

The Real Option to Lapse a Variable Annuity: Can Surrender Charges Complete the Market?

Moshe A. Milevsky and¹ Thomas S. Salisbury
York University, Toronto
Ontario, M3J 1P3, Canada

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¹Milevsky (the contact author) is an Associate Professor of Finance at the Schulich School of Business, and can be reached at: Tel: (416) 736-2100 ext: 66014, Fax: (416) 225-5034, or E-mail: milevsky@yorku.ca. Updated versions of this paper can be downloaded from www.yorku.ca/milevsky. Salisbury is a Professor of Mathematics and Statistics at the Faculty of Arts and can be reached at: salt@yorku.ca. This research is partially supported by grants from NSERC, SSHRC, and The Mathematics of Information Technology and Complex System (NCE-MITACS) project. The authors would like to thank Glenn Daily, Steven Posner, Steven Shreve, Yisong Tian, Hans Tuenter and Ken Vetzal for helpful conversations throughout the development of this research.

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ABSTRACT

Variable Annuity (VA), similar to their mutual fund cousins, are managed pools of investments whose gains are tax-deferred. In addition, they provide money-back guarantees on invested principal. These guarantees – which can be viewed as non-separable put options with a possibly increasing strike price – are structured to mature upon death of the contract owner and/or at a pre-specified time horizon. The protection is paid for by installments – as opposed to up front – and is funded with a proportional insurance charge that is deducted from the underlying fund on a periodic basis.

Of great importance – and critical to any formal analysis of this product – is the fact that the holder can *lapse the contract* and instantaneously repurchase an identical investment to reestablish a new basis for the guarantee. In the absence of transaction costs, this would be optimal each and every time the value of the account reaches a new high. We classify this strategy as the Real Option to lapse. In this paper we analyze this product, by focusing on the *optimal time* to exercise the Real Option to lapse.

Our paper's conceptual contribution lies in highlighting the critical importance of the deferred surrender charge (DSC) – essentially transaction costs – in *completing* the market and allowing the claim to be hedged. Technically, we formulate the valuation exercise as a *optimal stopping* problem to provide a closed-form analytic solution and complete analysis when hazard rates are constant. Some numerical examples are provided to confirm that most annuity vendors are substantially overcharging for this guarantee.

JEL Codes: G13, G23, D91, G22

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1 Introduction.

Most of the existing literature on derivative pricing – starting with Black-Scholes/Merton (1973) – tacitly assumes that the price of an a option will be paid in advance, at the time of acquisition. In practice, however, many financial and especially insurance contracts, that contain embedded options, are purchased by a series of installments. So, in fact, a very minor payment is made up front, and the remainder is due over an extended period of time. Sometimes, the magnitude of the installment payments themselves is proportional to the value of the underlying security on which the option is written. Clearly, it is incorrect to amortize the cost of the option over the (expected) life of the contract. This is because the installment plan endows the holder (long position) with an additional *Real Option*¹ to terminate payments, or lapse the contract without having paid the full value of the embedded option to the seller (short position). Most likely – and rationally – the contract will be lapsed, or terminated, when the embedded options are out-of-the-money. Exercising this option is akin to defaulting on a swap contract when it becomes a liability, or a corporation abandoning a mine when it becomes uneconomical, although with less distasteful legal (and ethical) consequences.

One of the most common example of this *personal abandonment option* occurs with (tax sheltered) Variable Annuities, in the U.S. and Segregated Funds in Canada. These insurance-based products are quite similar² to their mutual fund cousins, in all financial, legal and accounting aspects, except that they contain an additional money-back guarantee (to the estate or heirs) upon death, or at some fixed horizon³. If the fund has lost money – net of any withdrawals – the insurance company will refund the difference. In fact, it is now possible to purchase almost any well known mutual fund product, with, or without, the added insurance protection. The embedded protection is paid for by installments – as opposed to up front – and funded via a continuous insurance charge that is deducted from the underlying fund on a periodic basis. In other words, the management fees

¹We are using the term real option in the personal, as opposed to corporate finance, sense of word. This is strictly different from the classical use of the term in the literature. We refer the interested reader to the work by Berk (1999), Amram and Kulatilaka (1999), Trigeorgis (1996), Ross (1995), Ingersoll and Ross (1992) and Hubbard (1994) for additional information about real options.

²The *raison detre* of the Variable Annuity (VA) is the convenient deferral of taxation on all investment gains until the funds are withdrawn or annuitized. In this paper we will not concern ourselves with the tax aspects of the product, other than to mention that one can, in fact, lapse a VA contract, and purchase another one, without incurring any tax consequences.

³The insurance lingo for this feature is a guaranteed minimum accumulation benefit (GMAB) which is available as an additional rider on most variable annuity policies.

are higher on these products, compared to regular mutual funds.⁴ The size of this market is not trivial. According to recent estimates by Moody's, there is approximately \$1 trillion U.S. invested in variable annuity policies with explicit maturity guarantees. In fact, Moody's recently issued a 'special comment' which was entirely devoted to the risks of these products.⁵

The impetus for our paper, is that the holder of these products – collectively known as *Maturity-Protected Investments* (MPIs) – can at any time sell and then repurchase the investment to reestablish a new basis for the guarantee. This, in theory, would be optimal each and every time the value of the account reaches a new high. By continuously engaging in this out-and-in transaction, the individual could theoretically convert a money back guarantee (i.e. a vanilla put) into a highest-value-achieved (i.e. lookback put option) guarantee. Of course, transaction costs, in the form of loads and surrender charges complicate this simple scheme, which is the essence of our analysis. We classify this feature as the Real Option (RO) to lapse.

For example, assume that a contract owner, who is 55 years old, invests \$10,000 in a variable annuity with a basic money-back guarantee at death, or in ten years, whichever comes first. The guarantee is funded with an additional 100 basis points annual fee that is charged to assets on a daily basis. If, for example, the investment doubles in value to \$20,000 over the next year, the contract owner is still paying 100 basis points for a put option which is out-of-the-money by 50%. Clearly, there is a huge incentive to sell the fund, and then re-purchase the exact same investment, to re-establish the guarantee at the new \$20,000 level. The companies providing the guarantee are aware of this and therefore impose a contingent deferred surrender charge (DSC) to 'force' investors to stay in the fund, or at the very least recoup some of the costs if investors decide to lapse. Even so – and despite the transaction costs – it may be optimal to swap the old out-of-the-money put, in an exchange for the new at-the-money put. This paper will derive the optimal lapsation policy, in the presence of a particular DSC schedule as well as the appropriate continuous insurance charge that funds the maturity/death guarantee. Our paper links the DSC, the continuous insurance charge and the optimal lapsation strategy in a parsimonious and tractable manner.

From an academic point of view, a large and diverse body of research has been published over the last 20 years on the topic of maturity-protected investments. In the finance and insurance literature these products have been analyzed under the title of equity-linked, or unit-linked, insur-

⁴According to Morningstar Inc., the average expense ratio on the universe of 8,200 U.S. mutual funds is 137 basis points, while the 6,600 variable annuity sub-accounts have an average expense of 211 basis points. The 74 b.p. difference can be viewed as payment for the option.

⁵See *Bells and Whistles: Credit Implications of the New Variable Annuities*. Moody's Investors Service, October 2000.

ance policies. The landmark contribution in the field, was the Brennan and Schwartz (1976), or Boyle and Schwartz (1977) extension of the Black-Scholes/Merton formula (1973) to equity-linked insurance contracts. They assumed a market that is complete in both financial and mortality risk. Therefore, all derivative prices can be expressed as suitable expectations with respect to an appropriate probability measure. Building on the complete markets framework, various researchers extended the analysis to stochastic interest rates and exotic payoff structures. See, for example, Bacinello and Ortu (1993, 1996), Ekern and Persson (1996), Nielsen and Sandman (1996), Persson and Aase (1997), Aase and Persson (1994) as well as Miltersen and Persson (1999). Without exceptions, all of the above mentioned papers on equity-linked insurance have focused on locating the *single initial premium* that funds, or pays for, the maturity benefit. In practice, of course, the guarantee is always paid by installments, which, as we have argued, completely changes the nature of the problem. More recently, and more practically, Windcliff, Forsyth and Vetzal (2000), as well as Boyle, Kolkiewicz and Tan (1999) have looked at the ‘reset’ features available in some of the variable annuities using Monte Carlo and numerical PDE approaches. Milevsky and Posner (2000), provided theoretical and empirical evidence on the cost structure of variable annuity contracts.

Of course, within the context of insurance, a complete market assumption implies that vendors can completely diversify their mortality risk by selling enough policies. In contrast to these assumptions, *our main argument is that when option premiums are paid by installments – even in the presence of complete mortality and financial markets – the ability to ‘lapse’ de facto creates an incomplete market in which the contingent claim can not be hedged.* Therefore, to salvage the hedge, our theoretical contribution is to identify the contingent deferred surrender charges (DSC) – properly calibrated to the optimal lapsation policy – that will complete the market.

Usually, one thinks of transaction costs, commissions and trading fees as impediments to pricing *via* risk-neutral expectations. Indeed, theoretical research by Garman and Ohlson (1981) and Dermody and Rockafellar (1991) as well as empirical work by Ronn (1987) has shown that ‘frictions’ induce a multitude of non-unique valuation operators, which is the essence of incomplete financial markets. In contrast, we will demonstrate that our particular contingent claim can only be hedged in the presence of a ‘friction’, which is the contingent deferred surrender charge (DSC). This is not unlike the ideas introduced by Dammon and Green (1987), and Prisman (1986), where tax (frictions) can induce equilibrium in the bond market.

From a slightly different perspective, our research is similar to Geske’s (1979) *compound option*, or Carr’s (1988) *sequential exchange opportunity* where the holder is granted the right (but not the obligation) to acquire another option at some future point in time. In our case, the contract owner

has the right to continue holding the fund – while paying additional expenses – and maintaining the ‘old’ money back guarantee. Likewise, our problem can also be positioned within the context of the vulnerable (defaultable) options literature, for example Johnson and Stulz (1987), but where the default emanates with the buyer, instead of the seller.

The technical contribution of this paper is to formulate⁶ the personal lapsation decision as an optimal stopping problem. The Real Option is then valued as an American contingent claim, with the strike price being equal to the underlying account value net of any deferred surrender charges. From that point, standard American option pricing techniques, such as Jack (1991), Kim (1990), or Huang, Subrahmanyam and Yu (1996), or the more recent Ju (1998) or Carr (1998), can be applied to locate the optimal lapsation boundary, and by inversion, the proper continuous insurance charge.

Conveniently, when the population hazard rate is assumed constant over time (exponential death) the free-boundary problem can be simplified *a la* McKean (1965), and the corresponding ODE can be solved, to obtain closed form analytic expressions for all quantities of interest. We provide expressions for the optimal level at which to lapse the contract, assuming a fixed asset and deferred surrender charge.

By means of an example we pre-empt our numerical results with the following case study; assuming the current 6% interest rate environment. A 60 year-old individual, with a life expectancy of 20 years⁷, that purchases a death-protected mutual fund with a (historical broad-based U.S. equity) volatility of 20%, an additional *continuous insurance charge* of 30 basis points per annum, and a 2% contingent deferred surrender charge, should optimally lapse the contract as soon as the account appreciates by exactly 57%. The owner should sell the position – incurring the 2% DSC – and then repurchase the exact same contract, which obviously generates a new at-the-money guarantee. From the insurer’s point of view, if the population exercises too early, the insurer pockets more surrender charges than are needed to cover the hedge. If the population exercises too late, then its guarantee – at least for a time – is at a lower level than the insurer is hedged against. So, when people die, the insurer does not have to pay out as much.

⁶Optimal stopping techniques, *vis a vis* the decision to surrender or lapse an insurance contract, have been employed in the academic literature. See for example Grosen and Jorgensen (1997) as well as Albizzati and Geman (1994). However, the options analyzed in those papers involve explicit interest rate guarantees that are added to insurance products and therefore more properly treated as financial option, rather than Real Options. More importantly, in contrast to those papers, our guarantees are paid by installments, which is the *essence* of our market incompleteness problem.

⁷This is according to the U.S. Decennial Life Tables for 1989-1991, compiled by the U.S. Department of Health and Human Services in conjunction with the National Center for Health Statistics

Stated differently, a 30 basis point *continuous insurance charge* together with a 2% contingent deferred surrender charge will completely fund the money back guarantee. However, the same model also indicates that *if* the insurance company charges less than 16 basis points for the guarantee, the mortality-contingent claim is essentially un-hedgeable, regardless of how high the contingent deferred surrender charge is set. Finally, if the insurance company charges more than 167 basis points for the guarantee, regardless of how low the deferred surrender charge is set, a rational individual will never purchase the product.

The remainder of this paper is organized as follows. Section 2 will introduce notation, terminology and then derive the general model for the optimal lapsing time and the value of the American contingent claim. Section 3 continues by making some specific assumptions about hazard rates and maturity guarantees which then allows for a closed-form analytic solution. Section 4 provides some numerical examples, while Section 5 concludes the paper with some general remarks and directions for further research. All technical proofs are relegated to the appendix.

2 The General Model

2.1 Underlying Asset and Dynamics

The contract owner pays exactly \$1, at time $t = 0$, to acquire a long position in the mutual fund, together with a non-separable increasing-strike put option that matures at $\min[\tau, T]$, where τ is the stochastic time of death and T is the maturity of the guarantee, if any. In other words, the put option guarantees at least e^{gt} , $g \geq 0$ at time T , or at death, whichever comes first. Of course, we can let $T = \infty$, which implies a guarantee at death only. In the most general formulation, the physical price process for the underlying asset obeys:

$$dU_t = (\mu_t - \alpha_t)U_t dt + \sigma(U_t, t)dB_t, \quad U_0 = 1, \quad (1)$$

under the statistical (or actuarial measure), where B_t is a standard Brownian motion, μ_t is the drift rate, net of any non-insurance management fees, and α_t is the to-be-determined continuous insurance charge that ‘pays for’ the option. The variable α_t can be viewed as a dividend yield outflow. Of course, the dividend does not go to the fund holder, but to the insurance company. In fact, actual dividends are assumed to be completely re-invested in the fund, and are therefore absorbed in μ_t . Nevertheless, the continuous-time payment $\alpha_t U_t$ flows to the insurance company providing the maturity/death guarantee. In practice, the underlying fund consists of a collection of individual securities, each following its own diffusion process. Without any loss of generality, we

will simply focus on the mutual fund value U_t , and assume that it can be treated as a single asset.⁸

The risk neutral process for the mutual fund, which we use for our computations, is:

$$dU_t = (r_t - \alpha_t)U_t dt + \sigma(U_t, t)d\tilde{B}_t, \quad U_0 = 1, \quad (2)$$

where r_t is the risk-free short rate, that replaces the drift, and \tilde{B}_t , is the Brownian motion under the new (Girsanov-transformed) measure.

One can also think of the underlying process U_t as:

$$U_t = S_t A_t, \quad \text{where} \quad A_t = e^{-\int_0^t \alpha_s ds} \quad (3)$$

and S_t is the market value of the assets supporting the fund. In other words:

$$dS_t = r_t S_t dt + \sigma(S_t, t)d\tilde{B}_t, \quad S_0 = 1 \quad (4)$$

Finally, we let:

$$R_t = e^{\int_0^t r_s ds}, \quad (5)$$

denote the standard money market account, where r_s is defined as above.

2.2 Lapsation and Deferred surrender Charges.

The owner can lapse or surrender (read: sell) the contract at any time $t < \tau \leq T$, and immediately receive an amount $(1 - k_t)U_t$, where $0 \leq k_t < 1$. The *deterministic* function k_t represents the contingent deferred surrender charge (DSC), which also goes directly to the company insuring the contract. One can think of the deferred surrender charge k_t , as both an incentive to remain invested in the contract, and, more importantly, as a mechanism for funding the put option. Intuitively, one can think of k_t as back-up for α_t , in the event the owner lapses before the original option has been fully paid for. Clearly, at the extreme, if $k_t = 1$, the owner will never lapse and the ‘full’ α_t will be collected for the entire life of the product.

Practically speaking, our model will locate the minimal (suitably defined) DSC needed to cover a fixed (suitably defined) α_t continuous insurance charge, as well as the minimal α_t required to fund the guarantee, in the presence of a fixed DSC. Indeed, as we shall prove later, if we assume a $k_t = 0$, for all $t \geq 0$ – in other words, no deferred surrender charges – there is no viable continuous insurance charge α_t that will fund the put option. In other words, if $k_t = 0$, then the only α_t that

⁸Technically speaking, there is an element of a ‘passport option’ – see Shreve and Vecer (2000) for recent details on these option – since the holder can re-allocate the individual sub-accounts to increase the value of the guarantee. We ignore this problem at a first pass.

will fund the guarantee is so high that the buyer's rational policy will be to lapse immediately. Such a product is not economically viable. Likewise, there is no deferred surrender charge schedule that can compensate for an $\alpha_t = 0$. A policy of not lapsing will simply leave no revenue with which to fund the guarantee. Both the deferred surrender and the continuous insurance charge are critical for completing the market and to properly hedge the contingent claim.

In particular, in the case of exogenously imposed constant values for the surrender and continuous insurance charge, our model will identify the (α, k) 'curve' that completely funds the guarantee. Any combination of parameters along this curve will result in a viable product. This curve will also induce lower and upper bounds for α , denoted by α_L and α_H respectively, outside which the product is unsustainable. More on this in section 4.

2.3 The Maturity and Death Guarantees

Let $G(U_t) = \max[e^{gt}, U_t]$, $g \geq 0$ denote the guaranteed amount. The fund payoff can therefore be described as follows:

$$\text{Payoff} = \begin{cases} (1 - k_t)U_t & \text{if Lapsed.} \\ \max[e^{g\tau}, U_\tau] & \text{at Death } (t = \tau) \\ \max[e^{gT}, U_T] & \text{at Maturity } (t = T) \end{cases} \quad (6)$$

The formulation in equation (6) is general enough to include cases with no maturity guarantee – such as with variable annuity policies in the U.S. – in which case we set $T = \infty$. Nevertheless, the following stylized facts should be evident from the structure of equation (6) First, our model assumes that all possible surrender charges are waived upon death. In other words, it is never optimal to lapse 'an instant' before death⁹ since $\max[e^{g\tau}, U_\tau]$ is strictly greater than $(1 - k_\tau)U_\tau$, whenever $k_\tau > 0$. Second, in the event of a time- T maturity guarantee, the money back guarantee implies that $k_T = 0$ and all surrender charges are eliminated. Note that we do not require k_t to be continuous at $t = T$.

Finally, it is very important to note that the actual word 'lapsation' can imply two very different activities. Lapsation can be *rational*, when it is immediately followed by a re-purchase and solely conducted to re-establish the basis of the guarantee. And lapsation can be *irrational*, when the deferred surrender charge 'penalty' exceeds the value of the new option. Regardless of whether lapsation was rational, or not, the payoff (or value) at the time of lapsation will always be $(1 - k_t)U_t$. Our intention is to locate the situations where it is rational to incur the $k_t U_t$ 'penalty' for the sole

⁹Note, however, that in practice, estate taxes might create an incentive to lapse the contract 'an instant' prior to death. Needless to say, we will ignore taxes and other market imperfections for the time being.

purpose of re-acquiring the contract and resetting the level of the guarantee. Our model, as it currently stands, does not account for consumption or other liquidity needs that would induce people to lapse for reasons other than swapping an old option, for a new one. As such, we stress for one final time that our optimal lapsation strategy means the optimal strategy for re-establishing the basis. Clearly, though, there is nothing irrational about withdrawing funds from an investment account, in order to fund general consumption needs.

2.4 The Death Rate

Let $\lambda_t \geq 0$ be the (hazard) death rate for the insured population holding the variable annuities. In our current formulation of the problem, and throughout the paper, the hazard rate is assumed to be deterministic. Therefore, we let:

$$\beta_t = e^{-\int_0^t \lambda_s ds} \quad (7)$$

denote the probability of any individual within the group surviving to time t , conditional on being alive at time zero. This implies that $\beta_0 = 1$ and $\lim_{t \rightarrow \infty} \beta_t = 0$.

More importantly, and quite critical to our model, we assume a *very large* pool of insured fund owners – each of whom invests a relatively small amount in the protected mutual fund – so that a fraction $\beta_t \lambda_t dt$ of the population dies between time t and $t + dt$. This is another way of stating the classical (and simplifying) assumption that mortality risk is completely diversifiable, and therefore not priced by the market. As a result, the outflow (or payout) due only to death between time t and $t + dt$ is precisely:

$$c_t dt := v_0 G(U_t) \beta_t \lambda_t dt, \quad (8)$$

where v_0 is the (very large) originally invested capital of all the fund owners. We are tracking the dynamic evolution of the entire sum of money which was originally invested in the death-protected mutual fund at time $t = 0$. Therefore, to be absolutely precise, we will in fact be computing the optimal lapsation policy for the entire population, as opposed to any particular individual within that group. Rational behaviour will be for all individuals still alive at that time – who have the same investment guarantee – to lapse simultaneously.

It should be emphasized that we are not going to hedge any individual variable annuity account. Rather, we will hedge the seller's exposure to all such accounts in aggregate.

2.5 The Hedge Portfolio

The insurance company insuring the protected mutual fund hedges the guarantee by trading the underlying asset S_t (not U_t) and the money market account R_t , during the life of the product. Up to the optimal lapsation time, the hedge portfolio will be denoted by:

$$V_t = \varphi_t S_t + \psi_t R_t \quad (9)$$

where φ_t is the amount held in the underlying security, and ψ_t is the amount invested in the money market account. This formulation is intuitively consistent with the firm's commitments, since, if there were no guarantees, $V_t := v_0 U_t$, and $\varphi_t := v_0 A_t$, as per equation (3). The presence of the guarantee 'forces' a ψ_t term, and a more complex trading strategy φ_t .

The hedging portfolio will obey the following stochastic differential equation:

$$dV_t = \varphi_t dS_t + \psi_t dR_t - c_t dt, \quad (10)$$

The term $\varphi_t dS_t + \psi_t dR_t$ in equation (10) is a self-financing portion, while the quantity, $c_t dt$ represents the (consumption) outflow due to death. At this point we have not determined (optimized) α_t and k_t , and therefore V_0 does not necessarily equal v_0 , the original amount invested. In other words, recall that our objective is to find the $\{\alpha_t, k_t\}$ 'pair' that makes the hedge self-financing. In our context, self-financing implies that the initial (aggregate) cost of the hedge, V_0 , is exactly equal to the initial (aggregate) amount invested by the unit holders, denoted by v_0 .

By the definition of a hedging strategy, for the portfolio V_t to cover the guarantee, we must have that:

$$V_t \geq L_t = \begin{cases} v_0(1 - k_t)\beta_t U_t & \text{for } 0 \leq t \leq T \\ v_0\beta_T G(U_T) & \text{for } t = T \end{cases}, \quad a.s. \quad (11)$$

The value of the portfolio (assets) must always exceed the liability. The portion $v_0(1 - k_t)\beta_t U_t$ covers the 'lapsation value' of the fund, for the fraction (β_t) who are still alive, while $v_0\beta_T G(U_T)$ covers the maturity guarantee.

2.6 Formulation as American Contingent Claim

Using martingale pricing methodology, extensively described in Karatzas and Shreve (1998), we now locate V_0 , which is the initial cost of the hedge. First, notice that although V_t , from equation (10), is not a self-financing portfolio (SFP), as a result of $c_t dt$, it can obviously be converted to an SFP by adding an appropriate term. The portfolio $V_t + \eta_t R_t$, is an SFP if we construct $d\eta_t = (c_t/R_t)dt$,

and $\eta_0 = 0$. This, in turn, implies that:

$$M_t := \frac{1}{R_t} (V_t + \eta_t R_t) = \frac{V_t}{R_t} + \eta_t, \quad (12)$$

is a Martingale. Therefore, since $R_0 = 1$, and $\eta_0 = 0$, we have that:

$$V_0 = M_0 = \tilde{E}_0[M_s] = \tilde{E}_0 \left[\frac{V_s}{R_s} + \int_0^s \frac{c_t}{R_t} dt \right] = \tilde{E}_0 \left[\frac{V_s}{R_s} + \int_0^s \frac{v_0 \beta_t \lambda_t G(U_t)}{R_t} dt \right], \quad (13)$$

for any $s > 0$, where $\tilde{E}_t[\cdot]$ denotes the risk-neutral conditional (on time- t) expectation. In words, the discounted value of the hedging portfolio *plus* the discounted sum of all payments made at death, is a Martingale. If, for example, the death rate and the continuous insurance charge are set to zero ($\alpha_t = 0$, $\lambda_t = 0$), while the deferred surrender charge is set at 100%, (i.e. $k_t = 1$), equation (13), with $s = T$, collapses to: $V_0 = \tilde{E}_0[\max[1, S_T]/R_T]$. This is the Black-Scholes/Merton, risk-neutral expectation for an at-the-money European put option plus a position in the underlying security.

Finally, our paper's main theoretical result is as follows: The initial value (cost) of the hedging portfolio for the variable annuities must satisfy:

$$V_0 = v_0 \sup_{0 \leq s \leq T} \tilde{E}_0 \left[\frac{L_s}{R_s} + \int_0^s \frac{\beta_t \lambda_t G(U_t)}{R_t} dt \right], \quad (14)$$

where the supremum in equation (14) is taken over all possible stopping times, in the early exercise sense of McKean (1965). Most importantly, the value of $s^*(U)$ which maximizes the risk-neutral expectation in equation (14), is the optimal lapsation time for the death-protected mutual fund.

At first glance, the expectation in equation (14), and the free boundary problem it creates, is more 'complicated' than a standard American option pricing situation. This is primarily due to the path-dependent nature of the integral term. And, in the most general case, all we can hope for are numerical approximation, *a la* techniques described in detail by Kim (1990) or Ju (1998), for example. Fortunately, as we shall see in the next section, when we impose a particular structure on the hazard rate λ_t (and maturity guarantee) and in addition we assume a fixed $\{\alpha, k\}$ 'pair', the problem can actually be solved in closed-form. This is quite similar to the valuation of the perpetual American put – which is available in closed form – because the relevant PDE collapses to an easily-solvable ODE.

To complete the argument, the expectation in equation (14) will provide us with V_0 . We then set $V_0 = v_0$, to locate the $\{\alpha_t, k_t\}$ surface. To understand why this is the case, imagine the following. If we assume that k_t is small enough and the $\alpha_t = 0$, then it must be that for any fixed

v_0 , $V_0/v_0 > 1$, since the initial capital is not enough to fund the benefit. Also, if $\alpha_t = \alpha \rightarrow \infty$, then $V_0/v_0 < 1$, since, $U_t \rightarrow 0$ and in this case the guarantee collapses to:

$$\frac{V_0}{v_0} = \max \left[1 - k_0, \int_0^T \frac{\beta_t \lambda_t e^{gt}}{R_t} dt + \frac{\beta_T e^{gT}}{R_t} \right]. \quad (15)$$

Now, $\int_0^T \beta_t \lambda_t dt = 1 - \beta_T$, so the second term in equation (15) is less than one, as long as $r_t > g \forall t$. This should be intuitive since otherwise the guarantee exceeds the risk free rate which can not be sustainable.¹⁰ Likewise, the first term in equation (15) is less than one, provided that $k > 0$. And so, $V_0/v_0 < 1$, when $\alpha \rightarrow \infty$, provided that both conditions apply. Therefore, in either event, there must exist some intermediate value of $\hat{\alpha}$, that results in $V_0/v_0 = 1$, and that exactly funds the maturity/death guarantee.

2.7 Alternative Risk-Neutral Representation.

Define the (future lifetime) random variable, τ , with density $\lambda_t \beta_t$, such that:

$$P(\tau > t) = \int_t^\infty \lambda_s \beta_s ds = \beta_t. \quad (16)$$

We can now express equation (13) as:

$$V_0 = v_0 \sup_{0 \leq s} E_0^* \left[\frac{G(U_{\min[\tau, s]})}{R_{\min[\tau, s]}} \right], \quad (17)$$

which provides a risk-neutral pricing relationship. The sup in equation (17), is over all stopping times (unlike τ) with respect to the filtration of U_t . We should point out at this juncture, that equation (17), and equation (14), are general enough to include situations where the guarantee expires at a certain age. This is the exact opposite of a maturity guarantee, and is common in some of the variable annuity policies. This essentially implies that if death occurs *after* a certain age, (age 80 for example) the payoff from the fund is limited to U_τ at the time of death, and not $G(U_\tau)$.

2.8 The Discounted Value of the Continuous Insurance Charges

In some cases, it might be important to compute the discounted value of the continuous insurance charge. First, let I_t^o , denote the stochastic – discounted to time zero – value of fees collected until time t , on a simple account of initial investment \$1. By construction, we have that:

$$dI_t^o = \frac{\alpha_t U_t}{R_t} dt. \quad (18)$$

¹⁰The few companies who do offer accumulation rates greater than the risk free rate are obviously undertaking a large amount of credit risk.

The quantity $\alpha_t U_t dt$ can be viewed as the instantaneous cash flow to the insurance company, while the R_t^{-1} factor, discounts the quantity to time zero. We are interested in both the dynamics of I_t^o and its risk neutral expectation $\tilde{E}_0[I_\xi^o]$, where ξ is a general stopping time for the process U_t . First, by a simple chain rule, we have that:

$$\begin{aligned}
d\left(\frac{U_t}{R_t}\right) &= \frac{-r_t U_t}{R_t} dt + \frac{1}{R_t} dU_t \\
&= \frac{-r_t U_t}{R_t} dt + \frac{(r_t - \alpha_t) U_t}{R_t} dt + \frac{\sigma(U_t, t) U_t}{R_t} d\tilde{B}_t \\
&= \frac{-\alpha_t U_t}{R_t} dt + \frac{\sigma(U_t, t) U_t}{R_t} d\tilde{B}_t \\
&= -dI_t^o + \frac{\sigma(U_t, t) U_t}{R_t} d\tilde{B}_t.
\end{aligned} \tag{19}$$

Therefore, by rearranging equation (19), and recalling that $R_0^{-1} U_0 = 1$, we have that:

$$\begin{aligned}
I_\xi^o &= \int_0^\xi dI_t^o = - \int_0^\xi d\left(\frac{U_t}{R_t}\right) + \int_0^\xi \frac{\sigma(U_t, t) U_t}{R_t} d\tilde{B}_t \\
&= 1 - \frac{U_\xi}{R_\xi} + \int_0^\xi \frac{\sigma(U_t, t) U_t}{R_t} d\tilde{B}_t.
\end{aligned} \tag{20}$$

The discounted value of the insurance risk charge, up to a stopping time (subject to standard integrability conditions) ξ , is: $1 - R_\xi^{-1} U_\xi$ plus an Ito (martingale) integral term, whose expectation is clearly zero. This implies:

$$\tilde{E}_0[I_\xi^o] = 1 - \tilde{E}_0 \left[\frac{U_\xi}{R_\xi} \right] = 1 - \tilde{E}_0 \left[S_\xi e^{-\int_0^\xi (r_t + \alpha_t) dt} \right] \tag{21}$$

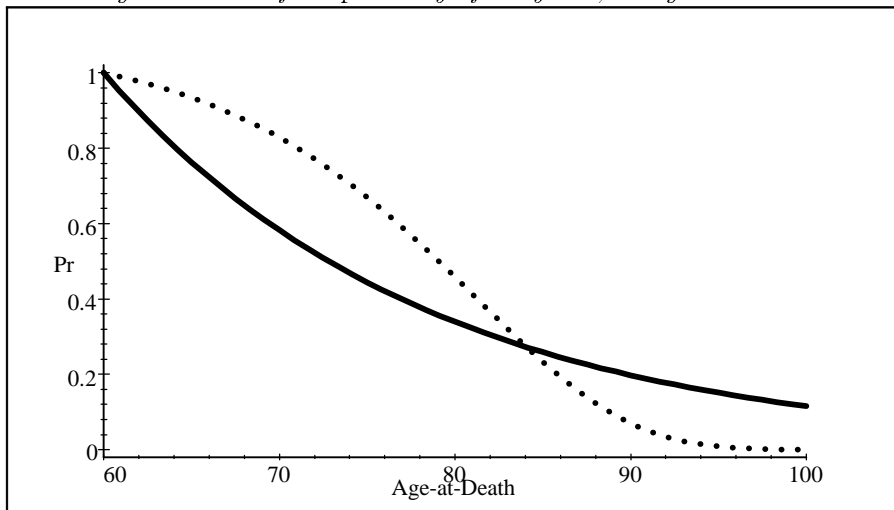
where the second equality comes from equation (3). In specific cases, equation (21) can be solved to provide the entire distribution of the discounted value of fees.

3 Analytic Solution.

3.1 Constant DSC and Hazard Rate with Guarantee at Death only.

In this section we make some assumptions on the structure of λ_t, α_t, k_t in order to derive complete analytic solutions for the optimal lapsation time. Specifically, we start by assuming that $\lambda_t = \lambda$, which implies that $\beta_t = \exp\{-\lambda t\}$. This is an exponential assumption for future lifetime, which can be calibrated to any mortality table by fixing the same life expectancy. For illustrative purposes, Figure 1 displays the relationship between the exponential distribution and a proper mortality table, assuming they share the same life expectancy, or first moment. Specifically, the graph shows the Cumulative Density Function (CDF) of the age-at-death random variable, under both assumptions.

Figure 1: The graph displays the relationship between the probability of survival under an exponential distribution assumption (solid line) and a realistic Society of Actuaries population mortality table (dashed line), assuming the same life expectancy of 20 years, at age 60.



In terms of ‘goodness of fit’ to population mortality, the exponential assumption kills too many people early on, and lets too many live later. As well, the exponential assumption allows for a finite probability of surviving to any age. This is clearly unrealistic, but, we claim it provides a reasonable first approximation for the optimal lapsation time.

Also, since most insurance companies charge the same insurance fee, α , regardless of the age of the account holder – which implies that the aggregate hazard rate is independent of age – we believe that an exponential assumption can be justified based on common practice in the industry.

In this section, we further assume that $T = \infty$, which means that the guarantee only applies at death. Likewise, we let $g = 0$, so that $G(U_t) = \max[1, U_t]$ and the deferred surrender charge is $k_t = k$ and the insurance charge is $\alpha_t = \alpha$. The constant assumption for both these variable is less problematic, since, in practice this is usually the case. Finally, we assume a simplified geometric Brownian motion economy with $\sigma(U_t, t) = \sigma U_t$, $r_t = r$, and $R_t = \exp\{rt\}$.

The main pricing equation, originally presented in equation (13), can be written as:

$$V_t = V(t, u) = v_0 \sup_{t \leq s} \tilde{E}_t \left[(1 - k) e^{-\lambda s} e^{-r(s-t)} U_s + \int_t^s \lambda e^{-\lambda q} e^{-r(q-t)} \max[1, U_q] dq \right], \quad (22)$$

where the first term in the expectation captures the surrender/lapsation value, and the second (integral) term captures the actual death benefit. Now, given the infinite maturity, we can re-write equation (22) as:

$$V(t, u) = e^{-\lambda t} v_0 W(u), \quad (23)$$

where

$$W(u) = \sup_{0 \leq s} \tilde{E}_0 \left[(1-k)e^{-(\lambda+r)s} U_s + \int_0^s \lambda e^{-(\lambda+r)q} \max[1, U_q] dq \right] \quad (24)$$

Using standard techniques obtained from the generator of the diffusion process – see Karatzas and Shreve (1998) for details – this leads to the following ordinary differential equation (ODE), satisfied by the valuation function $W(u)$;

$$(r - \alpha)uW'(u) + \frac{\sigma^2}{2}u^2W''(u) - (\lambda + r)W(u) = -\lambda \max[1, u], \quad (25)$$

on the interval $(0, L)$, and $W(u) = (1-k)u$, on the interval $[L, \infty)$, where L is the optimal lapsation boundary. Also, by construction, $W(1) = 1$.

The solution to this particular ODE, is:

$$W(u) = \begin{cases} \frac{\lambda}{\lambda+\alpha}u + b_1u^{a_1} + b_2u^{a_2} & 1 \leq u \leq L \\ \frac{\lambda+ru^{a_1}}{\lambda+r} & 0 \leq u \leq 1 \end{cases}, \quad (26)$$

where:

$$a_1 = \frac{-(r - \alpha - \frac{1}{2}\sigma^2) + \sqrt{(r - \alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)}}{\sigma^2}, \quad (27)$$

$$a_2 = \frac{-(r - \alpha - \frac{1}{2}\sigma^2) - \sqrt{(r - \alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)}}{\sigma^2} \quad (28)$$

and

$$b_1 = \frac{1}{a_1 - a_2} \left(\frac{a_1 r}{\lambda + r} - \frac{\lambda + \alpha a_2}{\lambda + \alpha} \right), \quad (29)$$

$$b_2 = \frac{1}{a_1 - a_2} \left(\frac{\lambda + \alpha a_1}{\lambda + \alpha} - \frac{a_1 r}{\lambda + r} \right) \quad (30)$$

The next step is to compute the optimal lapsation time, which is the level L , at which the fund should be lapsed, since the time dependency is irrelevant. This is similar to the process of computing the perpetual American put price by locating the lapsation level that maximizes the option value. In our case, the optimal lapsation boundary is at:

$$L = \left(\frac{b_2(1 - a_2)}{(a_1 - 1)b_1} \right)^{\frac{1}{a_1 - a_2}}, \quad (31)$$

and the corresponding¹¹ deferred surrender charge is:

$$k = 1 - \frac{\lambda}{\lambda + \alpha} - b_1 L^{(a_1 - 1)} - b_2 L^{(a_2 - 1)} \quad (32)$$

¹¹The details of this ‘proof’ are available from the authors upon request.

The feasible region for the continuous insurance is determined by the conditions $b_1 \geq 0$, and $L \geq 1$. This leads to a range of feasible values in between:

$$\alpha_L = \frac{\sigma^2}{2} \left[\frac{2r}{\sigma^2} - \frac{2(\lambda + r)}{a\sigma^2} - 1 + a \right], \quad (33)$$

where a is the unique root of the cubic equation,

$$a^3 - a^2 + a \frac{2(\lambda + r)}{\sigma^2} - \frac{(\lambda + r)^2}{r\sigma^2} = 0 \quad (34)$$

and

$$\alpha_H = \frac{\sigma^2 \lambda}{2r} \quad (35)$$

Naturally, a , the root of equation (34), must be obtained using (Newton) numerical methods. Note that when $\alpha = \alpha_L$, the DSC must be high enough to guarantee non-lapsation so that $L = \infty$. Likewise, when $\alpha = \alpha_H$, $L = 1$ and lapsation is instantaneous. The maximum contingent deferred surrender charge – that would make the product viable – is:

$$\bar{k} = 1 - \frac{\lambda}{\lambda + \alpha_L} = \frac{\alpha_L}{\lambda + \alpha_L}, \quad (36)$$

since $b_1 = 0$ and $L = \infty$. Stated differently, if the insurance company wants to levy the lowest possible fee on the variable annuity – denoted by α_L – they must charge \bar{k} so they have enough to cover the hedging cost. *A Deferred Surrender Charge of \bar{k} will complete the market.*

Finally, we conclude that the Total Expected (risk neutral) Discounted Fees consists of two portions. The first portion is the present value of fees collected until *the earlier* of death or lapsation. The second portion is the present value of the DSC, *if* lapsation occurs prior to death. Consequently,

$$\text{TEDF} = \left[1 - \tilde{E}_0 \left[R_{\min[\xi, \tau]}^{-1} U_{\min[\xi, \tau]} \right] \right] + \tilde{E}_0 \left[R_{\xi}^{-1} kL, \xi < \tau \right] \quad (37)$$

$$= 1 - (1 - k)L\tilde{E}_0 \left[R_{\xi}^{-1}, \xi < \tau \right] - \tilde{E}_0 \left[R_{\tau}^{-1} U_{\tau}, \xi \geq \tau \right] \quad (38)$$

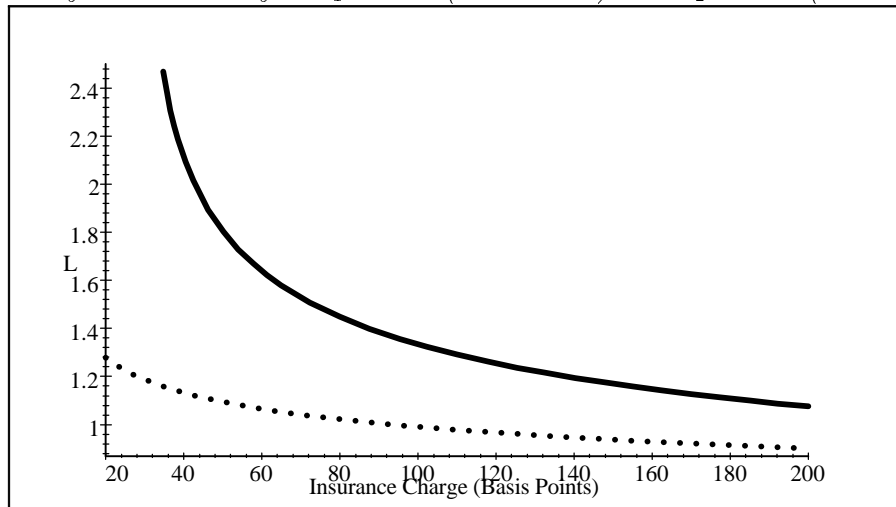
$$= \frac{\alpha}{\lambda + \alpha} (1 - L^{1-a_1}) + kL^{1-a_1} \quad (39)$$

In particular, if $\alpha = \alpha_L$ (which implies that $L = \infty$), we have that $\text{TEDF} = \alpha/(\lambda + \alpha)$.

4 Numerical Examples.

In the following example we will assume an $r = 0.06$ interest rate environment, and a population of individuals, each with a life expectancy of exactly $1/\lambda = 20$ years. According to equations (35,33), a variable annuity with a volatility of $\sigma = 15\%$, can ‘afford’ a continuous insurance expense charge of any number between $(10000\alpha_L =) 7.26$ and $(10000\alpha_H =) 93.7$ basis points. If, in fact, the

Figure 2: The graph displays the relationship between the continuous insurance charge (α), in basis points, and the optimal lapsation level (L). We assume an $r = 0.06$ interest rate, and a $1/\lambda = 20$ year life expectancy. The volatility is $\sigma_1 = 0.15$ (dashed line) and $\sigma_2 = 0.25$ (solid line).

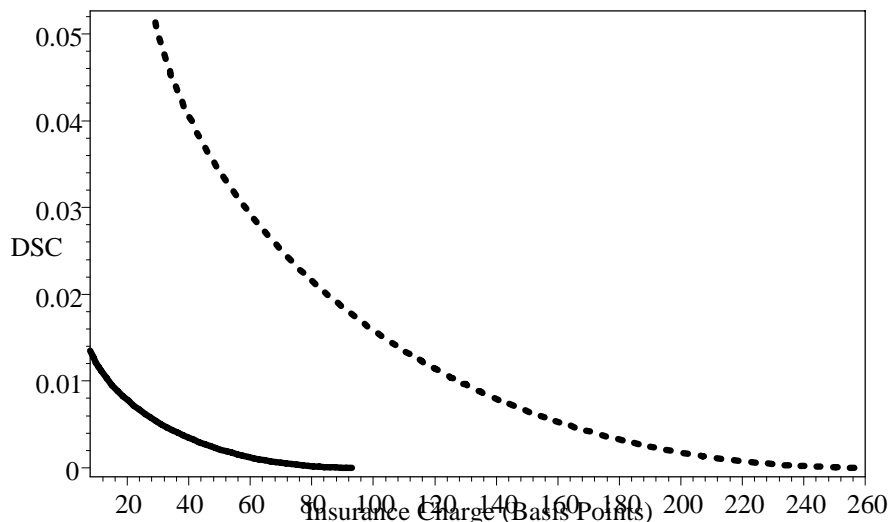


lowest insurance charge, of 7.26 basis points is levied on the fund, then the company must charge a DSC of exactly $k = 0.0143$ ($\approx 1.4\%$), according to equation (36). The optimal lapsation level, L , will depend on the particular ‘pair’ (α, k) chosen by the company. For example if $\alpha = 10$ basis point, then $L = 1.564$, and the individual should lapse as soon as the fund appreciates by 56.4%.

Figure (2) displays the graphical relationship between the continuous insurance charge (in basis points) and the optimal lapsation level. Specifically, we assume a 6% interest rate and a 20 year life expectancy. The two curves represent a volatility of 15% and 25% respectively. Intuitively, a higher curve indicates a higher volatility because the value of the embedded option is higher and therefore one should wait longer – i.e. higher price appreciation – before discarding the old option. Also, *ceteris paribus*, as the insurance asset charge increases, the optimal lapsation level decreases. This is because the costs of holding the fund are higher relative to the value of the guarantee as well as the exit costs. In fact, although it is hard to see from the figure, if the insurance charge is exactly equal to α_H , the optimal lapsation level ‘hits’ a value of one. This means that one should lapse the contract instantaneously after purchasing it, or in other words, the contract should never be purchased. Stated differently, if $\alpha \geq \alpha_H$, the contract is not viable since no rational individual will ever hold it.

In the same vein, Figure (3) displays the relationship between the continuous insurance charge (in basis points), and the contingent deferred surrender charge. Once again, we assume a 6% interest rate and a 20 year life expectancy. The two curves represent a volatility of 15% and 25%

Figure 3: The graph displays the relationship between the continuous insurance charge (α), in basis points, and the Deferred Sales Charge (k). We assume an interest rate of $r = 6\%$, and a life expectancy of $1/\lambda = 20$ years. The volatility is $\sigma_1 = 15\%$ (solid) and $\sigma_2 = 25\%$ (dashed).



respectively.

Perhaps at the risk of belaboring our main point, we emphasize that all combinations of (α, k) on the curve represent an appropriate charge for the embedded option. *In other words, there is no unique price for the contingent claim.* As the figure indicates – in the case of 25% volatility – the insurance company can pick a DSC of 5.3% and levy a continuous insurance charge of 28 basis points, or they can charge no DSC and impose a α of 260 basis points. Either of these pairs – or any combination on the curve in between – will create enough money to construct a self financing portfolio to hedge the option. Of course, the optimal lapsation policy on the part of the individual will depend on which one of these pairs is chosen by the company, as evidenced by Figure (2). Nevertheless, there is no right or wrong continuous insurance charge, rather, an entire range of values can be justified, provided the contingent deferred surrender charge is properly selected.

Finally, as a summary, Table #1 displays the relationship between volatility σ and the feasible region for the continuous insurance charges. For example, a mutual fund with a volatility of 30%, should, at the very least, levy a continuous insurance expense charge of 42.3 basis points. This (low) amount would fund the death benefit, *only* if the company imposes a DSC of 7.8%. If the DSC is set lower than 7.8%, when $\alpha = 42.3$ basis point, the death benefit is un-hedgeable. Obviously, as intuition would dictate, if the company wants to charge a higher α than 42.3 basis points, they can charge a lower DSC.

A few points are in order. First, when the life expectancy is greater than 20 years, i.e. the hazard rate $\lambda < 1/20$, the feasible region (α_L, α_H) uniformly moves lower. In other words, the company can charge less for both (α, k) . Intuitively, higher life expectancy is akin to lower volatility, which, as one can see in Table 1, has a dramatic effect on the values of α_L, α_H , and \bar{k} . For example, when $1/\lambda = 30$, which is roughly a 50 year old, we have that $\alpha_L = 22.1$ basis points, $\bar{k} = 6.2\%$ and $\alpha_H = 250$ basis points. In the same manner, a higher interest (discount) rate will also move the feasible region lower.

As a means of comparison, we contrast our numerical results to data supplied by Morningstar Inc., for the universe of 375 variable annuity policies (6,600 sub-accounts) sold in the United States. The median insurance charge levied on a simple *money back guarantee* variable annuity (fund) is 115 basis points. The median volatility for these funds is 18%, and the median surrender charge is 7%. According to equation (32), for a volatility of $\sigma = 0.18$, and $\alpha = 115$ basis points, with $r = 0.06$, and $1/\lambda = 20$, we get $k = 3.1 \times 10^{-4}$. If we take a more aggressive ten-year life expectancy ($1/\lambda = 10$), we get $k = 9.2 \times 10^{-3}$. Both are nowhere near the median 7% reported by Morningstar. However, if we let $1/\lambda = 4.4$ years, we obtain ‘fair’ pricing.

Stated differently, according to our model, the average insurance charge on a variable annuity sold in the United States can be justified, if one assumes that all contract owners are completely rational and when the typical contract owner is expected to die in exactly 4.4 years. Furthermore, since non-rational lapsation benefits the insurer, the reality of non-rational behaviour implies that even a lower average life expectancy makes observed prices “fair”. These results are consistent with the low numbers obtained in other estimates of the ‘value’ of the death benefit in variable annuities – see for example Milevsky and Posner (2000) and Windcliff, Forsyth and Vetzal (2000) – but fully accounts for lapsation in an economically parsimonious manner.

5 Conclusion.

In a recent issue (July 2000, page 48) of the *Dow Jones Investment Advisor*, a financial planner was quoted as saying:

“...with the guarantee on a variable annuity providing a protected floor...then if the investments pan out, we can always 1035-exchange¹² the client into another contract that allows us to establish a new higher death benefit. A few months ago we had a

¹²A 1035-exchange is the practitioner terminology for the section of the United States income tax act (ITA) which allows this transaction without inducing a taxable event.

| Assumptions: $r = 0.06, \lambda = 1/20$ | | | |
|---|-----------------|--------------|-----------------|
| Volatility (σ) | $10000\alpha_L$ | $100\bar{k}$ | $10000\alpha_H$ |
| 10% | 2.0 | 0.4 | 41.6 |
| 15% | 7.2 | 1.4 | 93.7 |
| 20% | 16.2 | 3.1 | 166.6 |
| 25% | 28.2 | 5.3 | 260.4 |
| 30% | 42.3 | 7.8 | 375.0 |
| 40% | 74.2 | 12.9 | 666.6 |

Table 1: The Effect of Volatility on the Feasible Region

73-year old client in tech stocks...when the account appreciated by 40% we did a 1035-exchange. The old contract had the old floor...we got a new contract with a 40% higher floor....This strategy only works...with products that contain no surrender charges...We like to structure it so it does not cost the client anything..."

As the above article indicates, it appears that Real Option to lapse is valuable and is quite popular. Although we obviously disagree with the second part of the quote, namely that it only ‘makes sense’ when there are no surrender charges. In any event, our paper has examined a very basic question faced by all investors in variable annuity contracts. The issue at hand is: *At what point should I lapse my contract and re-establish the basis of the guarantee?* We view the personal ability to lapse the investment as a Real Option that is analogous to the abandonment option, or the option to shut down, in classical corporate finance. The symmetric opposite side to this question is: *What is the optimal asset-based charge that funds the guarantee, assuming investors will lapse rationally?* Indeed, the answer to both questions lie in the structure of the deferred surrender charge.

Our paper’s main contribution lies in highlighting the critical importance of the deferred surrender charge (DSC) in *completing* the market and allowing the claim to be hedged. In some sense, one can say that transaction costs (frictions) complete the market and allow for the existence of a self-financing strategy. The self-financing strategy does not result in a unique price, per se, but rather a menu or schedule of charges that can support the claim. Technically, we formulated the problem of *when to lapse* as a free boundary problem and provided a closed-form analytic solution when hazard rates were constant.

Our model should enable users to answer any of the following questions:

1. By how much does the account value have to appreciate, before it is optimal to exercise the Real Option to lapse the contract, and re-establish a new (higher) basis?
2. Assuming a particular continuous insurance charge, what is the lowest contingent deferred surrender charge that will allow the company to recoup its hedging costs? Likewise, assuming a fixed DSC, what is the lowest continuous insurance charge that will allow the insurance company to recoup its hedging costs?
3. How high can the continuous insurance charge be set, while still maintaining a viable product? In other words, how expensive does the product have to be, for it to be optimal to *never purchase* the contract?

Further research will attempt to ‘solve’ the optimal lapsation policy for more general hazard rates, underlying processes and maturity guarantees. Also, the authors will examine some of the issues pertaining to hedging these guarantees, especially as it relates to using exotic products, such as Barrier options, that take advantage of lapsation behavior. Finally, the authors will use some of the recent work on minimizing Shortfall Risk, which is in the spirit of an actuarial approach, as an alternative to No Arbitrage valuation.

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