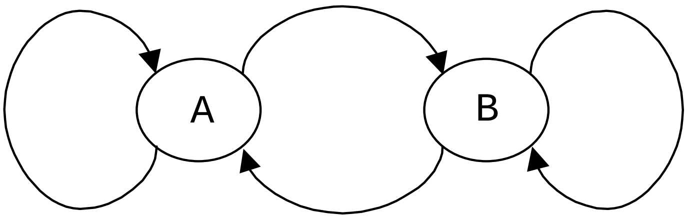


Figure 12: Markov chain of state A and B

We calculate initial probability distribution $\pi = (\pi_A \quad \pi_B)$, and then calculate T_m, the average interval of spot trading, by the use of the following formula:

$$T_m = m_{18}\pi_A + m_{22}\pi_B$$

Table 4 shows the result of calculation for spot model as well as spot and future model. The transition probability matrix of spot model (threshold state 22) and spot and futures model (threshold states 18) at that time is as follows, where the latter model does not consider rollovers that has little influence to the results.

[Transition probability matrix P for spot model]

$$\mathbf{P} = \left(\begin{array}{c|cc} 0.2909 & 0.3553 & 0.3538 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \quad (25)$$

where the first row represents the transition probability to the state 22, the second row represents the transition probability to the state A, the third row represents the transition probability to the state B.

[Transition probability matrix P for spot and futures model]

$$\mathbf{P} = \left(\begin{array}{ccccc|cc} 0.3017 & 0.2269 & 0.0941 & 0.0260 & 0.0039 & 0.3471 & 0.0005 \\ 0.2225 & 0.3005 & 0.2267 & 0.0943 & 0.0264 & 0.1252 & 0.0045 \\ 0.0988 & 0.2217 & 0.2993 & 0.2263 & 0.0948 & 0.0277 & 0.0314 \\ 0.0250 & 0.0998 & 0.2208 & 0.2982 & 0.2259 & 0.0032 & 0.1271 \\ 0.0031 & 0.0255 & 0.1007 & 0.2198 & 0.2971 & 0.0003 & 0.3536 \\ \hline 0 & & \cdots & & 0 & 1 & 0 \\ 0 & & \cdots & & 0 & 0 & 1 \end{array}\right) \quad (26)$$

where the first row represents the transition probability to the state 18, the second row represents the transition probability to the state 19,..., the 5th row represents the transition probability to the state 22, the 6th row represents the transition probability to the state A, the 7th row represents the transition probability to the state B.

Table 4: Comparison of the average interval of spot trading measured in the occurrences of cash flow

	spot model	spot and future model
the average interval of spot trading	1.41	4.41

Table 4 shows that spot is traded once per 1.4 occurrences of cash flow in the case of spot model, whereas spot is traded once per 4.4 occurrences of cash flow in the case of spot and future model. This illustrates that trading frequency would be reduced to 1/3 .

5. Conclusion

5.1 Average cost and robustness

Numerical experiments shows that threshold policy diminishes the average cost compared with negligence policy and constant policy in the cash management. Practitioners uses threshold policy through their experience. Our experiments support practitioners' this knowledge.

We then confirmed that the average cost of spot model is almost the same as the cost of spot and future model. Fund managers tend to refrain from using futures because of the larger tracking errors from the benchmark, and also because of the rollover cost. But our result may reduce their concern.

In practice the lower limit of cash position is often stipulated in the internal regulation or some agreement. In this case the threshold value may be larger than the optimal level, thus the average cost of spot model will be bigger than the cost for spot and future model.

The average cost of spot model varies considerably with the threshold value ,on the other hand the cost of spot and future model does not vary widely. This robustness of spot and future model means that the average cost does not change considerably even if the estimated threshold value is different from the optimal threshold value. The advantage is attractive to fund managers.

5.2 Average interval of spot trading

Next we confirmed that the average interval of spot trading in a spot and future model is longer than that in a spot model. Cost of office work is difficult to measure, but the trading frequency has strong relationship with the cost of office work. Reduction of trading frequency means reduction of office cost.

5.3 Further development

There remain several topics to be studied further. We does not proved that the threshold strategy is the optimal strategy in the discrete framework. We assumed that the cash flow occurs periodically. It is interesting to loosen the assumption. We did not assume the existence of fixed cost other than trading cost. If there exists some fixed cost, the treatment of threshold would be different. We assumed that the distribution of cash flow is independent. If the sequence of cash flow has some serial correlation, our model will have the room for improvement.

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(This paper is the translation from the Japanese version by Ken Sugita , an AFIR member and a coworker of Mr. Yamashita.)