The Effect of Risk Diversification on Price

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Abstract

The diversification property requires that the insurer's reduction in uncertainty arising out of risk diversification be accompanied by a reduction in the insurer's certainty equivalent price for the portfolio. Diversification is shown to alter the insurer's aversion to risk by reducing its risk aversion parameter. For a portfolio consisting of uncorrelated Normally distributed exposures, the risk margin for each exposure subsequent to diversification is proportional to the variance of the damages. Prices after diversification satisfy the property of value additivity. For a portfolio of Normally distributed exposures, the completely diversified component of risk is shown to be irrelevant for risk pricing. This principle is adopted for other than Normal distributions, and is used to demonstrate that the certainty equivalent price for any exposure after taking diversification into consideration is $\mu + \lambda\rho\sigma$, where μ and σ are the mean and standard deviation of the outcomes, ρ is the correlation coefficient between the exposure and the portfolio, and λ is a constant for all exposures within the portfolio. Provided that the insurer's expenses are not excessive, the insurer's price reduction obtained through diversification permits the insurer and the insured to find a mutually acceptable price for the transfer of risk. Market supply and demand curves, and hence the market price, are derived from the certainty equivalent prices for the market participants. In a perfectly competitive market, the insurer's fixed overhead expense is not included in the insurer's price. For investments in securities, the risk pricing model is shown to be consistent with the Capital Asset Pricing Model.

1. Introduction

This paper continues the examination of insurance risk pricing commencing in the author's "The Diversification Property." The development of the risk pricing model was based on the property that the certainty equivalent price for the combination of two perfectly correlated exposures is equal to the sum of their certainty equivalent prices, whereas diversification across less than perfectly correlated exposures reduces risk and hence reduces the certainty equivalent price for the portfolio. This paper will examine how the reduction to the risk margin for the portfolio is reflected in the prices for the individual exposures.

The standard approach for examining the price for a transfer of risk in a utility theory framework is to assume that the insurer has a lower aversion to risk than the insured, as described in Borch (1990). Assuming that an insurer has fixed policy transaction expenses of s, a mutually acceptable price P will exist for the transfer of damages X provided that the insurer's certainty equivalent price of $P_2(X + s)$, or $P_2(X) + s$, is no greater than the insured's certainty equivalent price $P_1(X)$. Any price P such that $P_2(X) + s <= P <= P_1(X)$ is acceptable to both participants in the transaction.

One shortcoming of the approach described above is that the insured may be less risk averse than the insurer. Even when the insurer is less risk averse, transaction expenses may result in a higher certainty equivalent price for the insurer than for the insured. That is, it may be the case that $P_2(X) < P_1(X)$ but that $P_2(X) + s > P_1(X)$. In this situation, a mutually acceptable price will not exist.

A second deficiency of the standard utility theory approach is that it does not explicitly recognize the insurer's ability to diversify its risk over a large number of independent or partially correlated exposures. Suppose that the insurer is able to obtain a premium $P = P_2(X) + s$ for each exposure. In this situation, the insurer's expected risk and return for each policy would be in balance for each individual exposure. However, the insurer's diversification of its risk over a portfolio of exposures would enable it to achieve an aggregate return in excess of its aggregate risk. In the absence of competition or regulation, there would be no impetus for the insurer to reduce its price since its original price is acceptable to both participants. In this situation, diversification would have no effect on the insurer's price. Or, competition may require the insurer to reduce its price until its aggregate risk and return are in balance. This would represent a situation of perfect diversification. A third possibility is that the insureds in a particular market segment may be unwilling to accept the price the insurer establishes without taking its risk diversification into account. This would be the case if the certainty equivalent prices for the entire population of insureds were lower than the insurer's proposed price. However, these insureds may be willing to accept a price that reflects the insurer's risk diversification within the individual market segment. Consequently, price reductions corresponding to diversification of risk across the insurer's entire portfolio may not be required. In this situation, the insurer could establish its risk margins for Wisconsin Homeowners exposures independently from its risk margins for Oregon Personal Automobile coverage.

Initially, this discussion will consider risk pricing solely from the perspective of the two participants in the transaction. No consideration will be given to external influences, such as market competition or regulation. The paper will examine how the reduction in the risk across the insurer's portfolio or market segment is reflected in reductions to the insurer's certainty equivalent price for the individual exposures. This will lead to an alternative approach for evaluating the existence of a mutually acceptable price for the insurance transaction. Finally, the discussion will describe how market prices reflecting the diversification of risk can be established.

2. Risk Diversification for the Normal Distribution

The certainty equivalent price after diversification will initially be considered for exposures having the Normal distribution. Consider an insurer with a portfolio of exposures X_i, each of which is Normal with $X_i \sim N(\mu_i, \sigma^2_i)$. The X_i are not required to be independent of one another. According to Hogg and Craig (1995, p. 228), the sum of Normal random variables is Normal, so that the portfolio W consisting of the sum of the X_i from 1 to n, i.e., $W = \Sigma^n X_i$, also has the Normal distribution with:

$$E(W) = \Sigma^n \mu_i$$

and:

$$V(W) = Cov(W,W) = Cov(W,\Sigma^n X_i) = \Sigma^n Cov(W,X_i)$$

Define ρ_i as the correlation coefficient of X_i with the portfolio W. Each ρ_i is less than or equal to 1 in absolute value, with:

$$\rho_i = Cov(X_i,W) / (\sigma_i \sigma_W)$$

or:

$$Cov(W,X_i) = \rho_i \sigma_i \sigma_W$$

so that:

$$V(W) = \Sigma^n \rho_i \sigma_i \sigma_W$$

Since $V(W) = \sigma^2_W$, this implies that:

$$\sigma_W = \Sigma^n \rho_i \sigma_i$$

The author has demonstrated in "The Diversification Property" that for Normally distributed exposures, the risk margin for W at the insurer's certainty equivalent price for the portfolio is proportional to the standard deviation. The risk margin for the portfolio is $\lambda\sigma_W$, with:

$$\lambda\sigma_W = \lambda\Sigma^n \rho_i \sigma_i$$

where λ is a risk aversion parameter corresponding to the insurer's risk aversion factor of $e/c - 1$ in the risk pricing model (Schnapp, equation (3)).

Since each X_i is Normal, the risk margin for each individual exposure prior to diversification is $\lambda\sigma_i$. As a result, the sum of the risk margins for the exposures in the portfolio prior to diversification is:

$$\lambda\Sigma^n \sigma_i$$

Recall that the diversification property requires only that diversification reduce price. It does not specify how the reduction in price for the portfolio is reflected in the prices for the individual exposures. The insurer is free to establish the price it charges for each exposure, provided that its price is less than the certainty equivalent price for the insured. However, the issue to be addressed here is the insurer's certainty equivalent price rather than its actual price. Based on the result shown above, diversification across Normal exposures reduces the insurer's total risk margin from $\lambda\Sigma^n\sigma_i$ *to* $\lambda\Sigma^n\rho_i\sigma_i$. This suggests that diversification reduces the risk margin for each individual exposure X_i from $\lambda\sigma_i$ to $\lambda\rho_i\sigma_i$, where the diversification factor ρ_i modifying the risk margin is the correlation coefficient of the exposure X_i with the portfolio W. However, this result cannot be demonstrated solely from the very minimal conditions of the diversification property itself. Instead, this discussion will require that all certainty equivalent prices be based on a single pricing model, rather than a risk pricing model for prices prior to diversification and a diversification model for prices subsequent to diversification. This implies that the standard deviation pricing formula (Schnapp, equation (2)) for Normally distributed exposures must also be valid for certainty equivalent prices after diversification. Consequently, the risk margin for X_i after diversification must be of the form $\lambda'\sigma_i$ for some λ'. Since the value λ' may vary for each exposure X_i, λ' will be represented as λk_i. In total, the risk margin for the portfolio is $\lambda\Sigma^n k_i\sigma_i$, which must be equal to the value of $\lambda\Sigma^n\rho_i\sigma_i$ derived above. The only reasonable value for k_i would appear to be ρ_i. Additional support for this assumption will be provided below.

In effect, the diversification rule developed above reduces the insurer's risk aversion parameter for exposure X_i from λ to $\lambda\rho_i$. Consequently, the insurer's certainty equivalent price after diversification for exposure X_i can still be derived from the standard deviation pricing formula, with $\lambda\rho_i$ replacing λ. Equivalently, this corresponds to a revised risk aversion factor $e/c - 1$ in the risk pricing model, so that the results from "The Diversification Property" still apply.

Based on the assumption that ρ_i is the appropriate diversification factor, the insurer has two certainty equivalent prices for each exposure. The certainty equivalent price for X_i prior to considering the effect of diversification is:

$$P(X_i) = \mu_i + \lambda\sigma_i$$

The insurer's certainty equivalent price for X_i subsequent to diversification, $P(X_i \mid W)$, is:

(1) $$P(X_i \mid W) = \mu_i + \lambda\rho_i\sigma_i$$

Since ρ_i is no greater than 1, diversification will never increase the price for an exposure.

One consequence of the reduction to the insurer's risk aversion parameter for X_i from λ to $\lambda\rho_i$ is that the parameter may differ for each exposure even though the insurer's inherent aversion to risk, λ, is unchanged. Consequently, the insurer's risk aversion parameter λ is relevant only with respect to the pricing of the entire portfolio and not to the pricing of the individual exposures within the portfolio. This also means that an insurer's risk aversion parameter cannot be evaluated solely from the price for an individual exposure. For example, consider two exposures X_1 and X_2, each of which is identically distributed as $N(\mu,\sigma^2)$ but having different diversification factors ρ_1 and ρ_2 with respect to the portfolio W. Even though the two distributions are identical, the certainty equivalent prices after diversification will differ for the two exposures. Furthermore, diversification across a different portfolio of exposures may produce a different set of certainty equivalent prices. In the special case of an exposure with a diversification factor between –1 and 0, the certainty equivalent price for the exposure will be less than its expected value, similar to what would be observed if the insurer had a negative aversion to risk.

For the portfolio $W = \sum^n X_i$, the diversification formula developed above results in additive prices for combinations of individual exposures after diversification. To show this, let $P(X_i \mid W)$ represent the price after diversification for each exposure X_i, so that $P(W) = \sum^n P(X_i \mid W)$. If Y is a combination of two exposures in the portfolio, say $Y = X_{n-1} + X_n$, then it can be demonstrated that the price for Y after diversification is equal to the sum of its components:

$$P(Y \mid W) = P(X_{n-1} \mid W) + P(X_n \mid W)$$

To show this, define the modified portfolio Z as $\sum^{n-2} X_i + Y$. Since $W = Z$, $P(W)$ is equal to $P(Z)$. Also, for $i < n - 1$, the correlation coefficient of X_i with Z is identical to the correlation coefficient of X_i with W. As a result, the certainty equivalent price after diversification for X_i, where $i < n - 1$, as an element of the portfolio Z, is identical to its original value, i.e., $P(X_i \mid Z) = \mu_i + \lambda\rho_i\sigma_i$, which is also equal to $P(X_i \mid W)$. Since all of the components of $P(W)$ and $P(Z)$ with $i < n - 1$ are identical, this requires that $P(Y \mid Z)$ must be equal to $P(X_{n-1} \mid W) + P(X_n \mid W)$.

One interesting aspect of the pricing formula $P(W) = \Sigma^n P(X_i \mid W)$ is its similarity to the formula for perfectly correlated exposures. It can be demonstrated that this implies that the completely diversified component of the risk is irrelevant for pricing. This can be shown by expressing X_i as $g_iW + U_i$, where g_i is defined as $\rho_i\sigma_i / \sigma_W$, so that U_i is uncorrelated with W. Mathematically, g_iW is the projection of the vector X_i on the vector W, where the covariance function corresponds to the inner product operator. The remainder, the vector U_i, is perpendicular to W. It should be observed that the decomposition of X_i into two perpendicular components requires that the diversification factor k_i must be equal to ρ_i.

Since $\Sigma^n g_i = 1$, then:

$$W = \Sigma^n X_i = \Sigma^n (g_iW + U_i) = W + \Sigma^n U_i$$

which implies that $\Sigma^n U_i$ is identically equal to 0. Consequently, the uncertainty inherent in the U_i is completely eliminated by diversification across the portfolio. In essence, the exposures U_i add nothing to the uncertainty of W, and hence add nothing to the risk margin for the portfolio.

Note that:

$$E(X_i) = E(g_iW) + E(U_i)$$

or, equivalently:

$$\mu_i = g_i\mu_W + E(U_i)$$

Also:

$$P(X_i \mid W) = \mu_i + \lambda\rho_i\sigma_i$$

so that:

$$P(X_i \mid W) = g_i\mu_W + \lambda\rho_i\sigma_i + E(U_i)$$

However, the certainty equivalent price for g_iW is:

$$P(g_iW) = g_iP(W) = g_i\mu_W + (\rho_i\sigma_i / \sigma_W) \times (\lambda\sigma_W) = g_i\mu_W + \lambda\rho_i\sigma_i$$

so that:

$$P(X_i \mid W) = P(g_iW) + E(U_i) \quad (2)$$

Hence, the risk margin for X_i after diversification is proportional to the risk margin for W:

$$P(X_i \mid W) – E(X_i) = P(g_iW) – E(g_iW) = g_i(P(W) – E(W)) \quad (3)$$

Equation (3) shows that the diversified component of the risk, represented by the uncertainty in U_i, is irrelevant to the risk margin for X_i after diversification. This result provides an explanation for the original formula that initiated this discussion, $P(W) = \Sigma^n P(X_i \mid W)$. Based on the observations that $P(X_i \mid W) = P(g_iW) + E(U_i)$ and $\Sigma^n E(U_i) = E(\Sigma^n U_i) = 0$, the formula for $P(W)$ simply represents the summation of the certainty equivalent prices for the perfectly correlated exposures g_iW:

$$\begin{aligned} P(W) &= \Sigma^n P(X_i \mid W) \\ &= \Sigma^n (P(g_iW) + E(U_i)) \\ &= \Sigma^n P(g_iW) \end{aligned}$$

Equation (2) also can be used to justify the value additivity property, since:

$$X_{n-1} + X_n = (g_{n-1} + g_n)W + (U_{n-1} + U_n)$$

so that:

$$P(X_{n-1} + X_n \mid W) = (g_{n-1} + g_n)P(W) + E(U_{n-1} + U_n)$$

which is also equal to $P(X_{n-1} \mid W) + P(X_n \mid W)$.

3. Diversification Over Uncorrelated Normal Exposures

Consider the special case in which the exposures are uncorrelated or independent. Each exposure is again assumed to have the Normal distribution. The correlation coefficient ρ_i can be evaluated as:

$$\rho_i = Cov(W,X_i) / \sigma_i\sigma_W = Cov(\Sigma^n X_j,X_i) / \sigma_i\sigma_W = V(X_i) / \sigma_i\sigma_W = \sigma_i / \sigma_W$$

Using this result, the risk margin after diversification for exposure X_i is:

$$\lambda\rho_i\sigma_i = \lambda\sigma^2_i / \sigma_W$$

If the portfolio is fixed so that σ_W can be considered to be constant, the risk margin for each X_i can be expressed as $\kappa\sigma^2_i$, where $\kappa = \lambda/\sigma_W$ is constant for all i. Under these conditions, the certainty equivalent price is equivalent to Pratt's variance pricing formula, as described in Robison and Barry (1987, p. 40):

(4) $$P = \mu + \kappa\sigma^2$$

Miccolis (1997) provides an example of the use of this formula in the development of increased limits factors for liability insurance.

While the variance pricing formula is valid for a portfolio consisting of uncorrelated Normal exposures, it is not appropriate when the exposures are correlated. For example, consider a set of exposures X_i taken from a common Normal distribution with the exposures perfectly correlated with one another. If the variance pricing formula were valid in this situation, the certainty equivalent premium to insure a single exposure X_i would be identical for all i:

$$P_i = \mu_X + \lambda\sigma^2_X$$

Consequently, the premium to insure all n exposures would be:

$$\Sigma^n P_i = n\mu_X + \lambda n\sigma^2_X$$

This result can be compared to the premium for the portfolio $Z = \Sigma X_i$. Z has an expected value of $\mu_Z = n\mu_X$ and a variance of $\sigma^2_Z = n^2\sigma^2_X$. The variance pricing formula would indicate a premium for Z of:

$$P_Z = \mu_Z + \lambda\sigma^2_Z$$

After substitution, this becomes:

$$P_Z = n\mu_X + \lambda n^2\sigma^2_X$$

The premium P_Z for the portfolio exceeds the sum of the individual premiums $\Sigma^n P_i$. Since this is inconsistent with the diversification property, the variance pricing formula is not valid under these circumstances.

4. The Mutually Acceptable Price

The certainty equivalent price P represents the amount that an individual would be willing to pay in order to transfer the exposure to uncertain damages. At this price, the individual is indifferent between retaining the exposure X or transferring the exposure for the price P. It will be assumed that the price for any risk transfer differs from the expected value of the outcomes solely due to the presence of uncertainty. The purpose for this assumption is to eliminate subjective elements as pricing considerations. Given this condition, the individual would not be willing to accept a price less advantageous than the certainty equivalent price.

A comparison of the certainty equivalent prices for the insurer and the insured has two possible outcomes. If the insurer's certainty equivalent price for the exposure is less than that of the insured, then any price between these two values will be acceptable to both participants. However, if the insurer's certainty equivalent price exceeds that for the person exposed to the uncertain outcome, a mutually acceptable price for the insurance transaction will not exist. An insurer may have a higher certainty equivalent price for the transaction due to a greater aversion to risk than the exposed individual, or due to the expenses it incurs in the transaction. This will be examined in the following discussion.

Consider a risk transfer between an insured and an insurer, each of which has an identical aversion to risk. To simplify the situation, all transaction expenses will initially be assumed to be zero. For the insurer, the certainty equivalent price P_2 is the solution to the equation:

$$EU_2(w_2 - X + P_2) = U_2(w_2) = 0$$

where U_2 is the insurer's utility function (in the sense of the risk pricing model) and w_2 is its initial wealth. Similarly, the certainty equivalent price P_1 for the insured can be expressed as the solution to the equation:

$$EU_1(w_1 - X) = U_1(w_1 - P_1) = 0$$

Letting $w_3 = w_1 - P_1$, this can be restated as:

$$EU_1(w_3 - X + P_1) = U_1(w_3) = 0$$

Since the two participants have identical aversion to risk, let $U_1 = U_2 = U$. Since the utility function represents the value of gains or losses relative to the initial wealth, the specific values of w_2 and w_3 are irrelevant. Replacing each with w gives:

$$EU(w - X + P_1) = 0 = EU(w - X + P_2)$$

Since each of the two equations has a unique solution, this gives the result $P_1 = P_2$, that is, the two participants' certainty equivalent premiums are identical. In this situation, a mutually acceptable price exists.

Next, consider the more realistic situation in which the insurer also incurs fixed policy transaction expenses of e. In order for the transaction to be acceptable, the insurer must obtain an appropriate price for the uncertain exposure of $Y = X + e$ rather than for just the exposure X. The insurer's certainty equivalent price for Y is $P(X + e) = P(X) + e = P_2 + e$, where $P_2 = P_1$ is the value determined above. In this situation, the insurer's certainty equivalent price would be $P_1 + e$. Since the insured is not willing to pay more than P_1, a mutually acceptable price will not exist.

When the ability of the insurer to diversify its exposure to risk is taken into consideration, the conclusion from the previous paragraph may no longer hold. Since diversification acts to reduce uncertainty, it also reduces the insurer's certainty equivalent price. Provided that the insurer's certainty equivalent price $P_2(X + e \mid W)$ after diversification over the portfolio W is no greater than the insured's certainty equivalent price $P_1(X)$, a mutually acceptable price for the insurance transaction will exist. This will be considered in more detail in the following example.

Consider the portfolio W which consists of n exposures with $X_i \sim \mathrm{N}(\mu_i, \sigma^2_i)$ with transaction expenses of e_i for exposure X_i. The insurer's certainty equivalent price for insuring exposure $Y_i = X_i + e_i$ is:

$$P_2(Y_i \mid W) \;=\; P_2(X_i + e_i \mid W) \;<=\; P_2(X_i \mid W) \;+\; P_2(e_i)$$

If the expenses e_i are constant, then $P_2(Y_i \mid W) \;=\; P_2(X_i \mid W) + e_i$. Also, since X_i is Normal:

$$P_2(Y_i \mid W) = \mu_i + e_i + \lambda_2 \rho_i \sigma_i$$

where ρ_i is the correlation coefficient of X_i with the portfolio W. For the sake of illustration, the exposures will be assumed to be independent. This reduces the formula above to:

$$P_2(Y_i \mid W) = \mu_i + e_i + \lambda_2 \sigma^2_i \,/\, \sigma_W$$

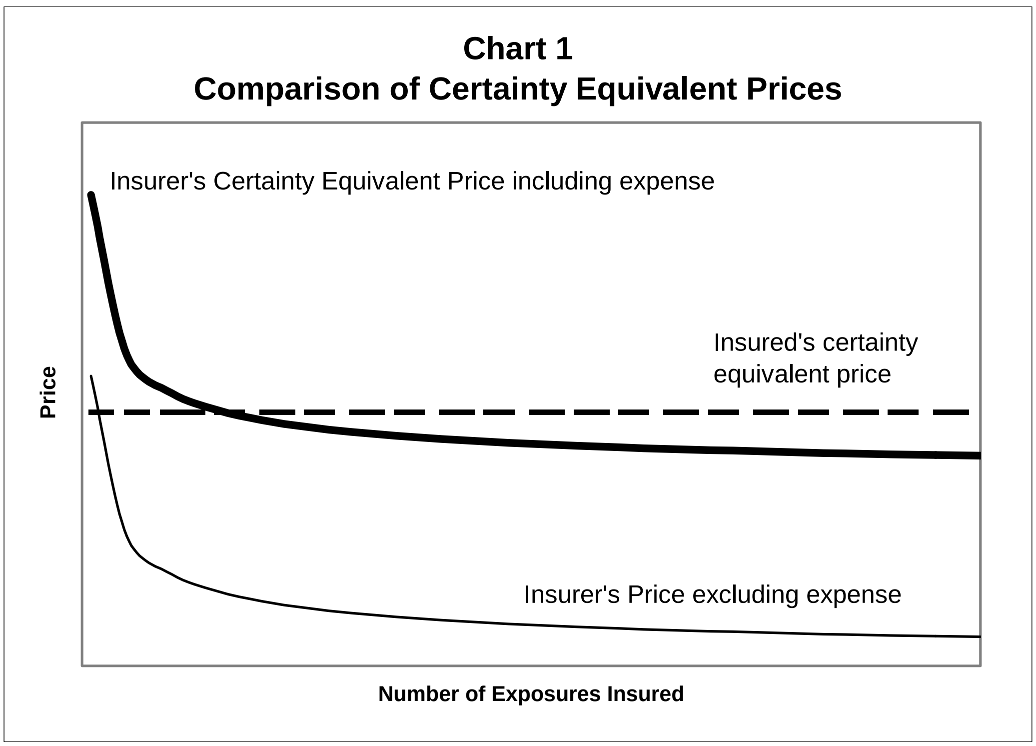
Furthermore, if the standard deviations are identical, $\sigma_i = \sigma$, this can be restated as:

(5) $$P_2(Y_i \mid W) = \mu_i + e_i + \lambda_2 \sigma / \sqrt{n}$$

This result can be compared to the insured's certainty equivalent price of:

(6) $$P_1(X_i) = \mu_i + \lambda_1\sigma$$

Based on equation (5), as the number of exposures in the portfolio increases, the insurer's certainty equivalent price $P_2(Y_i \mid W)$ for exposure Y_i will decrease. If, for some n, this price is less than the insured's certainty equivalent price, then a mutually acceptable price for the insurance will exist, as illustrated in the following chart.



In the example given above, a mutually acceptable price for the transaction will exist provided that:

$$\mu_i + e_i + \lambda_2\sigma / \sqrt{n} \;<=\; \mu_i + \lambda_1\sigma$$

Eliminating μ_i from both sides of this formula, this shows that a mutually acceptable price will exist if the insurer's combined risk margin and expense loading is less than or equal to the risk margin for the insured:

$$e_i + \lambda_2\sigma / \sqrt{n} \ <= \ \lambda_1\sigma$$

This requires, first, that the insurer's expenses must be less than the risk margin for the insured:

(7) $$e_i < \lambda_1\sigma$$

and second, that the insurer is able to diversify its exposure so that its risk margin is no greater than the difference between the insured's risk margin and the insurer's expenses:

(8) $$\lambda_2\sigma / \sqrt{n} \ <= \ \lambda_1\sigma - e_i$$

5. Insurer Pricing for Distributions Other Than Normal

The development of the insurer's certainty equivalent price for individual exposures in a portfolio as described above requires that each exposure have the Normal distribution. The following discussion considers methods for pricing exposures that are not Normally distributed.

The price for any exposure within a portfolio (or market segment) must be developed in two steps. First, the insurer must determine an adequate return for the portfolio as a whole. The portfolio would consist of a relatively homogeneous set of exposures with reasonably similar risk characteristics, including mean, variance, distribution of outcomes, and insurance coverage. For example, the portfolio may consist of all of the insurer's exposures for a single coverage within one state, such as Private Passenger Automobile Bodily Injury Liability exposures for Virginia evaluated at basic policy limits. The insurer can determine an appropriate risk margin for the portfolio in aggregate by applying the risk pricing model (Schnapp, equation (3)) to the historical or simulated experience of the portfolio.

Second, the insurer must determine prices for the individual exposures in the portfolio. Two methods will be considered. The first method treats each exposure as though it were from the Normal distribution. The second method allocates the portfolio's aggregate risk margin to the individual exposures. The first method recognizes the differences in risk between each exposure, while the second method does not.

Define the portfolio W as:

$$W = \Sigma Y_i = \Sigma(X_i + e_i)$$

Suppose that the insurer has used the risk pricing model to determine that its risk margin for the portfolio is M. The insurer's certainty equivalent price for the portfolio would be:

$$P(W) = \mu_W + M$$

For a Normal distribution, M is equal to $\lambda\sigma_W$, where λ is the insurer's risk aversion factor and σ_W is the standard deviation of the outcomes for the portfolio. The value of λ is a constant that is independent of the parameters of M. However, if the distribution of the portfolio is not Normal, λ_W can be defined as M/σ_W. In this situation, the parameter λ_W may depend on the distribution. For convenience, λ_W will be denoted as λ so that $P(W)$ can be expressed as:

$$P(W) = \mu_W + \lambda\sigma_W$$

Even though the individual exposures within W may not have the Normal distribution, the first pricing method will apply the standard deviation pricing formula to each exposure Y_i. This allows the price for Y_i to be established as:

$$P(Y_i) = \mu_i + \lambda\rho_i\sigma_i$$

where μ_i and σ_i refer to the distribution of Y_i and ρ_i is the correlation coefficient of Y_i with W. This develops a risk margin for Y_i that takes into consideration the variability of its outcomes and its correlation with the portfolio. Also, the sum of the premiums for the individual exposures Y_i equals the premium for the entire portfolio.

If the expense e_i is a fixed amount for each exposure i, then since $Y_i = X_i + e_i$:

$$P(Y_i) = P(X_i) + e_i$$

so that:

$$P(Y_i) = \mu_i + \lambda\rho_i\sigma_i + e_i$$

where μ_i and σ_i now refer to the distribution of X_i rather than Y_i and ρ_i is the correlation coefficient of X_i with W.

Typically, the expenses for each exposure are variable rather than fixed. In general, expenses can be decomposed into those that are proportional to the premium, such as commissions and premium taxes, and those that are proportion to the damages, such as loss adjustment expenses. The expense ratios c and d will be assumed to be identical for every policy in the portfolio so that:

$$e_i = cP(Y_i) + dX_i \text{ for all } i$$

Since:

$$Y_i = X_i + e_i$$

then:

$$Y_i = X_i + cP(Y_i) + dX_i$$

or:

$$Y_i = (1 + d)X_i + cP(Y_i)$$

This implies that:

$$P(Y_i) = (1 + d)P(X_i) + cP(Y_i)$$

or:

$$P(Y_i) = (1 + d)P(X_i) / (1 - c)$$

so that:

$$P(Y_i) = (1 + d)(\mu_i + \lambda\rho_i\sigma_i) / (1 - c)$$

Provided that X_i already includes all loss adjustment expenses, this formula can be simplified by setting d to 0. Also, if the exposures are independent, then $\rho_i = \sigma_i / \sigma_W$ so that:

$$P(Y_i) = (\mu_i + \lambda\sigma^2_i / \sigma_W) / (1 - c)$$

If the exposures have identical standard deviations, this can be reduced to:

$$P(Y_i) = (\mu_i + \lambda\sigma_W / n) / (1 - c)$$

or:

$$P(Y_i) = (\mu_i + M / n) / (1 - c)$$

where n is the number of exposures insured. In this situation, each insured contributes a fixed amount, M / n, to the insurer's total risk margin of M. Summarized across the portfolio, this is:

$$P(W) = (\Sigma^n\mu_i + M) / (1 - c)$$

or:

$$P(W) = \Sigma^n\mu_i + M + cP(W)$$

This result states that the portfolio premium consists of the expected damages $\Sigma^n\mu_i$ plus the expected profit M plus the expenses $cP(W)$. Notice that the price for an individual exposure can be expressed as:

$$P(Y_i) = \mu_i + m_i + cP(Y_i)$$

where m_i is the selected risk margin for the exposure. Under the assumption that all exposures are independent and have identical standard deviations, the technique described above develops risk margins of $m_i = M / n$.

The second method for determining the price for individual exposures allocates the aggregate risk margin M to each exposure in proportion to the premium. This method disregards the differences in risk between exposures. For exposure Y_i, select the risk margin m_i as:

$$m_i = rP(Y_i)$$

where:

$$r = M / P(W)$$

This results in:

$$P(Y_i) = \mu_i + rP(Y_i) + cP(Y_i)$$

or:

$$P(Y_i) = \mu_i / (1 - r - c)$$

This is the standard pricing formula used for Property/Casualty ratemaking.

6. The General Formula for Certainty Equivalent Prices

The earlier results in this discussion have been based on a particular conjecture regarding the certainty equivalent price for Normally distributed exposures within a portfolio. That conjecture has been expressed as $k_i = \rho_i$. No comparable result has been developed for distributions other than the Normal. The following discussion will introduce an additional assumption that will be used to develop the certainty equivalent price for any exposure within a portfolio.

Consider again the decomposition of each X_i into $g_iW + U_i$ as discussed above, where $g_i = \rho_i\sigma_i / \sigma_W$ and U_i is uncorrelated with W. Since $\sum^n U_i$ is equal to 0, the uncertainty inherent in the exposures U_i is completely eliminated by diversification across the portfolio. Consequently, the uncertainty in the U_i adds nothing to the risk margin for the portfolio. The principle to be adopted is that any uncertainty that is completely eliminated by diversification across the portfolio, such as the uncertainty in the U_i, is not only irrelevant to the risk margin of the portfolio but is also irrelevant to the risk margin of any exposure within the portfolio. In essence, U_i can be treated as a constant. This implies that the risk margin for X_i after diversification is identical to the risk margin for g_iW, so that:

$$P(X_i \mid W) - E(X_i) \;=\; P(g_i W) - E(g_i W) \;=\; g_i(P(W) - E(W))$$

When W has a Normal distribution, the risk margin $P(W) - E(W)$ is equal to $\lambda\sigma_W$, where λ is constant regardless of the parameters of W. Otherwise, the risk pricing model can be used to determine the risk margin M for the single exposure W. In this situation, define a value $\lambda = \lambda_W$ such that $\lambda\sigma_W = M$. Then:

(9) $$P(X_i \mid W) - E(X_i) \;=\; g_i M \;=\; g_i \lambda \sigma_W$$

Since $g_i = \rho_i\sigma_i / \sigma_W$, the risk margin for X_i after diversification can be reduced to $\lambda\rho_i\sigma_i$. Hence, the certainty equivalent price for X_i after diversification for any distribution is:

(10) $$P(X_i \mid W) = \mu_i + \lambda\rho_i\sigma_i$$

Since $E(X_i) = E(g_i W) + E(U_i)$, this can also be expressed as:

$$P(X_i \mid W) \;=\; P(g_i W) + E(U_i)$$

7. Market Price

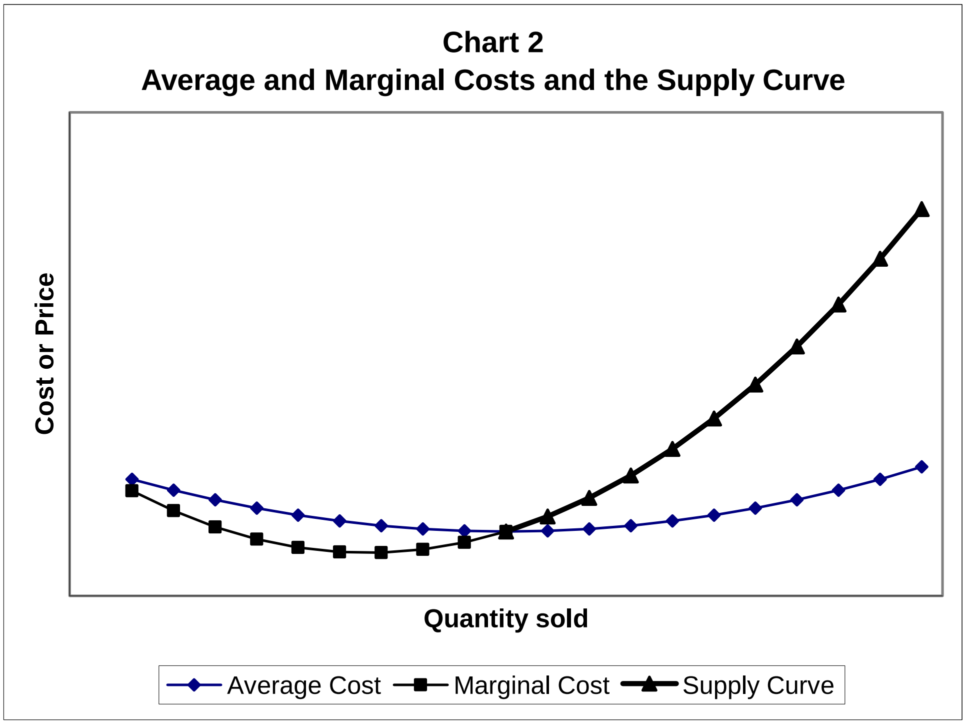
The risk pricing model provides a means for establishing market prices for the transfer of insurance liabilities without consideration of the cost of capital requirement of the Actuarial Standards Board (1997) as described in Actuarial Standard of Practice No. 30. This can be accomplished by applying the model to an analysis of market supply and demand.

The first step in the analysis is to develop a demand function for insurance. Suppose that the distribution of risk aversion parameters for the individuals exposed to a particular set of perils is known. Since the risk aversion parameter determines each individual's certainty equivalent price, the number of policies insuring these perils that would be purchased at each price can be established. This information represents the market demand curve.

The next step is to determine the supply curve for each insurer under conditions of perfect competition. These conditions mean that the insurer has no control over the market price and that identical exposures would be charged the same premium. According to Thompson and Formby (1993), the supply curve is based on the insurer's cost function. The insurer's total cost for any particular number of exposures insured consists of the sum of the expected damages, the fixed overhead expense, the transaction expenses, and the insurer's cost of risk. This is simply the insurer's certainty equivalent price for the portfolio of exposures plus its fixed overhead expense. The insurer's cost of risk for the market segment may be considered to include partially diversifiable process risk as well as undiversifiable parameter risk, as described in Meyers (1991). Note that while parameter risk is undiversifiable over a single market segment, the insurer can reduce this risk by diversification over several market segments.

The insurer's average cost curve can be considered to have a shape similar to the uppermost curve shown in Chart 1. However, as the number of exposures increases beyond the number currently insured, the insurer will eventually reach a point of diminishing returns, i.e., a point where its average costs begin to increase. These increased costs may arise due to the need to hire less experienced staff to handle the increased work loads, to less stringent underwriting criteria, or for other reasons. Regardless of the reason, the insurer's short-run average cost curve may be assumed to be "U" shaped. Given this shape for the average cost curve, the insurer's marginal cost curve will intersect the average cost curve at its minimum and remain above the average cost curve for all greater quantities sold, as indicated in Chart 2 below.

In order to maximize its profits, an insurer will insure that number of exposures such that its marginal cost matches its marginal revenue. This result is based on differentiation of the formula *Total Profit = Total Revenue – Total Cost*. Since the market is assumed to be perfectly competitive, the insurer's marginal revenue is equal to the market price. Hence, the insurer can maximize its profits by insuring the number of exposures indicated by its marginal cost curve. However, the insurer would not provide coverage along that portion of its marginal cost curve below its average cost curve, since this would result in the insurer obtaining a premium less than its average cost. Consequently, the insurer's supply curve is that portion of its marginal cost curve that lies above its average cost curve, as illustrated in Chart 2.



Recall that the risk pricing model includes transaction expenses but excludes the insurer's overhead expense. The supply curve helps to resolve the previously unaddressed issue of the inclusion or exclusion of the insurer's overhead expense in its price. According to the discussion above, the insurer's supply curve is a portion of its marginal cost curve. Since the insurer's marginal cost is independent of its overhead expense, the overhead expense is not explicitly included in the insurer's price. The only effect of the insurer's overhead expense is based on its effect on the average cost curve. Since the supply curve is above the average cost curve, any increase in the insurer's overhead expense will increase its average cost curve and consequently will truncate the lower portion of its supply curve.

Even though the insurer cannot include its overhead expense in its price, it can be demonstrated that the insurer will insure enough exposures to recover its overhead expense. To see this, assume that the market price is above the minimum point of the insurer's average cost curve. At this price, the insurer will be willing to participate in the market by insuring the number of exposures indicated by its supply curve. Since the market price is above its average cost curve, the insurer will recover all of its costs, including its overhead expense.

The final step in the analysis of the market price is to compare supply and demand for the market as a whole. Given the supply function for each insurer, the market supply curve is the simply the total of the number of exposures that each insurer is willing to supply at each particular price. Under conditions of perfect competition, the market price is the price at which the market supply and market demand curves intersect.

8. The Capital Asset Pricing Model

The diversification property can be shown to be consistent with the Capital Asset Pricing Model (CAPM). The one period CAPM formula states that:

$$E(r_c - r_f) = \beta(r_M - r_f)$$

where:

$$\beta = Cov(r_c, r_M) / \sigma^2_M$$

and r_c and r_M represent the returns on the security and the market, respectively. Stated in terms of the current price P_0 and the future price P_1 one period in the future, the return is defined as $r_c = P_1 / P_0 - 1$. The subtraction of the risk-free rate r_f from r_c and r_M in the CAPM formula represents an adjustment for the time value of money.

In order to apply the risk pricing model to a security, prices must be stated in terms of their present values. Since the risk pricing model permits future uncertainty to be discounted to present value at the risk-free rate, the present value of the future price is $PV(P_1) = P_1 / (1 + r_f)$. The real return on the security is then defined as $r = PV(P_1) / P_0 - 1$. The return r for the risk pricing model is related to the CAPM return r_c by the formula $r = r_c - r_f - r_c r_f$. Disregarding the relatively small third term, this shows that the return r for the risk pricing model corresponds to the $r_c - r_f$ term in the CAPM. Similarly, the definition of r_M for the risk pricing model corresponds to the $r_M - r_f$ term in the CAPM.

Over relatively brief periods of time, the distribution of returns from an individual security can be assumed to have a Normal distribution, as described in Neftci (1996). Initially, consider the situation in which the security is priced according to its own risk, without reference to its correlation with the market return. Based on the standard deviation pricing formula, the expected return $E(r)$ should equal the security's risk margin of $\lambda\sigma$, where λ represents the risk aversion of market participants, i.e., $E(r) = \lambda\sigma$. Similarly, the expected return $E(r_M)$ for the entire market should be the market's risk margin of $\lambda\sigma_M$, i.e., $E(r_M) = \lambda\sigma_M$. Taking into consideration the correlation of the security with the market, the risk margin for the security after diversification should be $\lambda\sigma\rho$, where the diversification factor ρ is the correlation coefficient of the return on the security with the market return. Since:

$$\rho = Cov(r, r_M) / (\sigma\sigma_M)$$

the risk margin for the security after diversification is:

$$E(r) = \lambda\sigma\rho = (\lambda\sigma)Cov(r, r_M) / (\sigma\sigma_M)$$

Referring back to the definition of β, this equation can be restated as:

$$E(r) = \beta\lambda\sigma_M$$

or:

$$E(r) = \beta E(r_M)$$

This differs from the standard CAPM formula due to the expected value operator on the right hand side of the equation. However, the formula suggests that the return r on the security can be considered to be the sum of the market return βr_M and an independent error term e, where $E(e) = 0$, so that:

$$r = \beta r_M + e$$

Taking the conditional expectation, a variation of the CAPM formula is obtained:

$$E(r \mid r_M) = \beta r_M$$

9. Conclusion

The purpose of this discussion has been to examine the effect on diversification on pricing. This is considered from the perspective of a cost of risk model. For less than perfectly correlated exposures, diversification reduces the risk per exposure and hence reduces the certainty equivalent price. The standard deviation pricing formula for Normal distributions has been used to demonstrate the validity of the variance pricing formula for certainty equivalent prices subsequent to diversification, provided that the exposures are independent. A general formula for the certainty equivalent price of any exposure within a portfolio has also been developed. Mutually acceptable prices for insurance will exist provided that the insurer's expenses are not excessive and the insurer diversifies its risk across a sufficiently large number of exposures. Market prices can be justified on the basis of market supply and demand. The risk pricing model has also been used to develop a variation of the Capital Asset Pricing Model formula.

The remaining issue with this analysis is the extent to which risk diversification should be reflected in insurer pricing decisions. Insurers may be able to offer prices below their customers' certainty equivalent prices by diversifying their risk over individual market segments rather than over their entire risk portfolios. If there is no need for the insurer to diversify its risk over its entire portfolio in order to provide an acceptable price to its insureds, it should be expected that the insurer would maximize its profits by charging the highest acceptable price. While each market segment would have its risk and return in balance, the insurer's aggregate return would be in excess of its aggregate risk, i.e., it would earn a real return over its entire portfolio. Even though each market segment could be profitable or unprofitable in any particular year, the insurer's profits in aggregate would be positive and relatively stable. This outcome appears to be consistent with what is generally observed for insurance companies, but may require further testing. However, any such study must recognize the influence of previous actuarial pricing techniques, particularly those that are based on a total return approach, on insurance company profitability.

One consequence of the theory of risk pricing based on the principle of risk diversification is that it eliminates the cost of capital as the primary consideration in risk analysis. Instead, the critical issue becomes the diversification of the insurer's risk across a portfolio of independent or partially correlated market segments in order to achieve the maximum return for a selected amount of risk. Markowitz (1991) has previous examined a similar issue with regard to investment portfolios. In addition, the risk pricing model suggests that if the insurer has an inadequate return on capital, it cannot necessarily increase its rates to obtain a greater return as the cost of capital approach might suggest. Instead, the insurer would improve its return on capital by increasing its return on premium. This can be accomplished by reducing the insurer's expenses or by participating in higher risk and hence more profitable market segments. The insurer can also reduce its aggregate risk by diversifying over additional independent market segments.

A second important consideration for the risk pricing model is the design of optimal reinsurance programs. For instance, an insurer may choose to insure more policies in a market segment than it has the financial capacity to accept. The insurer could address its capacity issue by recognizing its need for a larger risk aversion factor. However, if competition does not permit the insurer to raise its rates to fully offset its risk, the insurer may instead reinsure a portion of the exposure. Assuming that the reinsured exposure is small enough that the reinsurer is willing to accept a pro-rata portion of the exposure at the pro-rata price, the insurer may be able to price the policies using its normal risk aversion factor. In this situation, the insurer is borrowing capacity from a reinsurer in order to maintain competitive rates. Borch (1990) has examined the design of optimal reinsurance programs, but without recognition of the effect of risk diversification on price.

As a final observation, Brealey and Myers (1996, p. 146) review the real returns that have been obtained historically for various forms of investments. The risk pricing model can be used to evaluate the relationship of the return and the risk for investments. This examination would indicate whether investment returns correspond to complete diversification of risk across the entire market portfolio, as indicated by the CAPM, or whether diversification is less than complete.

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