

The Financial Economics of Universal Life: An Actuarial Application of Stochastic Calculus

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Abstract

This paper uses standard financial economic tools to analyze the structure of a simple equity indexed universal life contract. The analysis separates the equity investment risk from the interest rate risk inside the policy by determining the replicating portfolio of bonds and equities that matches the cash flows generated by the insurance contract. The main result is that the insurance contract can be broken into a 'linear part', for which the replicating portfolio can be constructed in essentially closed form, and a 'non-linear part' which requires numerical or other methods for solution. A related result is that the linear part of the solution is independent of the stochastic process driving the equity asset but the non-linear part does depend on the details of the stochastic process.

Because an important element of the analysis can be carried out in closed form the model is a candidate for use as a case study in the application of stochastic calculus to actuarial problems. The paper's style of presentation therefore reflects the case study format.

1 Introduction

Universal Life has been an important element of the North American life insurance market for over 20 years. For many companies in the industry today the product is the core of their new business operations. Despite the product's importance relatively little has been written about the fundamental financial economics of universal life. This paper is a contribution to the filling of that gap.

One possible reason for this situation is that none of the valuation methods currently used in the United States (US STAT & US GAAP) focus on the product's financial economics directly. However, new Canadian GAAP valuation techniques are more cash flow based and a financial economic viewpoint can yield significant insight into the Canadian financial reporting process. The financial economic viewpoint is also directly relevant to the approach being developed by the International Actuarial Association for use as part of the emerging International Accounting Standards.

Readers who are familiar with the existing accounting models will find something new in the financial economic viewpoint outlined in this paper. A key result is that the policyholder account balance is not necessarily the correct measure of the insurer's financial exposure to the investment risk being passed through to the policyholder.

Following this introduction the paper has three main sections followed by a conclusion.

The three main sections are.

- Formulation of the problem: Stochastic differential equations for the account balance, persistency bonus and policy reserve are developed. The valuation problem is formulated as a partial differential equation (PDE) with appropriate boundary conditions.
- The linear solution: The PDE for policy reserves admits a linear solution which satisfies the boundary conditions at maturity but not at fund exhaustion. The linear solution is important for understanding the dynamics of the replicating portfolio even though it is only an approximation. Properties of the linear solution are discussed in considerable detail.
- Dealing with Fund Exhaustion: Girsanov's Theorem is used to get a theoretical expression for the adjustment required to get from the linear solution to the exact solution of the valuation problem. This representation allows us to conclude that the linear solution is most accurate in the large account balance limit.

2. Formulation of the Universal Life Valuation Problem

This section assumes the reader is familiar with both stochastic calculus concepts and life insurance in continuous time. It is necessary to start by introducing a significant amount of notation.

2.1 Evolution of the Account Value

Let $I(t)$ be an external equity index, which the insurer wishes to credit to its Universal Life policy. It is assumed that the index follows the standard log normal return model i.e.

$$dI = I(mdt + \sigma dz)$$

A fundamental characteristic of Universal Life is the maintenance of an account value $R(t)$ to which the insurer credits premiums and interest and deducts expense and insurance charges. The account value is used to determine death benefits, surrender benefits and lapse benefits. In this paper $R(t)$ will be assumed to follow the Ito stochastic differential equation

$$dR = R(mdt + \sigma dz - \tilde{\Delta}dt) - \tilde{\mu}(F + \varepsilon R - R)dt + (g - \tilde{e})dt.$$

In the first term $\tilde{\Delta} = \tilde{\Delta}(t)$ is a deterministic spread that is deducted from the index return to determine the amount of interest credited to the policyholder.

The second term is the product of a Cost of Insurance (COI) rate $\tilde{\mu} = \tilde{\mu}(t)$ and the net amount at risk. The net amount at risk assumes the death benefit is a fixed Face Amount F plus ε times the account value $R(t)$. The parameter ε will be 0 or 1 depending on the type of contract.

The final term in the stochastic differential equation for R represents the crediting of premiums the deduction of non-spread related expenses. It will be assumed that premiums are a known deterministic function of time i.e. $g = g(t)$ and that the expense charge is also a deterministic function of the form $\tilde{e} = \tilde{\lambda}g(t) + \tilde{f}(t)$.

Collecting terms in the equation above we can write

$$dR = R(m - \tilde{\Delta} + \bar{\varepsilon}\tilde{\mu})dt + (g - \tilde{e} - \tilde{\mu}F)dt + \sigma R dz \quad (1)$$

where we have used the notation $\bar{\varepsilon} = 1 - \varepsilon$.

2.2 Evolution of the Accrued Bonus

Most Universal Life contracts in North America today offer a persistency bonus. This is a random variable $B(t)$ which accrues during the life of the policy and is credited to the account value at specified policy durations (τ_1, \dots, τ_n) . The bonus is a product feature

that encourages favorable policyholder behavior and it also plays an important role in the sales illustration process.

For this study we will assume that the bonus evolves according to a stochastic differential equation of the form

$$dB = \alpha_t R dt + \beta_t [R(m - \tilde{\Delta}) dt + \sigma dz] + \gamma_t \tilde{\mu} (F - \bar{\epsilon} R) dt .$$

The first term represents a bonus accrual that is proportional to the account value. This bonus behaves somewhat like an additive increase to the credited interest rate. The second term accrues a bonus proportional to the credited interest while the third term is designed to refund a proportion of the Cost of Insurance (COI) charges. By collecting terms this equation can be rewritten as

$$dB = [R(\alpha_t + \beta_t(m - \tilde{\Delta}) - \gamma_t \bar{\epsilon} \tilde{\mu}) + \gamma_t \tilde{\mu} F] dt + \sigma \beta_t R dz. \quad (2)$$

It would be unusual for any one policy form to offer more than one bonus structure at the same time. However, the different bonus structures will affect the replicating portfolio in different ways so we will carry all three types forward in the analysis.

Bonus accruals are often set to zero during the first 10 policy years and then start to pay at regular intervals, such as every five years, thereafter. Typical values for the accrual parameters might be $\alpha_t = 0$ for years 1-10 followed by accrual rates of 1%-2% with the first bonus actually being paid at duration 15. When proportional bonuses are used the bonus parameter β is often in the 10%-15% range once it becomes non zero. When the return of COI bonus structure is used the γ parameter usually varies between 0 and 20%. These are not the only bonus structures in use in the market place.

To summarize the discussion so far, we have a system of stochastic differential equations for the joint evolution of the account value R and bonus B given by equations (1) and (2) respectively. When the account value is positive these equations will hold at all times except those times τ when the bonus is credited. On a bonus payment date we have

$$\begin{aligned} R(\tau) &\leftarrow R(\tau) + B(\tau) \\ B(\tau) &\leftarrow 0 \end{aligned}$$

If the account balance becomes negative we say that the policy has entered fund exhaustion. What happens then depends on the insurer's administrative practices. In this paper we will assume that when funds exhaust

- all accrued bonuses are forfeited and

- life insurance coverage terminates unless the policyholder elects to pay sufficient premium to cover the ongoing cost of insurance and expense charges that keep the account value at 0 i.e. $dR=0$ and $dB=0$ from that point on.

2.3 Partial Differential Equation for the Reserve V

This section will use the principles of stochastic calculus, financial economics and traditional actuarial science to derive a partial differential equation for a reserve $V=V(t,R,B)$. In this context the word ‘reserve’ means the value of a portfolio of debt and equity index investments whose cash flows allow the insurer to meet the insurance contract’s cash flow obligations. When an appropriate level of conservatism is included in the assumptions many actuaries would call the quantity V the ‘fair value’ of the insurance liability.

What follows is a three step process where we consider

- the change in liability due to actuarial contingencies
- the change in liability due to financial issues
- the change in assets due to financial issues and policy mechanics

When these three steps are complete we will arrive at an equation which essentially says that the expected change in liability is equal to the expected change of the assets in the replicating portfolio.

The analysis begins by looking at how the reserve changes when the actuarial contingencies of death, surrender and partial withdrawal are considered. For simplicity we will ignore other product features such as policy loans.

Death Benefits: We will assume a force of mortality $\mu^d = \mu^d(t)$ and a death benefit equal to the face amount plus ϵ times the account value $DB = F + \epsilon R$. The expected change in liability due to mortality for a time increment dt is therefore equal to the change in liability $(F + \epsilon R - V)$ times the probability of death $\mu^d dt$

$$(F + \epsilon R - V)\mu^d dt .$$

Surrender Benefits: We will assume a force of surrender $\mu^s = \mu^s(t)$ and a surrender benefit equal the account value less a duration dependent surrender charge i.e. the cash value is of the form $CV(t, R) = \max(0, R(t) - SC(t))$. The contribution of the surrender contingency to the expected change in liability for a time increment dt is then

$$(\max(0, R - SC) - V)\mu^s dt .$$

Partial Withdrawal: Most Universal Life contracts allow for some withdrawal from the account value without causing the policy to terminate or incurring a surrender penalty.

This is often a significant benefit at later policy durations if amounts can be withdrawn, tax free, to help finance retirement.

A full analysis of all the issues surrounding partial withdrawal and policyholder behavior is very complex but a reasonable starting point is to assume

- there is a force of withdrawal $\mu^w = \mu^w(t)$ which drives the incidence of partial withdrawal events and
- the severity of partial withdrawal is an amount xR where x has a probability distribution $f(t,x)$ on $[0,1]$.

Under these assumptions the expected change in liability due to partial withdrawal, in a time increment dt , is

$$\left[\int_0^1 \{xR + V(t, R - xR, B)\} f(t, x) dx - V(t, R, B) \right] \mu^w dt$$

This is still too complex to work with so we will make the additional simplifying assumption that the expression above can be approximated by using a Taylor expansion

$$V(t, R - xR, B) = V(t, R, B) - xR \frac{\partial V}{\partial R} + \frac{1}{2} (xR)^2 \frac{\partial^2 V}{\partial R^2}.$$

When this approximation is used the expected change in liability due to partial withdrawal becomes

$$[\kappa R - \kappa R \frac{\partial V}{\partial R} + \frac{1}{2} v^2 R^2 \frac{\partial^2 V}{\partial R^2}] \mu^w dt$$

Where

$$\kappa(t) = \int x f(t, x) dx$$

and

$$v^2(t) = \int x^2 f(t, x) dx$$

are the first and second moments of x respectively. Estimates of these quantities would have to be developed from experience studies.

The second step in the analysis of the liability to consider what happens if none of the insurance contingencies occur, an event with probability $(1 - (\mu^d + \mu^s + \mu^w)dt)$. The change in liability will be driven by the evolution of the account value R and accrued bonus B . The change in liability is $V(t + dt, R + dR, B + dB) - V(t, R, B)$. We can now use the tools of stochastic calculus to write this change in liability as

$$dV = \left(\frac{\partial V}{\partial t} + [R(m - \tilde{\Delta} + \bar{\varepsilon}\tilde{\mu}) + (g - \tilde{e} - \tilde{\mu}F)] \frac{\partial V}{\partial R} + [R(\alpha_i + \beta_i(m - \tilde{\Delta}) - \gamma_i \bar{\varepsilon}\tilde{\mu}) + \gamma_i \tilde{\mu}F] \frac{\partial V}{\partial B} \right) dt + \frac{1}{2} (\sigma^2 R^2 \frac{\partial^2 V}{\partial R^2} + 2\beta\sigma^2 R^2 \frac{\partial^2 V}{\partial R \partial B} + \beta^2 \sigma^2 R^2 \frac{\partial^2 V}{\partial B^2}) dt + \sigma \left(\frac{\partial V}{\partial R} + \beta \frac{\partial V}{\partial B} \right) dz$$

When we combine this with the actuarial analysis done earlier we conclude that the total change in liability ΔL has a mean given by

$$E(\Delta L) = [(F + \varepsilon R - V)\mu^d + (\max(0, R - SC) - V)\mu^s + (\kappa R - \kappa R \frac{\partial V}{\partial R} + v^2 R^2 \frac{\partial^2 V}{\partial R^2})\mu^w] dt + (1 - (\mu^d + \mu^s + \mu^w) dt) E(dV)$$

and the sensitivity to a change in the random element driving the equity index is

$$\sigma R \left(\frac{\partial V}{\partial R} + \beta \frac{\partial V}{\partial B} \right) dz.$$

We will combine all these results into the one statement, accurate to first order in dt ,

$$\Delta L = [(F + \varepsilon R - V)\mu^d + (\max(0, R - SC) - V)\mu^s + (\kappa R - \kappa R \frac{\partial V}{\partial R} + v^2 R^2 \frac{\partial^2 V}{\partial R^2})\mu^w] dt + \left(\frac{\partial V}{\partial t} + [R(m - \tilde{\Delta} + \bar{\varepsilon}\tilde{\mu}) + (g - \tilde{e} - \tilde{\mu}F)] \frac{\partial V}{\partial R} + [R(\alpha_i + \beta_i(m - \tilde{\Delta}) - \gamma_i \bar{\varepsilon}\tilde{\mu}) + \gamma_i \tilde{\mu}F] \frac{\partial V}{\partial B} \right) dt + \frac{1}{2} (\sigma^2 R^2 \frac{\partial^2 V}{\partial R^2} + 2\beta\sigma^2 R^2 \frac{\partial^2 V}{\partial R \partial B} + \beta^2 \sigma^2 R^2 \frac{\partial^2 V}{\partial B^2}) dt + \sigma R \left(\frac{\partial V}{\partial R} + \beta \frac{\partial V}{\partial B} \right) dz$$

or

$$\Delta L = [(F + \varepsilon R - V)\mu^d + (\max(0, R - SC) - V)\mu^s + \kappa R \mu^w] dt + \left(\frac{\partial V}{\partial t} + [R(m - \tilde{\Delta} + \bar{\varepsilon}\tilde{\mu} - \kappa \mu^w) + (g - \tilde{e} - \tilde{\mu}F)] \frac{\partial V}{\partial R} + [R(\alpha_i + \beta_i(m - \tilde{\Delta}) - \gamma_i \bar{\varepsilon}\tilde{\mu}) + \gamma_i \tilde{\mu}F] \frac{\partial V}{\partial B} \right) dt + \frac{1}{2} ((\sigma^2 + v^2 \mu^w) R^2 \frac{\partial^2 V}{\partial R^2} + 2\beta\sigma^2 R^2 \frac{\partial^2 V}{\partial R \partial B} + \beta^2 \sigma^2 R^2 \frac{\partial^2 V}{\partial B^2}) dt + \sigma R \left(\frac{\partial V}{\partial R} + \beta \frac{\partial V}{\partial B} \right) dz$$

This equation captures the mean of the change in liability and random fluctuations due to market movements. It does not capture the random fluctuations due to variations in actuarial experience or policyholder behavior.

We now examine the asset dynamics. We will assume total assets backing the liability in the amount V with an amount QR invested in the equity index and the balance $(V - QR)$ invested in a debt asset which earns a force of interest δ after allowance for default and investment expense. The change in asset value ΔA that corresponds to the change in liability ΔL can then be written as

$$\Delta A = QR(mdt + \sigma dz) + \delta(V - QR)dt + (g - e)dt$$

The term $(g - e)dt$ represents the increase in assets due to the receipt of gross premiums g (the same premium that goes into the account value) less an expense allowance e . The expense allowance will generally be different from the expense load $\tilde{e} = \tilde{\lambda}g + \tilde{f}$ deducted from the account value. Here we will assume a combination of spread related and fixed expenses of the form

$$e = \Delta(t)R + \lambda g + f$$

where $\Delta(t)$ is a spread related expense which will allow for items such as asset based commissions, Canadian investment income tax, or any other expense which the insurer chooses to recover through the spread mechanism.

The asset and liability analyses are now complete. The final step is to demand that the change in asset match the change in liability for any equity return i.e.

$$\Delta L = \Delta A$$

or

$$\begin{aligned} & [(F + \varepsilon R - V)\mu^d + (\max(0, R - SC) - V)\mu^s + \kappa R\mu^w]dt \\ & + \left(\frac{\partial V}{\partial t} + [R(m - \tilde{\Delta} + \tilde{\varepsilon}\tilde{\mu} - \kappa\mu^w) + (g - \tilde{e} - \tilde{\mu}F)]\right)\frac{\partial V}{\partial R} + [R(\alpha_i + \beta_i(m - \tilde{\Delta}) - \gamma_i\tilde{\varepsilon}\tilde{\mu}) + \gamma_i\tilde{\mu}F]\frac{\partial V}{\partial B}dt \\ & + \frac{1}{2}((\sigma^2 + v^2\mu^w)R^2\frac{\partial^2 V}{\partial R^2} + 2\beta\sigma^2R^2\frac{\partial^2 V}{\partial R\partial B} + \beta^2\sigma^2R^2\frac{\partial^2 V}{\partial B^2})dt + \sigma R\left(\frac{\partial V}{\partial R} + \beta\frac{\partial V}{\partial B}\right)dz \\ & = QR(mdt + \sigma dz) + \delta(V - QR)dt + (g - e)dt. \end{aligned}$$

Equating the coefficient of the random element dz on both sides of the this equation yields,

$$\sigma R\left(\frac{\partial V}{\partial R} + \beta\frac{\partial V}{\partial B}\right)dz = \sigma QRdz,$$

so that the **equity ratio** Q is given by

$$Q = \left(\frac{\partial V}{\partial R} + \beta\frac{\partial V}{\partial B}\right).$$

This is a standard technique of financial economics for determining the appropriate hedging strategy.

With this result we can now equate the means and write $E(\Delta L) = E(\Delta A)$ as

$$\begin{aligned} & \frac{\partial V}{\partial t} + [R(\delta - \tilde{\Delta} + \bar{\epsilon}\tilde{\mu} - \kappa\mu^w) + (g - \tilde{e} - \tilde{\mu}F)] \frac{\partial V}{\partial R} + [R(\alpha_i + \beta_i(\delta - \tilde{\Delta}) - \gamma_i\bar{\epsilon}\tilde{\mu}) + \gamma_i\tilde{\mu}F] \frac{\partial V}{\partial B} \\ & + \frac{1}{2}[(\sigma^2 + \nu^2\mu^w)R^2 \frac{\partial^2 V}{\partial R^2} + 2\beta\sigma^2 R^2 \frac{\partial^2 V}{\partial R\partial B} + \beta^2\sigma^2 R^2 \frac{\partial^2 V}{\partial B^2}] \\ & + [(F + \epsilon R - V)\mu^d + (\max(0, R - SC) - V)\mu^s + \kappa R\mu^w] = \delta V + g - (\Delta R + \lambda g + f). \end{aligned} \quad (3)$$

This is the Fundamental Partial Differential Equation that we have been working to formulate. It is the statement that the total expected rate of change in the liability, due to both actuarial and financial issues, is equal to the expected rate of change in the asset, conditioned on the investment risk being hedged according to the strategy

$$Q = \left(\frac{\partial V}{\partial R} + \beta \frac{\partial V}{\partial B} \right).$$

We note that when that risk is hedged the expected return on the equity index m drops out of the analysis and is replaced by the force of interest δ . This is an example of the risk neutral valuation principle at work.

It is also useful to note that the fundamental equation allows for uncertainty in both the economic environment and policyholder behavior.

2.4 Boundary Conditions

The fact that a function $V=V(t,R,B)$ satisfies equation (3) derived above does not necessarily mean that it is the solution to the valuation problem. There are an infinite number of functions which satisfy any given differential equation. The way to isolate the solution that solves the problem is to specify boundary conditions. As the name implies boundary conditions refer to the function's behavior at certain extreme points of time or space. For this problem there are four relevant boundary situations to discuss

- *Value at Maturity*: If the contract matures or endows at policy duration T (e.g. age 100) then the reserve must accrue the maturity benefit. We will assume that the policy pays the face amount plus the account value plus any accrued bonus at maturity. The relevant boundary condition is then

$$V(T, R, B) = F + R + B$$

- *Value before and after a persistency bonus is paid*: If a persistency bonus is credited at duration τ then the reserve just before the payment is $V(\tau, R, B)$ while the value just after the payment is $V(\tau, R + B, 0)$. For consistency these values must be equal so at bonus payment dates

$$V(\tau, R, B) = V(\tau, R + B, 0)$$

- *Value at Fund Exhaustion $R=0$:* It is possible that a combination of investment returns, partial withdrawals and premium payments leads to a scenario where the account value exhausts before the maturity date. When this occurs, or even before, most insurers notify the policyholder that the insurance coverage will lapse unless additional premiums are paid. The policyholder now has an option to continue the contract or not.

For the proportion $0 < \pi(t) < 1$ of policyholders who elect to continue the contract we will assume they pay a premium \hat{g} just sufficient to keep the policy in force. This would imply

$$\hat{g} = \tilde{e} + \tilde{\mu}F = \tilde{\lambda}\hat{g} + \tilde{f} + \tilde{\mu}F$$

Given a suitable force of lapsation μ^l , the reserve for this situation can be written as

$$\int_t^\infty e^{-\int_t^s (\delta + \mu^d + \mu^l) ds} (\mu^d F + e - \hat{g}) dt = \int_t^\infty e^{-\int_t^s (\delta + \mu^d + \mu^l) ds} [(\mu^d F - \tilde{\mu}F) + (e - \tilde{e})] dt$$

For the remaining proportion $(1 - \pi)$ which lapse an appropriate liability to hold would be $(F - \hat{g}/12)\mu^d/12$. This liability reflects the industry practice of allowing the policyholder a 30 day grace period to pay the premium. Thus, if death occurs in that period, the death benefit paid is the Face Amount adjusted for the premium $\hat{g}/12$ that would have been sufficient to keep the policy in force for 30 days.

Putting these two pieces together we get the boundary condition for $R=0$ as $V(t, 0, B) = X(t)$ where the exhaust value $X(t)$ is given by

$$X(t) = \pi \int_t^\infty e^{-\int_t^s (\delta + \mu^e + \mu^l) ds} [(\mu^d F - \tilde{\mu}F) + (e - \tilde{e})] ds + (1 - \pi)[(F - \hat{g}/12)\mu^d/12].$$

In many practical situations this boundary condition would not be materially different from zero. However, there are some situations, such as Canadian products with level cost of insurance scales, where the choice of assumptions governing fund exhaustion can be very important.

- *Value in the large account balance limit.* Both US and Canadian tax authorities put limits on the amount of money that can build up inside an insurance policy without triggering adverse tax consequences. This usually forces the policyholder to withdraw funds, reduce premiums or increase the death benefit so that the account value stays

within the allowed range. We will not attempt to address this issue any further in this paper because the details vary by tax jurisdiction.

This completes the formulation of the partial differential equation and boundary conditions for the equity indexed Universal Life problem. The remainder of this paper focuses on techniques for extracting useful information from this formulation.

3 The Linear Solution

3.1 Formulation

This section simplifies the problem of solving the fundamental differential equation (3) by assuming a solution of the form $V(t, R, B) = H(t)R + L(t)B + K(t)$ which is linear in the variables R and B . This simplification is justified by the fact that the fundamental equation (3) is, almost, linear in those variables and hence should have a linear solution. The only non linear term in (3) arises from the fact that cash values can not be negative. We will call $H(t)$ the **hedge ratio**, $L(t)$ the **bonus ratio** and $K(t)$ the **insurance reserve** respectively.

Associated with a solution of this form is the equity exposure

$$QR = \left(\frac{\partial V}{\partial R} + \beta \frac{\partial V}{\partial B} \right) R = (H + \beta L)R.$$

The total reserve then decomposes as into equity and debt components as follows:

$$V(t, R, B) = H(t)R + L(t)B + K(t) = [H(t) + \beta(t)L(t)]R + [L(t)(B - \beta(t)R) + K(t)]$$

We see immediately that the multiplicative bonus structure ($\beta \neq 0$) requires a loan in the amount βLR from the debt side to equity side of the contract.

The analysis that follows is designed to determine the functions H , L and K so that the asset portfolio above is capable of replicating the policy cash flows, under the simplifying assumptions we have made. When this program is complete we will see that the A/L M problem has been analyzed into three different portfolio requirements

- An equity fund in the amount $QR = (H + \beta L)R$,
- A fund invested (or borrowed) short in the amount $L(B - \beta R)$,
- A portfolio of longer debt instruments in the amount $K(t)$.

We will refer to these as the equity fund, the short fund and the long or insurance fund respectively. The cash flows associated with the long fund will turn out to be very similar to the cash flows associated with traditional insurance products.

We start by rewriting equation (3) as

$$\frac{\partial V}{\partial t} + [R(\delta - \tilde{\Delta} + \bar{\varepsilon}\tilde{\mu} - \kappa\mu^w) + (g - \tilde{e} - \tilde{\mu}F)] \frac{\partial V}{\partial R} + [R(\alpha_t + \beta_t(\delta - \tilde{\Delta}) - \gamma_t\bar{\varepsilon}\tilde{\mu}) + \gamma_t\tilde{\mu}F] \frac{\partial V}{\partial B} + [(F + \varepsilon R - V)\mu^d + (R - SC - V)\mu^s + \kappa R\mu^w] = \delta V + g - (\Delta R + \lambda g + f).$$

The linearity of the solution has allowed us to drop all the second order terms and we are also ignoring the cash value floor of zero. Substituting $V(t, R, B) = H(t)R + L(t)B + K(t)$ we find

$$\begin{aligned} & \frac{\partial(HR + LB + K)}{\partial t} + [R(\delta - \tilde{\Delta} + \bar{\varepsilon}\tilde{\mu} - \kappa\mu^w) + (g - \tilde{e} - \tilde{\mu}F)]H + [R(\alpha_t + \beta_t(\delta - \tilde{\Delta}) - \gamma_t\bar{\varepsilon}\tilde{\mu}) + \gamma_t\tilde{\mu}F]L \\ & + [(F + \varepsilon R - HR - LB - K)\mu^d + (R - SC - HR - LB - K)\mu^s + \kappa R\mu^w] \\ & = \delta(HR + LB + K) + g - (\Delta R + \lambda g + f). \end{aligned}$$

Collecting all the terms proportional to R , B and 1 respectively this becomes

$$\begin{aligned} & R\left[\frac{dH}{dt} - H(\tilde{\Delta} - \bar{\varepsilon}\tilde{\mu} + \kappa\mu^w + \mu^d + \mu^s) + L(\alpha_t + \beta_t(\delta - \tilde{\Delta}) - \gamma_t\bar{\varepsilon}\tilde{\mu}) + (\kappa\mu^w + \mu^d + \mu^s + \Delta)\right] \\ & + B\left[\frac{dL}{dt} - L(\delta + \mu^d + \mu^s)\right] \\ & + \left[\frac{dK}{dt} - K(\delta + \mu^d + \mu^s) + (F\mu^d - HF\tilde{\mu}) + (e - H\tilde{e}) - (g - Hg) + \gamma_t F\tilde{\mu}L - SC\mu^s\right] = 0. \end{aligned}$$

If this equation is to hold for all values of R, B then each square bracket in the equation above must vanish itself. This leads to the following system of ordinary differential equations for the quantities H , L and K .

$$\begin{aligned} \frac{dH}{dt} &= H(\tilde{\Delta} - \bar{\varepsilon}\tilde{\mu} + \kappa\mu^w + \mu^d + \mu^s) - [L(\alpha_t + \beta_t(\delta - \tilde{\Delta}) - \gamma_t\bar{\varepsilon}\tilde{\mu}) + (\kappa\mu^w + \mu^d + \mu^s + \Delta)] \\ \frac{dL}{dt} &= L(\delta + \mu^d + \mu^s) \quad (*) \\ \frac{dK}{dt} &= K(\delta + \mu^d + \mu^s) - [(F\mu^d - HF\tilde{\mu}) + (e - H\tilde{e}) - (g - Hg) + \gamma_t F\tilde{\mu}L - SC\mu^s] = 0. \end{aligned}$$

The boundary conditions for the original partial differential equation (3) can be interpreted for this system of ordinary differential equations as follows:

Value at Maturity: $V(T, R, B) = F + R + B$ implies $H(T) = L(T) = 1$ and $K(T) = F$.

Value on Bonus Payment Dates: The requirement that the reserve be continuous at bonus payment dates (τ_1, \dots, τ_n) is $H(\tau)R + L(\tau)B + K(\tau) = H(\tau)(R + B) + K(\tau)$ which clearly implies $L(\tau) = H(\tau)$.

Value at R=0: The linear solution is not flexible enough to fit the boundary condition $V(t,0,B) = X(t)$. This is the main source of error in the linear model. We can expect that the difference between the required value $X(t)$ and the value $L(t)B+K(t)$ attained by the linear solution is a measure of the error in the linear model. This issue will be discussed in more detail in the section 4 of this paper.

3.2 Analysis of the Bonus Ratio L

We can use the boundary condition on bonus dates, and equation (*) to understand the behavior of the bonus ratio between bonus dates. If t lies between two bonus dates (τ_{k-1}, τ_k) then (*) and the boundary condition allow us to write

$$L(t) = H(\tau_k) e^{-\int_t^{\tau_k} (\delta + \mu^d + \mu^s) ds}.$$

The bonus ratio at time t is therefore equal to the hedge ratio at the next bonus date discounted to time t with interest and contract persistency. This is a logical result because it tells us that the liability $L(t) B(t)$, for an accrued bonus $B(t)$, is equal to the reserve $H(\tau_k) B(t)$ which must be established on the bonus payment date discounted for interest and persistency. Note that while this choice of bonus ratio guarantees that the reserve is continuous across bonus payment dates the bonus ratio itself will usually have a discontinuity at bonus payment dates. (See the chart that follows.)

3.3 Analysis of the Hedge Ratio H

A significant insight into the nature of the hedge ratio $H(t)$ can be obtained by a simple change of variables. We will define the excess spread function $\xi(t)$ by

$$H(t) = 1 - \xi(t).$$

Note that the boundary condition $H(T)=1$ implies $\xi(T) = 0$. When we use this change of notation in the first of equations (*) we find that $\xi(t)$ satisfies the differential equation

$$\frac{d\xi}{dt} = \xi(\tilde{\Delta} + \kappa\mu^w + \mu^d + \mu^s - \bar{\varepsilon}\tilde{\mu}) - [\tilde{\Delta} - \Delta - L(\alpha_t + \beta_t(\delta - \tilde{\Delta}) - \gamma_t\bar{\varepsilon}\tilde{\mu}) + \bar{\varepsilon}(\mu^d - \tilde{\mu})]$$

Solving this differential equation we get the result which shows why the excess spread function bears that name

$$\xi(t) = \int_t^T e^{-\int_t^s (\tilde{\Delta} - \bar{\varepsilon}\tilde{\mu} + \kappa\mu^w + \mu^d + \mu^s) ds} [\tilde{\Delta} - \Delta - L(\alpha_s + \beta_s(\delta - \tilde{\Delta}) - \gamma_s\bar{\varepsilon}\tilde{\mu}) + \bar{\varepsilon}(\mu^d - \tilde{\mu})] ds.$$

When $\bar{\varepsilon} = 0$ we easily see that the excess spread function is the present value of the spread related loads in the contract less the present value of the spread related costs with discounting taking place using account value persistency and spread load interest. The

spread related costs are the explicit spread expense $\Delta(t)$ and the cost of accruing the persistency bonus $L(\alpha_s + \beta_s(\delta - \tilde{\Delta}))$. These costs are offset by the explicit spread related load $\tilde{\Delta}(t)$ built into the rate crediting strategy.

It is also interesting to note that the excess spread is affected by the interest rate assumption $\delta(t)$ only indirectly through its impact on the cost of bonus accrual. If the product had no persistency bonus the excess spread function would be independent of the interest assumption.

When $\bar{\epsilon} = 1$ there are a number of additional issues affecting the excess spread $\xi(t)$.

- The analysis tells us that experience mortality is acting like a spread-related load and the cost of insurance mortality behaves as a spread-related cost. That experience mortality is a spread related load is reasonable because $\bar{\epsilon} = 1$ implies that the account value R is not paid on death so the equity portion of the contract experiences a gain on death.

We will explain why this load is offset by the cost of insurance mortality acting as a spread related expense when we examine the insurance fund cash flows. It will be seen that this is an internal transfer between the equity and debt portions of the contract which is required to hedge the equity risk.

- The term $\bar{\epsilon}\tilde{\mu}$ appears as a spread related load when $\bar{\epsilon} = 1$ and the cost of insurance refund bonus structure is in place. This is also part of an internal transfer required to hedge the equity risk.
- Finally, the term $-\bar{\epsilon}\tilde{\mu}$ appears in the list of quantities used to discount the spread differentials. This reflects the fact that account values accumulate with both interest and cost of insurance mortality when $\bar{\epsilon} = 1$.

We can now understand the nature of the equity exposure required to match the equity indexed Universal Life insurance policy. The total required equity exposure is

$$QR = (1 - \xi + \beta L)R$$

The first part of the requirement $(1 - \xi)$ reflects the spread structure of the product. If spread related loads exceed the spread-related costs then the required exposure is less than unity. Similarly if the spread costs exceed the spread related loads then an amount greater than the account value must be held.

The second part of the requirement βL is connected to the particular requirements of a persistency bonus which is proportional to the credited interest. There are two aspects to hedging the equity risk associated with this bonus structure

- By borrowing an amount βLR from the debt side of the contract and investing this amount in equities the insurer will generate an incremental investment return of

$$\beta LR(mdt + \sigma dz)$$

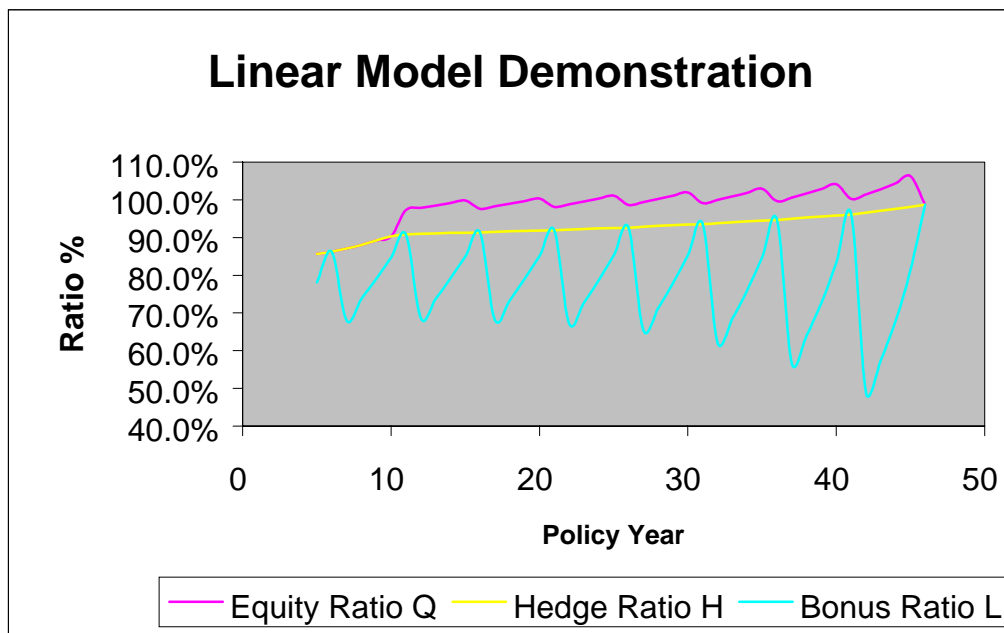
- In addition, the spread structure provides for a release of $\beta L(\delta - \tilde{\Delta})Rdt$

Adding these two pieces together we find the total amount released from the equity side of the contract is

$$\beta LR(mdt + \sigma dz) + \beta L(\delta - \tilde{\Delta})Rdt = \beta LR(mdt + \sigma dz - \tilde{\Delta}dt) + \delta \beta LRdt$$

The first term on the right represents the cost of accruing the bonus while the second term is the interest required to pay for the ‘loan’ taken from the debt side of the contract.

The chart below plots a numerical example of the functions Q , H and L using realistic actuarial assumptions for mortality, surrender and interest. The accrued bonus in this example is being driven by the assumption that $\beta = 0$ for the first 10 policy years and $\beta = 10\%$ thereafter. Other key assumptions are a spread load of $\tilde{\Delta} = 2.5\%$, a spread cost of $\Delta = 1.0\%$ and an interest rate environment of $\delta = 6.0\%$.



As the plot shows, the hedge ratio H grades smoothly from an amount just over 80% at the beginning of the policy to 100% at the end. This is consistent with the understanding of H as 1 less the excess spread function. Also note that the slope of the hedge ratio curve changes abruptly at duration 10 when the persistency bonus starts to accrue.

Bonus ratios exhibit a saw tooth pattern with an amplitude that grows over time as the impact of increasing mortality affects the discounting process.

The equity ratio is simply the combination $Q = H + \beta L$ except on bonus payment dates where the plot reflects the values just after the bonus has been paid. Note that the required equity exposure is less than the account value for the first 20 years of the policy.

It is also interesting to note that if a policy used the additive bonus structure with $\alpha = \beta(\delta - \tilde{\Delta}) = 10 * (.06 - .025) = 45bp$ in years 10 and later it will have the same hedge ratios as the example in Chart 1. However, it would have $Q=H$ as the matching investment strategy. Holding equities equal to 100% of the account value in this case would constitute a significant over exposure.

3.4 Analysis of the Insurance Reserve K

The final element of the linear model is the insurance reserve $K(t)$. From equation (*) it satisfies the differential equation

$$\frac{dK}{dt} = K(\delta + \mu^d + \mu^s) - [(F\mu^d - HF\tilde{\mu}) + (e - H\tilde{e}) - (g - Hg) + \gamma_t F\tilde{\mu}L - SC\mu^s].$$

This quantity can therefore be understood as a traditional actuarial present value

$$K(t) = \int_t^T e^{-\int_t^s (\delta + \mu^d + \mu^s) ds} [(F\mu^d - HF\tilde{\mu}) + (e - H\tilde{e}) - (g - Hg) + \gamma_t F\tilde{\mu}L - SC\mu^s] ds + Fe^{-\int_t^T (\delta + \mu^d + \mu^s) ds}$$

The first term of this result $F\mu^d - HF\tilde{\mu}$ represents the difference between the death benefits that must be paid and the hedge ratio multiplied by the cost of insurance charges deducted from the account value. When the net amount at risk is equal to F this is reasonable because the reserve released, from the equity fund when the COI charges are deducted from the account value is $HF\tilde{\mu}$. This is not inconsistent with the fact that the entire amount $F\tilde{\mu}$ is a revenue.

When the Net Amount at Risk is equal to $F-R$ the reserve released by the cost of insurance draw is only $H(F-R)\tilde{\mu}$ whereas the formula for K requires a transfer of $HF\tilde{\mu}$. The difference $HR\tilde{\mu}$ is the spread related cost that appeared in the analysis of the excess spread function when $\bar{\epsilon} = 1$. Note that without this adjustment the cash flows for the insurance fund would not be deterministic.

The term $e - H\tilde{e}$ represents the difference between actual, non spread related expense and the reserve released when the expense load is taken from the account value. Similarly, the term $g - Hg$ represents a continuation of the pattern although in this case we are concerned with a cash flow going into the account value rather than a charge coming out.

The last two terms in the definition of K represent the cost of accruing a cost of insurance refund persistency bonus offset by the income expected to come from surrender charges. Note that the insurance reserve must provide for the cost of the refund based on the full face amount even when $\bar{\epsilon} = 1$. This is consistent with the earlier result that $\bar{\epsilon}\tilde{\gamma}\tilde{\mu}$ is spread related income to the investment side of the contract.

The last term in the formula for K is the value of the endowment benefit.

3.5 Summary of the Linear Model

The following table summarizes the results of the linear model and shows how the policy benefits and expenses have been split between the three fund types described in the introduction to this section.

Linear Model: Summary Cash Flow Analysis

	Investment Fund	Insurance Fund	Bonus Fund	Total
Direct Cash Flows				
Death Benefit	$\varepsilon R\mu^d dt$	$F\mu^d dt$	0	$(F + \varepsilon R)\mu^d dt$
Endowment	R	F	B	$F+R+B$
Surrender Benefit	$R\mu^s dt$	$-SC\mu^s dt$	0	$(R - SC)\mu^s dt$
Withdrawal	$R\kappa\mu^w dt$	0	0	$R\kappa\mu^w dt$
Expense	ΔRdt	$(\lambda g + f)dt$	0	$\Delta Rdt + (\lambda g + f)dt$
Gross Premium	$-Hgdt$	$-(1-H)gdt$	0	$-gdt$
Inter Fund Transfers				
Expense Charges	$H(\tilde{\lambda}g + \tilde{f})dt$	$-H(\tilde{\lambda}g + \tilde{f})dt$	0	0
COI Charges	$H\tilde{\mu}Fdt$	$-H\tilde{\mu}Fdt$	0	
Bonus Accrual Charges	$L[\alpha + \beta(\delta - \tilde{\Delta})]Rdt$ $-L\bar{\varepsilon}\gamma\bar{\mu}Rdt$ $+L(mdt + \sigma dz)R$	$\gamma L\tilde{\mu}Fdt$	$-L[\alpha + \beta(\delta - \tilde{\Delta})]Rdt$ $-\gamma L\tilde{\mu}(F - \bar{\varepsilon}R)dt$ $-L(mdt + \sigma dz)R$	0
New Borrowing	$-(\beta LdR + \frac{d\beta}{dt}LRdt)$	0	$\beta LdR + \frac{d\beta}{dt}LRdt$	0
Bonus Payments	$-L(\tau)B = -H(\tau)B$	0	$L(\tau)B = H(\tau)B$	0
Reserve	$(H + \beta L)R$	$K(t)$	$L(B - \beta R)$	$HR + LB + K$

There are a number of observations that can be drawn from this table.

- The total cash flow of the policy has been decomposed into three funds. The cash flow for a given fund consists partly of interaction with the outside world (the direct cash flows) and partly of transfers to or from the other funds.
- The cash flows for the insurance fund are completely determined by the actuarial assumptions and are not affected by random movements in the equity index. From a financial perspective these cash flows are essentially fixed so the interest rate assumption is the crucial element in determining the value of the insurance fund.

To the extent that the hedge ratio is close to one and the account value deductions for mortality and expense are close to the actual amounts required the reserve build up in the insurance fund might not be large. However, if there is a material difference

between the slopes of the actual and COI mortality scales, as is often the case, the insurance reserve can be materially different from zero.

- The cash flows in the equity fund are a combination of stochastic benefits combined with deterministic transfers between it and the insurance fund together with stochastic transfers to or from the bonus fund. The crucial elements driving the valuation of this component are the spread structure and account value persistency.

The interest rate assumption affects the value indirectly through the cost of bonus accrual. Apart from this issue the linear model successfully splits the interest rate and equity risks into two distinct elements.

- The cash flows moving in and out of the bonus fund are highly stochastic yet it is part of the debt side of the contract. In the case where $\beta = 0$ the cash flows into the bonus fund consist of bonus accrual charges in the amount $L[\alpha R + \gamma(F - \bar{E}R)]dt$ flowing into the fund which accumulate, at the interest rate δ , to the bonus payment date. When $\beta \neq 0$ the mechanics are quite complicated with the borrowing cash flows and interest payments moving back and forth.

Since the future bonus fund cash flows are unknown at the valuation date the insurer can not lock in a yield curve at that time and be sure that he will always be able to pay the bonus. A rigorous discussion of the bonus fund issues therefore requires a stochastic model of both equity returns and interest rates which is beyond the scope of this paper.

A less rigorous but more pragmatic approach to understanding the bonus fund is to note that we could use a different interest assumption r to value the bonus fund. It will be conservative to use a low rate r when the bonus fund is positive (e.g. when $\beta = 0$) and a high rate r when the bonus fund is negative which would usually be the case when $\beta \neq 0$.

3.6 Reconciliation to the Revenue Model

Most actuaries are familiar with the revenue model of Universal Life which expresses the liability for a Universal Life contract as the policyholder account value less a present value of future margins. This is the model which FAS 97 takes as a starting point.

The linear cash flow model developed above can be reconciled to the revenue viewpoint by using the identity $H = 1 - \xi$ to rewrite the linear cash flow reserve as

$$\begin{aligned}
 V &= HR + LB + K \\
 &= (1 - \xi)R + LB + Fe^{-\int_t^T (\delta + \mu^d + \mu^s) ds} \\
 &\quad + \int_t^T e^{-\int_t^s (\delta + \mu^d + \mu^s) ds} [(F\mu^d - HF\tilde{\mu}) + (e - H\tilde{e}) - (g - Hg) + \gamma_t F\tilde{\mu}L - SC\mu^s] ds.
 \end{aligned}$$

Which can be further expanded by writing

$$V = R + B + Fe^{-\int_t^T (\delta + \mu^d + \mu^s) ds} - \left(\begin{aligned} & \int_t^T e^{-\int_t^s (\delta + \mu^d + \mu^s) ds} [F(\tilde{\mu} - \mu^d) + (\tilde{e} - e) + SC\mu^s] ds \\ & \xi(t)R + \int_t^T e^{-\int_t^s (\delta + \mu^d + \mu^s) ds} \xi(s)[g - \tilde{e} - F\tilde{\mu}] ds \\ & + (1-L)B - \int_t^T e^{-\int_t^s (\delta + \mu^d + \mu^s) ds} \gamma_t F\tilde{\mu} L ds \end{aligned} \right)$$

The first line in the large bracket above can be interpreted as the present value of mortality, expense and surrender charge margins. The second line is the present value of future interest spread margins while the last line represents the interest and persistency gain on the current accrued bonus offset by the cost of those future bonus accruals which are not built into the spread structure.

Note that the mortality, expense and surrender margins are discounted using the insurance fund interest rate as are the future COI bonus accrual charges.

The process for discounting the future spread margins starts with the present value of all spreads for amounts currently in the account value $\xi(t)R$ which is then adjusted for the spread impact of future premiums, expense loads and COI charges.

A high level summary of the above discussion would be the statement that revenues are not cash flows but there is a set of cash flows with the same present value as the set of revenues. The discounting process suggested by the linear model does not agree with the discounting process mandated by FAS 97 for US GAAP accounting.

3.7 Actuarial Risk in the Linear Model

The linear model described above is designed to hedge away the risk associated with random movements in the equity return. It does not hedge the risk that policyholders may not behave as assumed.

This section briefly looks at the issue of actuarial experience gains and losses in the linear model due to uncertain policyholder behavior. For example, if we have an unanticipated withdrawal ΔW the loss incurred is essentially the present value of all future spreads associated with that withdrawal.

$$\Delta W + V(t, R - \Delta W, B) - V(t, R, B) = \Delta W(1 - H) = \xi \Delta W.$$

A practical implication of this calculation is that if we want to build some conservatism into the expected withdrawal assumptions we should do so in such a way that

$E(\xi(t)\Delta W(t)) < 0$. Thus if $\xi(t) > 0$ we should upwardly bias the valuation withdrawal rate so that $E\Delta W(t) < 0$.

Similarly, an unanticipated premium payment Δg , results in a loss equal to

$$\Delta g[\lambda - 1 + H(1 - \tilde{\lambda})] = \Delta g[\lambda - \tilde{\lambda} - \xi(1 - \tilde{\lambda})].$$

The premium payment assumption will therefore have an element of conservatism if $E\Delta g[\lambda - \tilde{\lambda} - \xi(1 - \tilde{\lambda})] < 0$. This means that the sign of the bias in Δg is determined by the sign of $[\lambda - \tilde{\lambda} - \xi(1 - \tilde{\lambda})]$.

In general it is not possible to hedge both the financial risk and the actuarial risk at the same time unless $\xi(t) = 0$ and $\lambda = \tilde{\lambda}$.

It is possible to develop more sophisticated models of policyholder behavior within the context of the linear model but we will not do so in this paper.

4 Fund Exhaustion and the Girsanov Theorem

This section deals briefly with the errors in the linear model due to the fact that it fails to satisfy the correct boundary conditions when the fund exhausts at $R=0$. We start by recalling that the solution to the fundamental partial differential equation (3)

$$\begin{aligned} & \frac{\partial V}{\partial t} + [R(\delta - \tilde{\Delta} + \bar{\epsilon}\tilde{\mu} - \kappa\mu^w) + (g - \tilde{e} - \tilde{\mu}F)] \frac{\partial V}{\partial R} + [R(\alpha_t + \beta_t(\delta - \tilde{\Delta}) - \gamma_t\bar{\epsilon}\tilde{\mu}) + \gamma_t\tilde{\mu}F] \frac{\partial V}{\partial B} \\ & + \frac{1}{2}[(\sigma^2 + v^2\mu^w)R^2 \frac{\partial^2 V}{\partial R^2} + 2\beta\sigma^2 R^2 \frac{\partial^2 V}{\partial R\partial B} + \beta^2\sigma^2 R^2 \frac{\partial^2 V}{\partial B^2}] \\ & + [(F + \epsilon R - V)\mu^d + (\max(0, R - SC) - V)\mu^s + \kappa R\mu^w] = \delta V + g - (\Delta R + \lambda g + f) \end{aligned}$$

can be written as the mean of a stochastic integral

$$\begin{aligned} V(t, R, B) &= E e^{-\int_t^{\min(T, x)} (\delta + \mu^e + \mu^s) dr} V(\min(T, x), R, B) \\ &+ E \int_t^{\min(T, x)} e^{-\int_t^r (\delta + \mu^e + \mu^s) dr} [(F + \epsilon R)\mu^d + \max(0, R - SC)\mu^s + \kappa R\mu^w + \Delta R + \lambda g + f - g] dt \end{aligned} \quad (4)$$

Here the expectation is taken with respect to a measure in which the accrued bonus and the account value satisfy the stochastic differential equations

$$dB = [R(\alpha_t + \beta_t(m - \tilde{\Delta}) - \gamma_t\bar{\epsilon}\tilde{\mu}) + \gamma_t\tilde{\mu}F]dt + \sigma\beta_t R dz.$$

and

$$dR = R(\delta - \tilde{\Delta} + \bar{\epsilon}\tilde{\mu})dt + (g - \tilde{e} - \tilde{\mu}F)dt + \sigma R dz - (\kappa\mu^w R dt + v\sqrt{\mu^w} R dz')$$

respectively. The random element dz' driving the partial withdrawal process is independent of dz .

The integral considers the present value of death benefits, surrender benefits, withdrawal benefits and expenses offset by gross premiums. The upper bound on the integration indicates that the discounting stops at the earlier of contract maturity T or the time of fund exhaustion x .

In general it is not possible to carry out the stochastic integration in closed form so that numerical methods, such as monte carlo simulation, must be used when precise results are needed. However, it is possible to understand the relationship between the exact result (4) above and the linear model described in section (3) of this paper by noting that the linear model itself can be viewed as the mean of a stochastic integral

$$HR + LB + K = Ee^{-\int_t^{\min(T,x)} (\delta + \mu^e + \mu^s) dr} V^{Linear}(\min(T, x), R, B) \quad (5)$$

$$+ E \int_t^{\min(T,x)} e^{-\int_t^{\min(T,x)} (\delta + \mu^e + \mu^s) dr} [(F + \varepsilon R)\mu^d + (R - SC)\mu^s + \kappa R\mu^w + \Delta R + \lambda g + f - g] dt$$

Subtracting equation (5) from (4) we get

$$V(t, R, B) = H(t)R + L(t)B + K(t) + Ee^{-\int_t^{\min(T,x)} (\delta + \mu^e + \mu^s) dr} [X(x) - L(x)B(x) - K(x)] \quad (6)$$

$$+ E \int_t^{\min(T,x)} e^{-\int_t^{\min(T,x)} (\delta + \mu^e + \mu^s) dr} [\max(0, SC - R)\mu^s] dt.$$

This result shows that there are two sources of error in the linear solution

1. The error which arises from the boundary condition at $R=0$.
2. The error which arises from the cash value floor.

Both of these errors become smaller as the account value rises so we can conclude that **the linear model is the large account balance limit of the exact result**. Issues such as fund exhaustion and cash value floors make the exact solution of the valuation problem somewhat non-linear but these issues become less relevant as the account balance gets larger and the problem becomes linear. Under these conditions the insights gained from the study of linear model will apply.

When the account balance is small the nature of the replicating portfolio can change. From equation (6) we can see that a key issue will be the relationship between the exhaust value $X(x)$ and the amount naturally provided by the linear model $K(x)$ at the point of fund exhaustion. If $X(t) > K(t)$ for all policy durations we can expect that the

linear model will under reserve at low account values and if $X(t) < K(t)$ the linear model will be conservative. These conclusions can be validated with numerical examples.

5 Conclusions

The paper has studied the financial economics of a simple equity indexed Universal Life policy. The main result is the discussion of the linear model which we have shown to be a good approximation in the large account balance limit. The analytic tractability of the linear model allows us to understand the key issues underlying the construction of a replicating portfolio of equities and debt instruments.

The main issue driving the equity requirement is the spread structure or pattern of spread related loads and expenses. Debt instruments are required to deal with any mismatch between actual mortality and expense and the amounts released from the investment side of the contract when charges are deducted from the policyholder account value.

Universal Life products were originally designed to pass the investment risk of a life insurance through to the policyholder. When viewed from the perspective of a retrospective accounting system it would appear that an insurer can achieve this pass through goal by investing so as to match policyholder account balances. The prospective financial economic viewpoint taken in this paper shows that this need not be the case.

Since most products are designed to have more spread related loads than spread related expenses they usually have equity requirements that are less than the account value. The common industry practice of matching account values therefore leads to favorable mismatches when the equity markets rise and economic losses when the markets decline.

Finally, it should be noted that while the discussion in this paper has focused on equity indexed products the main conclusions remain valid for any form of universal life. The results of the linear model do not depend on any statistical property of the return generation process. Thus the insights regarding the replicating portfolio in the large account balance limit are valid for any form of universal life.