The Diversification Property

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Abstract

A relationship between risk diversification and price is introduced as a requirement of expected utility theory. The diversification property, as applied to insurance, requires diversification across perfectly correlated exposures to have no effect on risk and hence to have no effect on the certainty equivalent price, whereas diversification across less than perfectly correlated exposures is assumed to reduce both the insurer's risk and its certainty equivalent price. For Normally distributed damages, the risk margin is shown to be proportional to the standard deviation of the damages. The diversification property requires a particular form for the utility function, which allows the certainty equivalent price to be determined for any distribution. The expected utility can be decomposed into the expected risk and return for the transaction, with the certainty equivalent price being the point at which the expected risk and return are in balance. Federal income taxes are shown to be irrelevant to insurance pricing. The analysis demonstrates that the certainty equivalent price for uncertain future cash flows can be evaluated using the present value of the outcomes, discounted at the risk-free rate. This method differs from the opportunity cost of capital approach in that it considers the entire distribution of outcomes rather than simply the expected outcome for each future period. The proposed risk pricing model is shown to be consistent with the Arbitrage Theorem.

1. Introduction

Risk pricing is closely related to the concept of diversification. One of the more familiar examples of this is the use of risk diversification by Markowitz (1991) as the basis for portfolio selection theory. This technique determines the portfolio of securities having the minimum risk for a selected expected return. Portfolio selection theory does not develop prices for individual securities; instead, prices are considered to be extraneous to the model. The model measures risk based on the expected variance of the return on the portfolio, recognizing the correlation of returns within the portfolio. Markowitz also considered the semi-variance *S*, the expected square of the negative returns, as an alternative measure of risk. While this tended to produce better portfolios than the variance *V*, he noted that "variance is superior with respect to cost, convenience, and familiarity…In an analysis based on *V*, only means, variances, and covariances must be supplied as inputs; whereas an analysis based on *S* requires the entire joint distribution of returns."

A related theory, the Capital Asset Pricing Model (CAPM), bases the competitive market price for individual securities on risk diversification. The expected return on an individual security is determined by its beta, which is a proxy for the covariance of the return on the security with the market return. The market price for each security can be estimated as the present value of the future cash flows of the company, discounted to present value at the CAPM rate of return.

A different approach to risk pricing is based on the concept of arbitrage. Arbitrage represents an opportunity to make a risk-free return greater than that of risk-free Treasury bills by taking positions in different assets. If these diversification opportunities do not exist, securities are considered to be correctly priced and the prices are described as being arbitrage-free. The Arbitrage Theorem states specific mathematical conditions which ensure that prices are arbitrage-free. The theorem provides a means for determining the present value of uncertain future cash flows through the use of synthetic probabilities. The shortcoming of the theorem is that it does not describe how to determine these probabilities. The Arbitrage Theorem is used in options pricing theory for developing prices for financial derivatives.

Property/Casualty insurance pricing techniques generally disregard the concept of risk diversification. For instance, increased limits pricing factors for liability insurance are based on a risk loading formula. Miccolis (1977) determines the price for each liability limit as $P = \mu + \lambda\sigma^2$, where μ is the expected severity at that coverage limit, σ^2 is the corresponding variance, and λ is an arbitrarily selected scale factor. Robison and Barry (1987) provide a development of the variance pricing formula using an expected utility theory argument.

Another risk pricing method commonly employed in Property/Casualty ratemaking is the discounted cash flow approach. These models fall into two categories: internal rate of return (IRR) and net present value (NPV) methods. IRR methods determine the discount rate for which the present value of the cash flows is zero. The resulting discount rate is then compared to the opportunity cost of capital for an investment of similar risk to determine whether the investment is worthwhile. In contrast, the NPV method discounts cash flows at the opportunity cost of capital. Those projects with positive NPV are considered worthwhile. Both methods generally require assumptions regarding the allocation and timing of inflows and outflows of corporate equity to individual policy or to business segment, as described in Feldblum (1992). For either method, the opportunity cost of capital is evaluated from the CAPM or from a review of market returns for investments of similar risk. These models are generally applied to individual market segments consisting of exposures having similar variability, i.e., having similar values of σ^2. For this reason, the variance pricing formula will not apply. In addition, these models address the issue of the present value of future risky cash flows, which the variance pricing formula does not. Discounted cash flow models and other cost of capital techniques have replaced return on premium methods for determining insurance profit margins.

This paper examines the price for a transfer of risk from the perspective of expected utility theory. The expected utility model is a general decision making technique used by economists to evaluate an individual's choices under uncertainty. Additional information can be found in Borch (1990) and Robison and Barry (1987). Expected utility theory is based on axioms describing an individual's preferences among simple and compound lotteries (see Appendix 1). The axioms are used to demonstrate the existence of a utility function that can be used to rank an individual's preferences. Since utility depends on preferences, each person may have a different utility function. An optimal decision is the one that maximizes a person's expected utility. The expected utility for an uncertain outcome is determined from the utility for each outcome in combination with the probability distribution of the potential outcomes, i.e., $U(X) = \sum U(x_i)p_i$, where p_i is the probability corresponding to outcome x_i. Utility is often defined in terms of final wealth rather than income in order to incorporate the individual's capital constraint into the evaluation of the optimal decision. However, consideration of the individual's wealth is not required by the axioms.

The primary limitation of the expected utility model is that the shape of the utility function *U* is not known. A concave utility curve is generally assumed since this implies that the individual is risk averse and hence should be willing to pay an insurance premium in excess of the expected damages. Also, it is generally assumed that a wealthier individual will have a lower aversion to risk. Since an insurer would be expected to be wealthier and hence less risk averse than an individual, the price it would demand for the acceptance of risk exposure should be less than the price an individual exposed to the uncertain damages would be willing to pay to transfer the exposure. Under these conditions, and provided that the transaction expenses are sufficiently low, a mutually acceptable price for the transaction can exist which would improve the utility of both participants in the transaction.

An additional deficiency of expected utility theory is that it does not reflect the reduction of risk that can be achieved through diversification. When insuring a group of similar independent exposures, an insurer's uncertainty will increase less rapidly than the number of exposures. In essence, the insurer's risk per exposure is reduced. This paper will show that the introduction of a pricing assumption regarding risk diversification permits the form of the utility function to be determined with the exception of a single risk aversion parameter. This result enables the risk margin in the price for a risk transfer to be based directly on the uncertainty in the transaction, without reference to the insurer's equity or to the opportunity cost of capital.

The primary focus of this discussion will be on insurance, that is, the transfer of uncertain liabilities. An insurance transaction will be described in terms of the price or premium paid for the insurance, the damages incurred, and the indemnity paid. Unless otherwise specified, it will be assumed that the insurance provides full coverage so that the indemnity paid is equal to the damages incurred. The insurer may also incur expenses in the sale or servicing of the policy. The return achieved by the insurer for a particular outcome will be defined as the premium minus the indemnity and expenses for the transaction, whether this is a positive or negative value. Capital letters such as X, Y, and Z will be random variables representing the uncertain damages. $P(X)$ will represent the certainty equivalent price for X, the price at which the individual exposed to uncertainty is indifferent to retaining or insuring the exposure to potential damages. The risk margin for the transaction will be defined as $P(X) - E(X)$, that is, the non-negative amount the individual is willing to pay in excess of the expected damages in order to eliminate uncertainty. Lower case letters such as a, b, and c will be constants. The mean, standard deviation, and variance of the damages will be denoted as μ, σ, and σ^2, respectively.

Since the focus for this paper is insurance pricing, it will be convenient to consider damages to be non-negative and bounded, $0 <= X <=$ max(X). References to unbounded distributions such as the Normal should be considered as approximations to non-negative and bounded distributions. Initially, it will be assumed that all uncertain outcomes are realized and indemnified immediately in order to avoid the issue of the present value of uncertain future outcomes. The price for risk transfers with uncertain future outcomes will be considered at a later point.

2. Basic Results from Expected Utility Theory

Several results will be needed in the subsequent analysis. These include:

- $P(c) = c$
- $P(X + c) = P(X) + P(c) = P(X) + c$
- $P(X) >= E(X)$ for insurance exposures
- If $X <= Y$, then $P(X) <= P(Y)$

The first formula is the observation that a certain outcome has no risk. The second formula indicates that a certain outcome has no effect on the risk of a portfolio. This result is also valid for the subtraction of a constant. Since $Y = X - c$ can be restated as $X = Y + c$, this can be used to show that $P(X - c)$ is equal to $P(X) - c$.

The utility function $U(w)$ is assumed to be a continuous and increasing function of wealth, w. In addition, the assumption of a risk averse individual requires that U be concave downward. Given these conditions, the certainty equivalent price for an individual exposed to uncertain damages X is defined to be the unique value $P(X)$ such that:

$$U(w - P(X)) = EU(w - X)$$

The first pricing formula can be obtained by replacing X with c:

$$U(w - P(c)) = EU(w - c) = U(w - c)$$

Since U is an increasing function, the values $w - P(c)$ and $w - c$ must be equal, so that $P(c) = c$.

The second result can be obtained by replacing X by $X + c$:

$$U(w - P(X + c)) = EU(w - (X + c))$$

Let $w^* = w - c$:

$$U(w - P(X + c)) = EU(w^* - X)$$

Suppose that the risk taker has initial wealth of w^* rather than w. Let the certainty equivalent price for the transfer of the exposure X be defined as $P^*(X)$, so that:

$$EU(w^* - X) = U(w^* - P^*(X))$$

Under the assumption that the certainty equivalent price is insensitive to small changes in the risk taker's initial wealth, $P^*(X)$ will be essentially equal to $P(X)$ for small c. While this conclusion would also be true if the certainty equivalent price were independent of the insurer's wealth, this condition is stronger than is necessary for this result. Hence:

$$U(w - P(X + c)) = U(w^* - P^*(X)) = U(w^* - P(X))$$

But,

$$U(w^* - P(X)) = U(w - c - P(X))$$

so that:

$$U(w - P(X + c)) = U(w - c - P(X))$$

Since U is increasing, the values inside the parentheses must be identical. This yields the second result:

$$P(X + c) = P(X) + c$$

This discussion should not be given the interpretation that the individual's wealth is irrelevant to pricing. The issue of risk aversion in relation to wealth will be considered at a later point in the discussion.

The third formula is obtained from an application of Jensen's inequality:

$$U(w - P(X)) = EU(w - X) <= U(E(w - X)) = U(w - E(X))$$

Since U is increasing:

$$P(X) >= E(X)$$

For the final result, suppose that $X(\omega) <= Y(\omega)$ for each outcome ω in the sample space, so that:

$$w - X(\omega) >= w - Y(\omega)$$

Since U is increasing:

$$U(w - X) >= U(w - Y)$$

Hence:

$$EU(w - X) >= EU(w - Y)$$

Using the definitions of $P(X)$ and $P(Y)$, this is equivalent to:

$$U(w - P(X)) >= U(w - P(Y))$$

so that:

$$P(X) <= P(Y)$$

This result also implies that:

$$\text{If } X <= c, \text{ then } P(X) <= c$$

and:

$$\text{If } X >= c, \text{ then } P(X) >= c$$

3. Diversification and Price

The previous section demonstrated that the certainty equivalent price for a portfolio consisting of uncertain damages X and a certain outcome c is equal to the sum of their certainty equivalent prices, $P(X + c) = P(X) + c$. Next, the effect of multiplying the damages X by a constant will be considered. It will be shown that $P(aX) = aP(X)$ for non-negative constants a, provided that a is within a reasonable range of 1. Since the axioms of expected utility theory address lotteries rather than portfolio pricing, this discussion will introduce the additional requirement that diversification reduces risk. In particular, the diversification property, as applied to insurance, is the requirement that certainty equivalent prices have the following property:

- $P(X + Y) <= P(X) + P(Y)$
- $P(X + Y) = P(X) + P(Y)$, if and only if X and Y are perfectly correlated

The formula $P(X + Y) = P(X) + P(Y)$, without the restriction of perfect correlation, is the familiar concept of value additivity in perfect capital markets. Borch (1990) noted that this formula is consistent with expected utility theory only for a linear utility function. For this reason, he limited his investigations to the functional forms that would be consistent with value additivity of market prices. This observation will not be a concern in the current discussion since the diversification property requires value additivity only for perfectly correlated exposures.

The first point of the diversification property states that diversification reduces uncertainty and hence reduces the certainty equivalent price. The second point states that diversification will not reduce the uncertainty if the outcomes for the two exposures are perfectly correlated. Fundamentally, this means that additional units of a single exposure have the same price as the original units. These two points are analogous to the formula for the standard deviation for the sum of two random variables:

$$\sigma_{X+Y} <= \sigma_X + \sigma_Y$$

with equality holding if and only if the two variables are perfectly correlated with one another.

The diversification property will be used to develop the result $P(aX) = aP(X)$, first for integer values of a, and finally for any positive constant. Consider the damages X and Y for two perfectly correlated exposures. Since there is no diversification of risk, $P(X + Y) = P(X) + P(Y)$. If Y is selected to be equal to X, this result simplifies to:

$$P(2X) = 2P(X)$$

Similarly, for any positive integer a:

$$P(aX) = aP(X)$$

This result can be extended to any positive rational value a. Let a be represented as the ratio of two integers b and c, with $a = b/c$. Define Y as $(b/c)X$, so that $cY = bX$ and $P(cY) = P(bX)$. Since b and c are integers:

$$cP(Y) = P(cY) = P(bX) = bP(X)$$

or:

$$P(Y) = (b/c)P(X) = aP(X)$$

Since $Y = aX$:

$$P(aX) = aP(X)$$

For irrational values a, select two positive rational values b and c such that $b < a < c$. Then:

$$P(bX) <= P(aX) <= P(cX)$$

so that:

$$bP(X) <= P(aX) <= cP(X)$$

Since b and c can be selected to be arbitrarily close to a:

$$aP(X) <= P(aX) <= aP(X)$$

which demonstrates that $P(aX) = aP(X)$ for $a > 0$.

Given further consideration, it should be evident that the multiplication formula $P(aX) = aP(X)$ will not be valid for very large values of a due to the insurer's capital constraints. For example, suppose that X is either \$0 or \$1,000 with a 50% probability of a total loss, and $P(X)$ has been determined to be \$667. The insurer may be willing to accept a reasonable number of perfectly correlated exposures at this price. However, the same insurer may be unwilling to accept one million perfectly correlated exposures at this price due to the potential for incurring severe losses. The insurer's willingness to accept risk may depend not only on its current wealth but also on its potential wealth subsequent to the loss. In this situation, $P(1{,}000{,}000X)$ will not equal $1{,}000{,}000P(X)$. Consequently, the formula $P(aX) = P(aX)$ should be understood as being appropriate for reasonable values of a and for exposures X that are small in relation to the insurer's total wealth.

The multiplication formula, which has been derived from the second point of the diversification property, can be combined with the earlier result for the addition of a constant. This leads to the following linearity rule for any reasonable value of $a >= 0$:

$$P(aX + b) = aP(X) + b \tag{1}$$

It can also be shown that equation (1) implies the second point of the diversification property. To show that the price for the combination of two perfectly correlated exposures X and Y is equal to the sum of the two individual prices, define the random variable T by the formula:

$$Y - \mu_Y = a(X - \mu_X) + T$$

where the value of a is selected to be Cov(X,Y)/V(X). Notice that the expected value of T is 0. Multiplying both sides of the equation by $(X - \mu_X)$ and taking the expectation yields the result Cov(T,X) = 0. Multiplying both sides of the equation by $(Y - \mu_Y)$, taking the expectation, and recognizing that X and Y are perfectly correlated yields the result Cov(T,Y) = 0. Multiplying both sides of the equation by T and taking the expectation implies that V(T) = 0. This requires that T equal 0 and hence that $Y - \mu_Y = a(X - \mu_X)$ except on a set of probability measure 0.

To complete the proof, let $b = \mu_Y - a\mu_X$ so that $Y = aX + b + T$. By definition, $P(Y)$ is the price that makes the expected utility $EU(w - Y + P(Y))$ equal to 0. Since T is 0 almost everywhere, it can be disregarded in taking the expectation, so that:

$$0 = EU(w - Y + P(Y)) = EU(w - (aX + b + T) - P(Y)) = EU(w - (aX + b) - P(Y))$$

Since $P(aX + b)$ is the unique price that results in an expected utility of 0 for the last term of this equation, this shows that $P(Y)$ is equal to $P(aX + b)$, which equals $aP(X) + b$ according to equation (1). Similarly, letting $Z = X + Y = (a + 1)X + b + T$, a similar calculation shows that $P(Z) = (a + 1)P(X) + b$. Combining the two results gives the desired conclusion, $P(X + Y) = P(X) + P(Y)$. This proves that the linearity formula $P(aX + b) = aP(X) + b$ is equivalent to the second point of the diversification property.

4. The Certainty Equivalent Price for the Sum of Normal Random Variables

The formula developed above for the price of the linear transformation of a random variable will now be applied to random variables from the Normal distribution. According to Hogg and Craig (1995, p. 228), the Normal distribution has the property that the sum of any two Normal random variables has the Normal distribution. This makes it possible to evaluate the certainty equivalent price for a combination of exposures in relation to the certainty equivalent prices for the individual exposures.

Consider two independent exposures X and Y from the same Normal distribution with identical means $\mu_X = \mu_Y = \mu$ and variance $\sigma^2_X = \sigma^2_Y = \sigma^2$, denoted as $N(\mu,\sigma^2)$. Let the damage random variable Z be defined as the sum $X + Y$. Then Z has a Normal distribution with mean $\mu_Z = 2\mu$ and variance $\sigma^2_Z = 2\sigma^2$, so that $Z \sim N(2\mu,2\sigma^2)$. Next, define the random variable W as:

$$W = (X - \mu)(\sigma_Z/\sigma) + 2\mu$$

Since W is also $N(2\mu,2\sigma^2)$, W and Z have the same certainty equivalent price. Using the linearity rule from equation (1):

$$P(Z) = P(W) = (P(X) - \mu)(\sigma_Z/\sigma) + 2\mu$$

so that:

$$P(Z) - E(Z) = (P(X) - E(X))(\sigma_Z/\sigma)$$

This formula states that the risk margin for the portfolio Z is a multiple of the risk margin for the individual exposure X, where the value of the multiplier is $\sigma_Z/\sigma = \sqrt{2}$.

This result can be extended to the sum of n independent Normal random variables from the same distribution $N(\mu,\sigma^2)$. The random variable Z representing the portfolio of exposures ΣX_i has the Normal distribution, $Z \sim N(n\mu,n\sigma^2)$. Define W as:

$$W = (X - \mu)(\sigma_Z/\sigma) + \mu_Z$$

Since W has a Normal distribution with the same mean and standard deviation as Z, it has the same certainty equivalent price, $P(W) = P(Z)$. Using the linearity rule, the risk margin for Z is proportional to the risk margin for X:

$$P(Z) - E(Z) = (P(X) - E(X))(\sigma_Z/\sigma)$$

where $\sigma_Z/\sigma = \sqrt{n}$.

A different perspective on this result can be obtained by selecting one specific Normal random variable and relating the risk margin for any other Normally distributed random variable to the risk margin for this base distribution. For example, let X be a Normal random variable with mean μ and standard deviation 1 and suppose that the risk margin for X is a known value λ. Let Z be any other Normally distributed random variable with $Z \sim \mathrm{N}(\mu_Z, \sigma^2_Z)$. Since $W = (X - \mu)\sigma_Z + \mu_Z$ has the same distribution as Z, both W and Z have the same certainty equivalent price. The risk margin for Z is:

$$P(Z) - E(Z) = P(W) - E(W) = (P(X) - E(X))\sigma_Z = \lambda\sigma_Z$$

This demonstrates that for the uncertain damages Z, where Z is Normal, the risk margin is a constant multiple λ of the standard deviation of Z. Restating this result, the standard deviation pricing formula for the certainty equivalent price of a Normal random variable Z is:

$$P(Z) = \mu_Z + \lambda\sigma_Z \tag{2}$$

Provided that the value of λ is known, the standard deviation pricing formula in equation (2) immediately determines the price for any other Normal distribution.

5. The General Result

The standard deviation pricing formula developed in the previous section applies only for exposures whose damages follow or can be approximated by the Normal distribution. This section will develop a general risk pricing formula for any distributional form by determining the shape of the utility function consistent with the diversification property. This will also provide a means for evaluating λ in the standard deviation pricing formula.

For an insurer, the certainty equivalent price $P(X)$ for uncertain damages X can be described in terms of the insurer's utility function as:

$$EU(w - X + P(X)) = U(w)$$

That is, the insurer is indifferent between accepting the premium with its associated uncertainty and not accepting the exposure to risk. Without loss of generality, the utility function can be selected such that $U(w) = 0$:

$$EU(w - X + P(X)) = 0$$

This result applies to all damage random variables X. Applying this to aX, and recalling that $P(aX) = aP(X)$:

$$EU(w - a(X - P(X))) = 0$$

for all values of $a > 0$. If the diversification property is to hold, the insurer's utility function must satisfy this formula. The objective will be to find the form of all utility functions U for which this result is valid.

The first functional form to be considered is a straight line. This is suggested by the linearity of the price function P for perfectly correlated exposures, as indicated in equation (1). For example, suppose that U is the straight line having slope c such that $U(w) = 0$. Then:

$$U(w + t) = ct$$

for all valid values of t, so that for the outcome x:

$$U(w - a(x - P(X))) = c(-a(x - P(X))) = ac(-(x - P(X))) = aU(w - (x - P(X)))$$

Since $EU(w - (X - P(X))) = 0$, this demonstrates that $EU(w - a(X - P(X)))$ is also 0, as desired. This functional form implies that the certainty equivalent price is equal to the expected damages:

$$0 = EU(w - (X - P(X))) = E(-c(X - P(X))) = -cE(X - P(X)) = -c(E(X) - P(X))$$

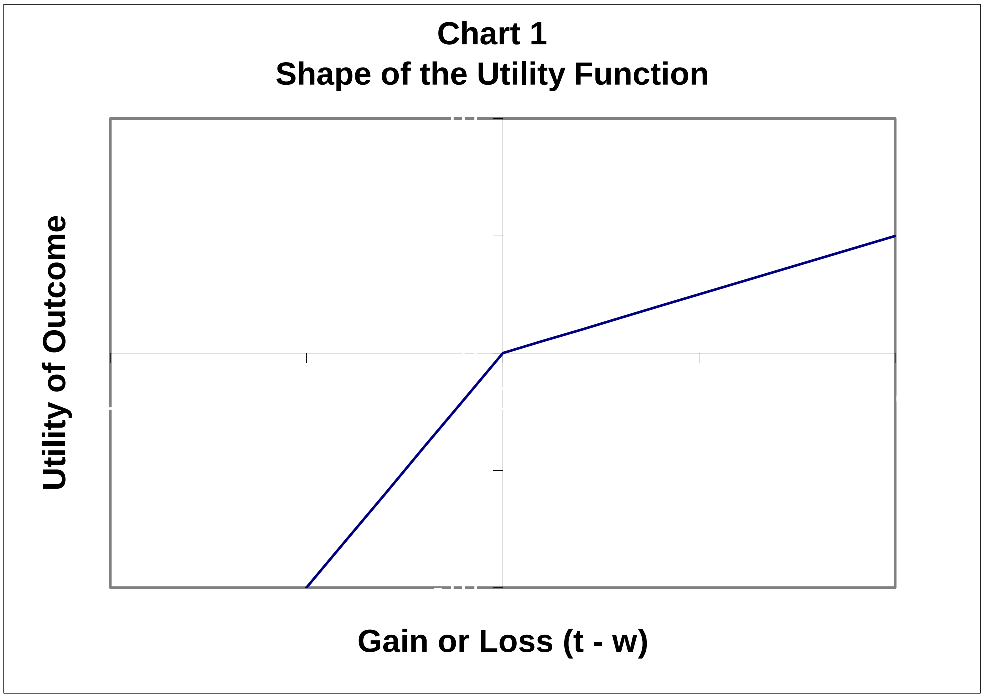
or:

$$P(X) = E(X)$$

However, this result is inconsistent with the diversification property. Applying this result to the uncorrelated random variables X and Y, then $P(X) = E(X)$. This also implies that $P(Y) = E(Y)$ and $P(X + Y) = E(X + Y)$. But this means that $P(X + Y) = P(X) + P(Y)$, which is true if and only if the damages are perfectly correlated. Since the damages are uncorrelated, this contradiction shows that the utility function cannot be a straight line.

The next functional form to be considered is the curve consisting of two rays meeting at a point, as shown in Chart 1. The initial step will be to evaluate the formula for the certainty equivalent price. The verification that this utility function satisfies the objective stated above will be postponed until the next section.

The utility function in Chart 1 will be defined as $U(t) = e(t - w)$ for $t < w$ and $U(t) = c(t - w)$ for $t >= w$, where w is the current wealth of the insurer. The value of t represents final wealth, so that $t - w$ represents the gain or loss arising from outcome t. Since a well-managed insurer should not provide coverage beyond its financial capacity, t is assumed to take only positive values. The values e and c can be understood to represent the marginal disutility of \$1 of loss and the marginal utility of \$1 of gain, respectively.



To meet the requirements that the utility function must be increasing and concave downward, both *c* and *e* must be positive values with $e > c$. Letting *F* represent the cumulative distribution function of *X*, the expected utility for a risk transfer can be expressed as:

$$EU(w - X + P(X)) = -e \int_{x>P(X)} (x - P(X))\, dF(x) + c \int_{x<=P(X)} (P(X) - x)\, dF(x)$$

Since the expected utility must be zero at the certainty equivalent price, this simplifies to:

$$(3) \qquad (e/c - 1) \int_{x>P(X)} (x - P(X))\, dF(x) = P(X) - E(X)$$

Notice that the individual values of *e* and *c* drop out of this formula. Instead, the insurer's risk aversion is completely characterized by the ratio $e/c > 1$. Equivalently, the utility function could have been defined with the marginal utility of $1 of gain being arbitrarily set to $1, i.e., $c = 1$, with the insurer's risk aversion being completely determined by *e*.

For a normal distribution $X \sim N(\mu,\sigma^2)$, *X* can be expressed as $\sigma Z + \mu$ where $Z \sim N(0,1)$. Using a change of variables from *x* to *z*, equation (3) can be simplified to show that:

$$P(X) = \mu_X + \lambda\sigma_X$$

This demonstrates that the utility function in Chart 1 is consistent with the standard deviation pricing formula in equation (2). The value of λ is $P(Z)$, where Z has the standard Normal distribution, $Z \sim N(0,1)$. Equation (3) can also be used to evaluate λ in terms of the risk aversion factor $(e/c - 1)$.

One potential criticism of the standard deviation pricing formula is its inconsistency with the variance pricing formula, which states:

$$P(X) = \mu_X + \lambda\sigma^2_X$$

Robison and Barry (1987, p. 40) provide support for this formula. However, the proof requires that the utility function be differentiable at w in order to develop a Taylor's series expansion of U. As shown in Chart 1, the utility function being considered here is not differentiable at w.

The risk pricing formula given in equation (3) needs to be examined in more depth. The interpretation of this formula is that the certainty equivalent price is that value $P(X)$ for which two sides of the formula are equal. The right hand side of equation (3) is the insurer's risk margin, which is the insurer's expected return for accepting the exposure to risk. Equivalently, this represents the insurer's expected profit, provided that the insurer has no expenses.

The left hand side of equation (3) consists of two components, an integral and a multiplier. The integration is performed over the range of outcomes $x > P(X)$, those outcomes for which the insurer incurs a loss and must use its own capital to indemnify the insured. For any such outcome, the integrand $x - P(X)$ is the amount of the insurer's capital consumed. Consequently, the integral represents the expected amount of the insurer's capital consumed due to accepting the transfer of risk. Disregarding the multiplier $(e/c - 1)$, this result states that at the certainty equivalent price for the transfer of risk, the insurer's expected return is equal to the cost associated with the risk transfer, where the cost is the expected financial loss that the insurer may incur.

It should be noted that equation (3) makes no reference to the insurer's wealth, w. The equation instead measures the insurer's utility of income, $P(X) - x$. Wealth is incorporated into the formula solely through the insurer's risk aversion factor, $(e/c - 1)$. Recall that this discussion has previously assumed that an insurer's certainty equivalent price, and hence its risk aversion factor, is constant for small changes in w. For larger changes in wealth, it may be reasonable to assume that an insurer's risk aversion factor will change. An insurer with a small capital base may have a high aversion to risk since a large financial loss could result in financial impairment or ruin. The same insurer with a large capital base may have a lower aversion to risk due to a lower probability of ruin. Since the subject is not essential for this discussion, differences in risk aversion as a function of wealth will not be given further consideration.

The product of the risk aversion factor $(e/c - 1)$ with the capital consumed for outcome x will be defined as the insurer's risk due to accepting the exposure to uncertainty. That is, for each outcome x, risk will be defined as:

$$\begin{aligned} Risk &= (e/c - 1)(x - P(X)) \qquad && \text{for } x > P(X) \\ &= 0 && \text{for } x <= P(X) \end{aligned}$$

Consequently, the insurer's expected risk is equal to the left hand side of equation (3):

$$E(Risk) = (e/c - 1) \int_{x>P(X)} (x - P(X))\, dF(x)$$

Based on this definition, equation (3) states that the certainty equivalent price $P(X)$ is the price at which the Expected Risk equals the Expected Return:

$$(4) \qquad E(Risk) = E(Return)$$

In other words, the price of the transaction should be established so that the expected risk and the expected return are in balance. Equation (4) expresses the concept that risk determines return. From a different perspective, the certainty equivalent price can be considered to be a reference point against which the individual evaluates the possible gains and losses, as described in Starmer (2000, p.351). The expected risk is related to the semi-variance introduced by Markowitz (1991) as a measure of risk in that both measures define risk solely in terms of the adverse outcomes.

It can be easily confirmed that the relationship between the utility function and the insurer's risk and return for each outcome x is:

$$U(x) = c(Return\ at\ (x) - Risk\ at\ (x))$$

This can be further simplified if *c* has been arbitrarily selected to be 1. Taking the expected value of both sides of the equation confirms that the price at which the expected utility is 0 is also the price at which the expected risk and return are in balance.

It should also be observed that equation (3) places an upper bound on *P(X)*. Since *X* is non-negative, the expected risk component is limited by:

$$\begin{aligned}(e/c - 1) \int_{x>P(X)} (x - P(X))\, dF(x) &<= (e/c - 1) \int_{x>P(X)} x\, dF(x) \\ &<= (e/c - 1) \int x\, dF(x) \\ &<= (e/c - 1)\, E(X)\end{aligned}$$

Based on equation (3), this means that $P(X) - E(X) <= (e/c - 1)\, E(X)$, or $P(X) <= (e/c)\, E(X)$. For example, if the insurer's risk aversion factor $(e/c - 1)$ is 1, then the expected risk is identical to the expected capital consumed. In this situation, the certainty equivalent price *P(X)* will be no greater than *2E(X)*. Note that this value for the risk aversion factor implies that *e*, the marginal disutility of $1 of loss, is twice the value of *c*, the marginal utility of $1 of gain. This is consistent with Starmer (2000, p. 365), who notes: "Benartzi and Thaler show that, assuming people are roughly twice as sensitive to small losses as to corresponding gains (which is broadly in line with experimental data relating to loss aversion), the observed equity premium in consistent with the hypothesis that investments are evaluated annually."

6. Shape of the Utility Function

This section will demonstrate that the utility function consisting of two rays, as shown in Chart 1, satisfies the second point of the diversification property. This will be demonstrated in two steps. First, it will be shown that the utility function satisfies the equivalent requirement that $P(aX) = aP(X)$. Second, it will be demonstrated that the utility function must be of this form.

For any damage exposure X, $P(X)$ is the unique value that satisfies the expected utility formula:

$$EU(w - X + P(X)) = 0$$

Using the definition of U, this can also be expressed as:

$$-e \int_{x>P(X)} (x - P(X))\, dF(x) + c \int_{x<=P(X)} (P(X) - x)\, dF(x) = 0$$

Consider the random variable $Y = aX$. The expected utility formula for Y is:

$$EU(w - aX + P(Y)) = -e \int_{ax>P(Y)} (ax - P(Y))\, dF(x) + c \int_{ax<=P(Y)} (P(Y) - ax)\, dF(x)$$

$P(Y)$ is the unique value that makes this result equal to zero. Consider the possibility that $P(Y) = aP(X)$. Substituting this on the right hand side of the formula permits a to be factored out, so that:

$$EU(w - aX + P(Y)) = aEU(w - X + P(X))$$

Since the expected utility formula on the right side of this equation is 0 by definition, this shows that $P(Y)$, or $P(aX)$, is equal to $aP(X)$.

The next objective will be to demonstrate that U must be of the form shown in Chart 1. Consider a random variable X with two outcomes, 1 and x, with $x < 0$. According to the continuity axiom of expected utility theory, there exists a probability p for which $EU(X) = 0$. Hence:

$$U(x)p + U(1)q = 0$$

or

$$U(x) = -U(1)q/p$$

Next, consider the random variable $Y = 2X$. Since $P(X)$ is the only solution to $EU(X - P) = 0$, and $EU(X) = EU(X - 0) = 0$, this shows that $P(X) = 0$. Consequently, $P(Y) = 2P(X) = 0$, so that $EU(Y - P(Y)) = EU(Y)$. But $EU(Y - P(Y)) = 0$ by definition. Hence $EU(Y) = 0$, so that:

$$EU(Y) = U(2x)p + U(2)q = 0$$

or

$$U(2x) = -U(2)q/p$$

Define $k = U(2) / U(1)$. Since U is concave and increasing, this requires that $1 < k <= 2$. Also, the value for $U(2x)$ is related to the value for $U(x)$ as follows:

$$U(2x) = kU(x)$$

Consider the specific value of $x = -1$, so that $U(-2) = kU(-1)$. Since $U(-1) < 0$ and $k <= 2$, this implies that:

$$U(-2) = kU(-1) >= 2U(-1)$$

However, if $U(-2) > 2U(-1)$, then U cannot be concave. Consequently, k must be equal to 2, so that for any $x < 0$:

$$U(2x) = 2U(x)$$

A similar result can be obtained for any other random variable $Y = tX$ for $t > 0$, so that:

$$U(tx) = tU(x)$$

The solution to this formula is a straight line ending at the origin:

$$U(x) = ex \qquad \text{for } x <= 0$$

A similar argument can be used to show that for $x >= 0$,

$$U(x) = cx \qquad \text{for } x >= 0$$

In order for U to be concave, it is necessary that $e > c$. This demonstrates that the utility curve must be of the form shown in Chart 1 in order to satisfy the requirement that $P(aX) = aP(X)$ for all $a > 0$.

7. Methods for Evaluating the Premium

In order to evaluate the certainty equivalent price for the uncertain damages, equation (3) must be solved for $P(X)$. This section reviews three techniques to obtain the solution.

For simple cases, it may be possible to evaluate the integrals directly. For example, let $(e/c – 1) = 1$, and suppose that X has the following distribution:

$X = 1{,}000$	$Prob(X = 1{,}000) = 0.50$
$X = 2{,}000$	$Prob(X = 2{,}000) = 0.50$

Since $1{,}500 < E(X) < P(X) < 2{,}000$, the expected risk component of equation (3) can be evaluated as $(2{,}000 – P(X)) * 0.50$, while the expected return is $P(X) – 1{,}500$. Equating the two results and solving gives $P(X) = 1{,}666.67$. This premium consists of expected damages of 1,500 plus a risk margin of 166.67.

$P(X)$ may also be obtained through a recursion process. Starting with an initial estimate P_1 of $P(X)$, a second estimate P_2 can be obtained by:

$$\int_{x>P1} (x - P_1)\, dF(x) = P_2 - E(X)$$

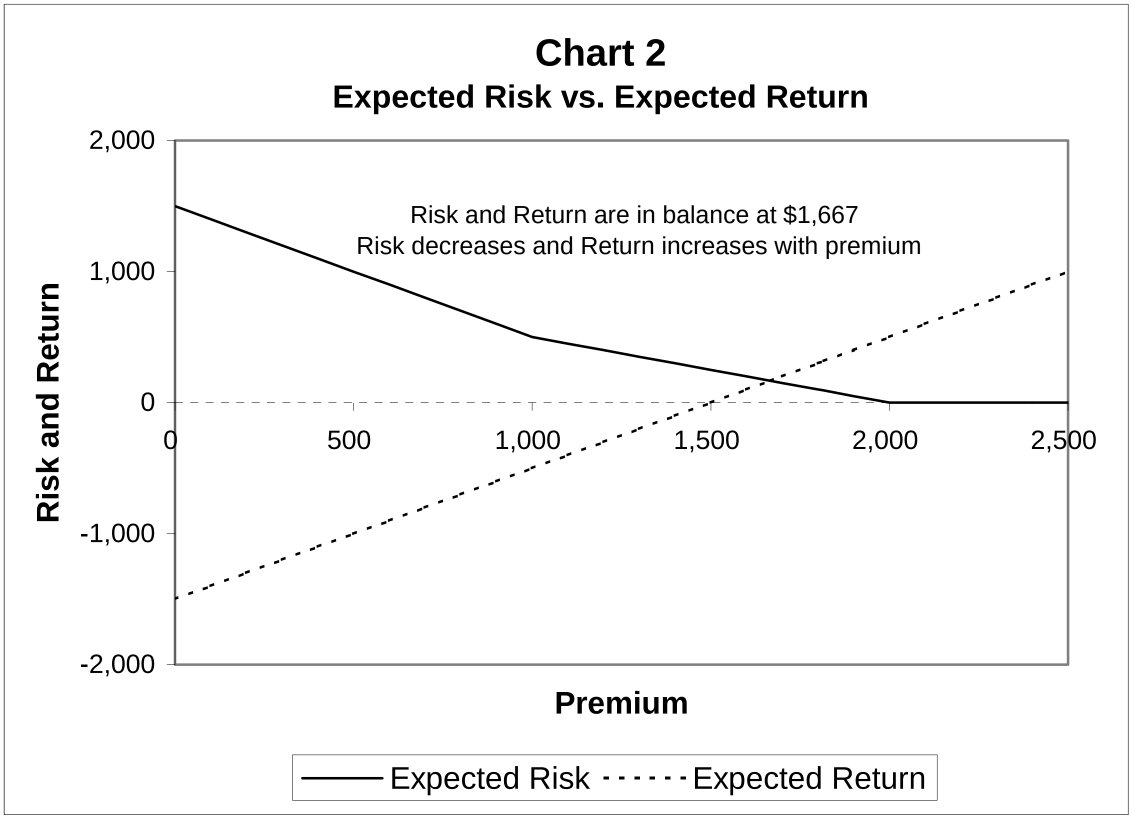
Using the two outcome example introduced above, select P_1 as $E(X) = 1{,}500$. Since the integral in the formula can evaluated as $(2{,}000 - 1{,}500) * 0.50 = 250$, this results in a second estimate of the premium of $P_2 = 1{,}750$. After several iterations, this converges to 1,666.67.

The most useful method for evaluating the premium uses Equation (4) to provide a graphical interpretation of the certainty equivalent price. Consider the two sides of equation (4) individually and define the expected risk and the expected return functions for an arbitrary premium P as:

(5) $$E(Risk;P) = (e/c - 1) \int_{x>P} (x - P)\, dF(x)$$

(6) $$E(Return;P) = \int (P - x)\, dF(x)$$

Notice that the expected risk is a decreasing function of P, while the expected return is an increasing function of P. When the value of P is equal to $P(X)$, the expected risk and expected return functions are equal. The expected risk and expected return functions for the two-outcome example are shown in Chart 2. The point of intersection of the two curves is the certainty equivalent price.



8. The Present Value of Future Uncertainty

The discussion up to this point has been limited to uncertain damages whose actual outcome is discovered immediately after the risk transfer has been completed. The model will now be extended to damages in which the outcome is realized at a future time.

Suppose that all of the potential outcomes for X are realized at a future time t instead of time 0. The certainty equivalent price $P(X)$ for the risk transfer can be evaluated at time t using the risk pricing model in equation (4). Since $P(X)$ evaluated at time t is a fixed amount, this value can be discounted to time 0 at the risk-free rate. The price P_0 that would be payable at time 0 to transfer the uncertain damages is:

$$P_0 = P(X) / (1 + r_f)^t$$

The certainty equivalent price can also be evaluated using a second approach, by discounting each outcome for X to time 0 at the risk-free rate. Consider the random variable Y, defined as the value of the uncertain outcome X discounted to time 0 at the risk-free rate:

$$Y = X / (1 + r_f)^t$$

The certainty equivalent price for Y can be evaluated using equation (1):

$$P(Y) = P(X / (1 + r_f)^t) = P(X) / (1 + r_f)^t = P_0$$

The consistency of the two results demonstrates that the certainty equivalent price for future outcomes can be obtained by discounting the uncertain outcomes to present value and then determining the certainty equivalent price.

This result can be extended to exposures having uncertain damages with multiple payments made over time. Let each x_i, $i = 1$ *to* m, represent a single outcome for the uncertain damages X. The damages paid at time $j = 1$ *to* n will be denoted as $x_{i,j}$. Each x_i consists of a sequence of payments:

$$x_i = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,n}) \text{ for } i = 1 \text{ to } m$$

For each outcome x_i, the individual payments $x_{i,j}$ can be discounted to present value at the risk-free rate. Replacing each sequence x_i with its present value $y_i = \Sigma\, x_{i,j} /(1+r_f)^j$, the certainty equivalent value for a stream of payments X will be defined as $P(Y)$.

The stream of payments $x_{i,j}$ can also be viewed in cross section. Let Z_j be set of a random variables representing the damages $x_{i,j}$ for each point in time, $j=1$ *to* n. This perspective leads to two other potential approaches for evaluating the certainty equivalent price. The first method is to total the discounted expected values $v^jE(Z_j)$ over all periods, $P = \Sigma v^jE(Z_j)$, where $v = 1/(1+r_f)$. The shortcoming of this approach is that it eliminates uncertainty from the analysis by its use of expected values in place of the values for each possible outcome. This results in an understatement of the certainty equivalent value of the damages. This understatement would need to be offset by the use of an interest rate i less than the risk-free rate, $i < r_f$, in order to produce the appropriate certainty equivalent value. However, it is not evident how to determine the correct interest rate without the use of the risk pricing model developed above.

The second method is similar, but totals the discounted certainty equivalent values over all periods, $P = \Sigma v^j P(Z_j)$. For any period j, the certainty equivalent price for the present values of the outcomes is identical to the present value of the certainty equivalent price, $P(v^j Z_j) = v^j P(Z_j)$. However, it is not generally true that the sum of the discounted certainty equivalent values, $\Sigma v^j P(Z_j)$, is equal to $P(X)$. An example is provided in the following table. Here, the outcomes for Z_j, $j > 1$, are dependent on the outcome for Z_1. This example has been constructed so that the four outcomes have identical total payments of 100 and identical present values of 90.81, based on a risk-free rate of 5%. Since all four present values are identical, the certainty equivalent price for this exposure is also 90.81. In comparison, the sum of the present value of the certainty equivalent prices over all periods is 96.19.

Table 1

Time	Outcomes				Certainty Equivalent	Present Value of C.E.
	A	B	C	D		
1	40	46.5	53	59.5	48.50	46.19
2	30	20	10	0	22.86	20.73
3	20	20	20	20	20.00	17.28
4	10	13.5	17	20.5	14.58	11.99
Total	100	100	100	100	105.94	96.19
Probability	0.40	0.30	0.20	0.10	---	---

The difference between the two calculations can be readily explained. The present value y_i of the series of payments $x_i = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,n})$ can be evaluated as:

$$y_i = x_{i,1} + vx_{i,2} + v^2 x_{i,3} + \ldots + v^{n-1} x_{i,n}$$

Each period of time j can be considered to have outcomes represented by the random variable Z_j. This permits the random variable Y of present values to be expressed as:

$$Y = Z_1 + vZ_2 + v^2Z_3 + v^3Z_4 + \ldots + v^{n-1}Z_n$$

This implies:

$$P(Y) = P(Z_1 + vZ_2 + v^2Z_3 + v^3Z_4 + \ldots + v^{n-1}Z_n)$$

However, the diversification property requires that:

$$P(Y) <= P(Z_1) + vP(Z_2) + v^2P(Z_3) + v^3P(Z_4) + \ldots + v^{n-1}P(Z_n) = \Sigma v^j P(Z_j)$$

which shows that the sum of the discounted certainty equivalent values for each period always equals or exceeds the certainty equivalent price. Similarly, the relationship between the certainty equivalent price and the sum of the discounted expected values can be obtained as:

$$P(Y) >= E(Y) = \Sigma v^j E(Z_j)$$

The combination of the two results places upper and lower bounds on the certainty equivalent price:

$$\Sigma v^j E(Z_j) <= P(Y) <= \Sigma v^j P(Z_j)$$

Another perspective on the present value of future uncertainty is presented in Appendix 2, which demonstrates the consistency of the risk pricing model with the Arbitrage Theorem.

9. The Effect of Expenses on Price

The risk pricing model developed above can be easily modified to incorporate the insurer's expenses. Suppose that in accepting the uncertain damages X in exchange for a premium of P, the insurer also incurs various expenses. These expenses represent the marginal transaction costs arising out of issuing and providing service on the policy. The insurer's fixed overhead expenses are not considered in this analysis. Let Y be the uncertain loss adjustment expenses for the transaction. Let f represent the fixed expenses for the transaction, i.e., those that are fixed dollar amounts independent of the premium. Suppose that the variable expenses such as commissions and premium taxes can be expressed as a percentage g of the premium, so that the amount in dollars is gP. Federal income tax payments will depend on the particular realized outcome for the insurer's pre-tax profit or loss of $P - (x + y + f + gP)$. Letting t represent the marginal tax rate, the federal income taxes are $(P - (x + y + f + gP))t$. Combining these results, the insurer accepts the uncertain outcome Z in exchange for a premium of P_Z, where Z represents the total of the insurer's costs:

$$Z = X + Y + f + gP_Z + (P_Z - (X + Y + f + gP_Z))t$$

Consider the simplest case, in which all expenses are fixed and taxes are disregarded. Setting Y, g, and t equal to 0 results in $Z = X + f$. Applying equation (1) to the formula above produces the result $P_Z = P(X) + f$.

More generally, the application of equation (1) to the formula for Z leads to the following result:

$$P_Z = (P(X + Y) + f) / (1 - g)$$

Notice that the federal income tax rate of t does not appear in this equation. This conclusion is inconsistent with the insurance ratemaking procedures described by Feldblum (1992) and Myers and Cohn (1987), as well as with the Actuarial Standards Board (1997) Actuarial Standard of Practice No. 30, which requires that income taxes be considered in the development of the insurer's profit and contingency provision. However, this formula can be given the simple interpretation that the taxation of underwriting gain and loss places the federal government in the role of a pro-rata reinsurer. The government can be considered to take a portion, t, of the premiums, damages, and expense components of the insurance transaction. The insurer takes the remaining portion, $1 - t$, of the various components of the transaction. The insurer's risk and return components for the transaction are both reduced by the same factor, $1 - t$, resulting in no change to the insurer's certainty equivalent price.

10. Application to Investments

Prior to this point, the focus of the discussion has been on insurance transactions, i.e., the transfer of a liability with uncertain outcomes. With a slight revision, the discussion can be modified to treat investments and insurance consistently. To do this, damages must be considered as negative amounts and investment gains as positive amounts. The diversification inequality also would need to be restated as $P(X + Y) >= P(X) + P(Y)$, i.e., that diversification always increases or has no effect on value. Given these modifications, the expected utility formula for an individual with initial wealth w who is considering the purchase of income producing asset X is:

$$EU(w + X - P(X)) = U(w) = 0$$

with $P(X) <= E(X)$. This differs from the earlier formula for insurance in that the signs on X and $P(X)$ are reversed. Note that X represents the present value of the uncertain outcomes, discounted at the appropriate risk-free rate.

The expected utility formula for an investment can also be expressed in terms of the expected risk and return for the transaction. The decision to transfer the asset can be considered as a choice between accepting and not accepting the exposure to uncertainty outcomes. For an individual accepting the uncertain asset, the return for each outcome is $x - P$, where P is the price for the asset and x is the present value of the cash flow from the investment:

$$Return = x - P$$

The individual is exposed to the possibility of a loss on the transaction if the actual outcome x is less than the amount invested. By analogy with insurance, the capital consumed for each outcome x is:

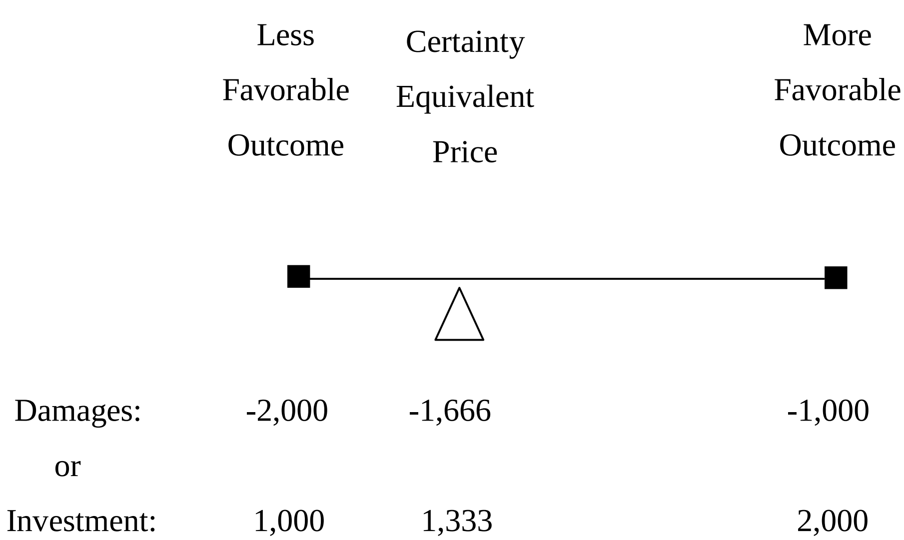
$$Capital\ consumed\ for\ outcome\ x \quad = P - x \qquad for\ x < P$$

$$= 0 \qquad\qquad \textit{for } x >= P$$

The risk for each outcome *x* can be defined as the capital consumed multiplied by the investor's risk aversion factor. With these definitions of risk and return, the certainty equivalent price for the asset is the price *P(X)* at which the individual's expected risk and return are in balance. This also represents the point at which the expected utility of the investment is 0 so that the individual is indifferent between the choices of accepting and not accepting the investment.

As an example, consider an individual with a risk aversion factor *(e/c – 1) = 1*. For an investment having the two possible outcomes of $1,000 and $2,000, it can be easily confirmed that the expected risk and return are in balance at a price of $1,333. In comparison, the certainty equivalent premium for insurance to protect against the two outcomes of –$2,000 and –$1,000 is –$1,666. The investment and insurance scenarios are essentially identical, as can be seen in the following diagram. Both risk transfers have a balance point one third of the distance from the less favorable outcome. This is a consequence of the less favorable outcome being assigned twice as much weight as the more favorable outcome.

Equivalence of risk pricing for insurance and investments for the two-outcome scenario



The risk pricing model differs from the opportunity cost of capital approach as described in Brealey and Myers (1996, p. 73) in one significant respect. The opportunity cost approach determines the value of an investment by discounting future cash flows to present value using the opportunity cost of capital. Only a single outcome, the expected cash flow for each period, is discounted. Since the use of the expected outcome eliminates the uncertainty from the analysis, discounting the expected cash flows at the risk-free rate overstates the value of the investment. To offset the overstatement, discounting must be performed using an interest rate in excess of the risk-free rate. This interest rate, the opportunity cost of capital, represents the appropriate rate of discount needed to reflect the risk of the investment. However, this can be considered to be a proxy for the direct evaluation of risk incorporated in the risk pricing model.

The opportunity cost of capital approach treats time and risk as two inseparable aspects of risk pricing. More specifically, this technique can be applied only if the uncertain outcomes are realized in future periods. In contrast, the risk pricing model regards time and risk as being distinct, and permits the uncertain outcomes to occur either in the present or in future periods. The key point for the risk pricing model is the cost of risk rather than the cost of capital. The model treats the expected risk as an expense element to be included in the price, i.e., *certainty equivalent price = expected damages + expected risk margin*. Consequently, an insurer's return should be expressed as its return relative to the price paid for the policy, that is, its return on premium. The risk pricing model implies that the insurer's return on equity is irrelevant for pricing. It should be noted that this conclusion conflicts with Actuarial Standards Board (1997) Actuarial Standard of Practice No. 30, which requires consideration be given to the insurer's cost of capital.

The application of the risk pricing model for an investment leads to a similar conclusion, that the relevant measure of return to the investor should be expressed as the return relative to the price paid for the investment. However, since the price of an investment is identical to the amount of capital invested, the return on price is equivalent to the return on investment.

11. The Cost of Conditional Risk Financing

The risk pricing model can also be interpreted in terms of the cost of conditional risk financing. Consider equation (3) for a damage distribution X with discrete outcomes, $x_1, x_2, \ldots, x_n$ having associated probabilities $p_1, p_2, \ldots, p_n$. The risk pricing model requires that the risk margin of $P - E(X)$ be equal to the expected risk. The expected risk is the product of the risk aversion factor of $(e/c - 1)$ and the expected capital consumed of $\sum(x_i - P)p_i$, with the summation defined over all outcomes for which $x_i > P$. At present, assume that the risk aversion factor is 1 so that the risk margin is equal to the expected capital consumed.

The objective in this discussion is to describe a self-insurance financing arrangement that functions similarly to the insurance contract. Rather than purchase insurance, an individual may choose to obtain a loan to cover the cost of the damages incurred. The loan would be conditional, in that it would be obtained only after damages had occurred. The individual would prefer insurance to this form of conditional risk financing only if the insurance is less expensive.

To evaluate the cost of the loan, assume that the individual is willing to contribute a maximum amount of L each year to pay for any damages incurred. If the actual damages x_i exceed L, a loan would be obtained for the amount of the damages in excess of L. Since the outcome x_i is expected to occur once each $1/p_i$ years, the annual payment on the loan of $(x_i - L)$ would be $(x_i - L)p_i$, disregarding any interest on the loan. Using a similar argument for the other potential outcomes, this would result in an expected loan payment of $\sum(x_i - L)p_i$. The expected loan payment would come out of the fixed amount $L - E(X)$ that the individual would contribute in excess of the expected damages, so that $\sum(x_i - L)p_i = L - E(X)$. This result is equivalent to equation (3) for the risk pricing model with a risk aversion factor of 1. Since insurance would be competitive with self-insurance only if the insurer's premium P is less than or equal to the self-insurance premium of L, the insured's risk aversion factor would not be greater than 1.

At this point, the interest rate on the loan needs to be considered. Suppose that the lender provides the loan at an interest rate equal to the risk-free rate. Also, recall that the present value of each loan payment is determined by discounting at the risk-free rate. For this reason, the present value of the future payments on the loan of $x_i - L$ is equal to $x_i - L$. Similarly, the present value of the expected future payments is $\sum(x_i - L)p_i$. Since $E(X)$ is the expected amount needed to pay for damages, the excess contribution of $L - E(X)$ can be used to make the payments on the loan. In order for the excess contribution to exactly match the expected loan payments, L must be the solution to $L - E(X) = \sum(x_i - L)p_i$. This result demonstrates that a risk averse individual who is able to obtain conditional risk financing at the risk-free rate would be willing to purchase insurance only if the risk aversion factor used in developing the premium is no greater than 1. A larger risk aversion factor would result in a premium that is not competitive with self-insurance.

If the individual cannot obtain conditional risk financing at the risk-free rate, the expected loan payment would be a larger amount such as $k\sum(x_i - L)p_i$ for some $k > 1$. Since the loan payments would be paid out of the excess contribution, this implies that $L - E(X) = k\sum(x_i - L)p_i$. This result is consistent with the risk pricing model in equation (3) with a risk aversion factor of k. For this individual, a competitive premium P must have a risk aversion factor less than or equal to k.

These results can be interpreted as placing an upper limit on an individual's risk aversion factor. As described above, an individual who is able to obtain conditional risk financing at the risk-free rate would purchase insurance only if the premium were based on a risk aversion factor, $e/c - 1$, that is less than or equal to 1. Another individual with a lower credit rating should be willing to pay a higher price for the insurance.

12. Conclusion

The purpose for this discussion has been to investigate the effect of diversification on risk pricing. The primary assumption used in this analysis is that the price for a portfolio of two perfectly correlated exposures should equal the sum of their individual prices. For exposures that are not perfectly correlated, the diversification property requires that the price for the portfolio should be less than the sum of the individual prices. The resulting risk pricing model balances the risk and return of the uncertain cash flows to determine the certainty equivalent price. Consequently, the risk of the exposure determines the return required for accepting the exposure. The certainty equivalent price can be considered as a reference point, with unfavorable outcomes on one side being balanced by the more favorable outcomes on the other. The unfavorable outcomes are weighted more heavily than the favorable outcomes by the use of the risk aversion factor. The model can be applied to the transfer of either asset or liability exposures.

The risk pricing model differs from techniques that rely on the opportunity cost of capital by taking into consideration the distribution rather than the expected value of the potential cash flows. In addition, the model resolves the issue of determining the present value of uncertain future outcomes. These results diminish, if not completely eliminate, the need to consider insurance risk pricing and profitability in a return on capital framework. Instead, an insurer can use equation (3) to determine its certainty equivalent price for each individual exposure, for entire market segments (e.g., all Pennsylvania Homeowners policies) treated as a single exposure, or for its entire portfolio of insurance policies. The only issue is which of these various levels of risk diversification is the most appropriate for establishing the insurer's profitability or price. This issue will be addressed in a subsequent paper on risk diversification.

The fundamental consideration for the risk pricing model is the uncertainty of the cash flows for the exposure. Other cash flows, such as the investment income the insurer can earn by investing the premium, may also be uncertain but are not relevant to pricing the transfer of risk. The risk pricing model also does not take into consideration the insurer's equity, its desired return on equity, or the return on equity of other insurers or other industries. Although the insurer's equity has no direct influence on pricing, it may have an influence on the quantity of exposures the insurer accepts and on its credit rating. This may have an indirect effect on the insurer's risk aversion factor since insurance must be competitive with the cost of conditional risk financing.

Other ratemaking considerations that have been discussed in the actuarial literature but are not relevant to the pricing of the transfer of risk according to the risk pricing model include the insurer's claim reserves on expired policies, the embedded yield in its investment portfolio, the uncertainty in the insurer's investment income or asset values, and its liquidity risk. The insurer's need for equity to support its anticipated growth, its legal organization as a mutual or a stock company, and the variation in the insurer's profitability arising from insurance industry competitive pricing cycles are also not considered. Restating this last point for emphasis, the risk pricing model indicates that insurers should not be rewarded with a larger profit margin simply due to variation in its profitability arising from insurance pricing cycles. In addition, the risk pricing model establishes a distinction between the insurer and the capital provider. The price for the risk transfer is evaluated solely from the perspective of the two participants in the risk transfer. An investor in the insurance company is not considered a participant in the transfer of risk. Consequently, the risk aversion or the marginal tax rate of an investor is not relevant for establishing the price of the risk transfer.

One benefit of the risk pricing model is that it provides a technique for evaluating risk and price without the assumption of perfect diversification across all exposures. Instead, the price for each individual exposure or any group of exposures treated as a single entity can be evaluated without reference to other exposures. This may be useful in situations that cannot be adequately explained by current theories. A potential application is suggested by Altman (1989), who observed that risky bonds achieve returns greater than should be expected based on their historical rates of default. Another application may be in an examination of the shape of the yield curve for risk-free bonds. By treating the risk-free rate as the certainty equivalent of the rate of future inflation, the short-term risk-free rate should be essentially identical to the expected rate of inflation. Since inflation is more uncertain over longer periods, this would justify a larger risk-free rate for longer-term bonds. To what extent this explains the shape of the yield curve would need to be determined.

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13. Appendix 1: The Axioms of Expected Utility Theory

Starmer (2000, p. 334) notes that expected utility theory (EUT) can be derived from three axioms: the ordering of preferences, continuity, and independence. The EUT model does not require that the outcomes be expressed in monetary terms or even numerically. The only requirement is that the individual must be able to identify a preference between any two uncertain prospects. Taking a more practical viewpoint, it will be assumed that all outcomes are expressed in monetary terms. For the two uncertain random variables *X* and *Y*, the notation *X* » *Y* will represent the individual's assessment that *X* is a more desirable prospect than *Y* (i.e., that *X* is preferred to *Y*).

The ordering axiom requires completeness, i.e., that either *X* » *Y* or *Y* » *X*, or both (denoted *X* ~ *Y*). Ordering also requires transitivity, so that if *X* » *Y* and *Y* » *Z*, then *X* » *Z*.

The continuity axiom requires that if *X* » *Y* and *Y* » *Z*, then there exists a probability *p* such that the compound lottery *(X,p,Z,1–p)* is equally preferred to *Y*. The notation *(X,p,Z,1–p)* represents uncertain damages which are equal to *X* with probability *p* or equal to *Z* with probability *(1 – p)*. The two axioms of ordering and continuity imply the existence of a preference function *U* which assigns a numerical value to each damage random variable, with *U(X) >= U(Y)* if and only if *X* » *Y*.

The third axiom, that of independence, requires that for all *X*, *Y*, and *Z*, if *X* » *Y* then *(X,p,Z,1–p)* » *(Y,p,Z,1–p)* for all probabilities *p*. Starmer (2000, p. 335) observes that "the independence axiom of EUT places quite strong restrictions on the precise form of preferences: it is this axiom which gives the standard theory most of its empirical content (and it is the axiom which most alternatives to EUT will relax)." Further discussion of the independence axiom and thhe alternatives to expected utility theory can be found in Machina (1987) and Starmer (2000). Given the three axioms, it can be shown that there exists a utility function *u* such that for all damage random variables *X*, the preference function *U* can be expressed as an expected utility:

$$U(X) = \sum p(x_i)u(x_i)$$

where $p(x_i)$ represents the probability of outcome x_i.

It should be observed that the diversification property differs from the axioms of EUT in that it is not based on an individual's preferences, but only on the behavior of the pricing function *P* for portfolios of exposures.

14. Appendix 2: The Arbitrage Theorem

The Arbitrage Theorem from the field of financial theory is basis for developing what is known as arbitrage-free pricing. Neftci (1996) describes the Arbitrage Theorem as providing a connection between risk and the time value of money. This result is in contrast to the risk pricing model, which does not consider time to be an element of risk. These two pricing techniques will be reconciled by demonstrating the relationship of the risk pricing model to the Arbitrage Theorem.

The Arbitrage Theorem can be described in terms of the following example. Consider three assets, each with a term of one year. The first asset is a bond with a current price of \$1 and having a risk-free yield of r_f. The second asset has a current price of *S* and pays either *T - \$1,000* with probability *p* or *T - \$2,000* with probability *q*, where $p + q = 1$. Both *S* and *T* are known values. The third asset is an insurance policy that has a current price of *P* and pays either \$1,000 or \$2,000, corresponding to the two outcomes for the second investment. The Arbitrage Theorem states that an arbitrage-free price *P* exists for the insurance if and only if there exist positive constants *u* and *v* such that:

$$\begin{bmatrix} 1 \\ S \\ P \end{bmatrix} = \begin{bmatrix} 1 + r_f & 1 + r_f \\ T - 1000 & T - 2000 \\ 1000 & 2000 \end{bmatrix} * \begin{bmatrix} u \\ v \end{bmatrix}$$

The matrix equation relates the present value of the three assets to their values one year in the future. The first row represents the ability of the individual to borrow money at the risk-free rate, and can be written as:

$$1 = (1 + r_f) * (u + v)$$

The values u and v are known as synthetic probabilities. Letting $y = (1 + r_f) * u$ and $z = (1 + r_f) * v$, it can be seen that y and z are positive values that resemble probabilities. That is:

$$1 = y + z$$

The second row of the matrix relates the current purchase price for an asset to the uncertain future outcomes. The third row represents the individual's ability to purchase insurance at a premium P to offset the uncertainty of the outcomes for the second asset. The uncertainty can be eliminated since the outcomes for the third asset are negatively correlated with those of the second asset.

Consider the equation describing the second asset:

$$S = (T - 1000) * u + (T - 2000) * v$$

Multiplying by $(1 + r_f)$, this becomes:

$$S * (1 + r_f) = (T - 1000) * y + (T - 2000) * z$$

The right hand side of the equation resembles an expected value calculation, but uses the synthetic probabilities y and z in place of the true probabilities p and q. The transformation of the true probability distribution into a synthetic probability distribution can be understood as a consequence of the risk aversion of the individual. If the individual were not risk averse, the synthetic probabilities would equal the true probabilities. For this reason, the current price S must correspond to the risk aversion factor for the individual. The risk aversion factor can be evaluated by testing various values in the risk pricing model. Assuming that a valid risk aversion factor is obtained (i.e., $e/c > 1$), the risk pricing model can then be used to determine the insurance price P. This result also guarantees the existence of the synthetic probabilities, as will be demonstrated below.

The following discussion will provide the construction of the synthetic probabilities corresponding to the risk pricing model. This will be done in the continuous case due to simplicity of notation. The initial step is to discount the uncertain outcomes to their present values X using the risk-free rate. Then, according to equation (1), the certainty equivalent price P for the exposure X can be expressed as:

$$EU(w - X + P) = 0$$

or:

$$\int U(w - x + P)f(x)dx = 0$$

where $f(x)$ is the probability density function of X.

The function $U(w - x + P)$ is:

$$c(P - x) \quad \text{for } P - x >= 0$$

and:

$$e(P - x) \quad \text{for } P - x < 0$$

In the earlier discussion, it was noted that c can be arbitrarily selected to be 1. For $x \neq P$, let $g(x)$ be defined as:

$$g(x) = [U(w - x + P) / (P - x)]f(x)$$

Using the definition of U, $g(x)$ can be expressed as:

$$f(x) \quad \text{for } x < P$$
$$ef(x) \quad \text{for } x > P$$

Define $g(x)$ at P as $g(P) = f(P)$. Then for all x:

$$U(w - x + P)f(x) = (P - x)g(x).$$

Since $g(x)$ is non-negative and $\int g(x)dx < e$, a synthetic probability density function $h(x)$ can be defined as:

$$h(x) = g(x) / \int g(x)dx$$

Based on these definitions, the expected utility can be restated as:

$$\int U(w - x + P)f(x)dx = \int (P - x)g(x)dx$$

Since the result on the left is 0:

$$\int xg(x)dx = P \int g(x)dx$$

Dividing both sides by $\int g(x)dx$ gives:

$$\int xh(x)dx = P$$

or:

$$E^{*}(X) = P$$

This result shows that the certainty equivalent price P can be determined as the expected value of the outcomes, where the expectation is based on a synthetic probability distribution h, with $h(x)$ equal to 0 only where $f(x)$ is 0. It should be noted that the construction of the synthetic probability density function $h(x)$ from $f(x)$ has relied on only two values: the certainty equivalent price P, and the risk aversion factor e. Since P is also a function of e, $h(x)$ can be considered to depend solely on the insured's risk aversion factor and the true probability distribution $f(x)$.