On the stock price model defined by the fractional Brownian semilinear stochastic differential equation: measure transformation and equilibrium of stock market

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Abstract

The existence and uniqueness conditions for solution of semilinear stochastic differential equations containing a differential with respect to fractional Brownian motion are considered in this paper. Also, for such fractional Brownian semilinear stochastic differential equations the conditions of measure transformation are established. The equilibrium conditions of stock market that described by the fractional Brownian semilinear stochastic differential equation are found.

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1. Introduction.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t, t \geq 0)$. We denote this family as $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Further throughout the paper the fractional Brownian motion (fBm) with Hurst index $H \in (1/2, 1)$ is denoted as $\left(B_t^H, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. This process is characterized by the following properties:

(1) B_t^H is a process with stationary increments;
(2) $B_0^H = 0$, and $\mathbb{E}\, B_t^H = 0$ for any $t > 0$;
(3) $\mathbb{E}\left(B_t^H\right)^2 = t^{2H}$ for any $t > 0$;
(4) B_t^H is a Gaussian process;
(5) the trajectories of stochastic process B_t^H are continuous.

We use the notion ([1]) of stochastic integral $\int_0^t f_s\, dB_s^H$, where f_s is a random measurable function, Hölder-continuous of order H (see, for example, [2], where it is proved that such integral can be considered as a limit of Riemann-Stieltjes sums). The main subject of our consideration are semilinear stochastic differential equations (SDE) containing stochastic differential with respect to (w.r.t.) fBm. Such equations represent a "pure fractional Brownian" stock price model. It should be noted that the "mixed fractional Brownian" stock price model, represented by the semilinear SDE containing stochastic differentials w.r.t. Wiener process and fBm, is studied in [3].

In a chapter 2 of the present paper we establish the conditions of existence and uniqueness of global solution of such equations under the local Lipschitzness of the drift coefficient. Similar questions were considered in [2] and [4], but in [2] only the existence and uniqueness of local solution were proved; in [4] the existence and uniqueness of global solution of equations containing only one differential of Hölder continuous process were considered under the global Lipschitzness of the drift coefficient.

In a chapter 3 we find the condition of differentiability of stochastic integrals with random Hölder-continuous integrand and fBm as an integrator. Using these differentiability conditions we establish the conditions of measure transformation for such semilinear SDE in a chapter 4. In a chapter 5 we establish equilibrium condition for pure fractional Brownian stock price model.

2. Existence and uniqueness of solution of semilinear stochastic differential equations containing differential with respect to fBm.

Let us consider the semilinear differential equation in complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{cases} dX_t = c\, X_t\, dB_t^H + b(t, X_t)\, dt, \quad t \geq t_0 \\ X_{t=t_0} = X_0 \text{ is } \mathcal{F}_{t_0}\text{- mesurable random value,} \end{cases} \tag{1}$$

where $b \in \mathcal{C}\left(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}\right)$ satisfies Lipschitz condition on $x \in \mathbb{R}$ uniformly on $t \in [0, T]$, that is $\forall T > 0\ \exists L_T > 0:$

$$\sup_{t\in[0,T]} |b(t, x_1) - b(t, x_2)| \leq L_T\, |x_1 - x_2| \tag{2}$$

Definition 1. *Solution of the equation (1) is a Hölder-continuous with index H random process that has a form*

$$X_t = X_0 + c \int_{t_0}^{t} X_s\, dB_s^H + \int_{t_0}^{t} b(s, X_s)\, ds, \quad t \geq t_0$$

It should be noted that integral $\int_{t_0}^{t} X_s\,dB_s^H$ exists if X_s is a Hölder-continuous with index H ([2]).

Theorem 1. *Let the function $b \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ satisfies (2) and growth condition*
$\forall T > 0\ \exists C_T > 0:$

$$|b(t,x)| \le C_T\,(1+|x|).$$

Then the equation (1) has a unique solution on $[0,+\infty)$, and moreover there exists $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that $\forall \omega \in \Omega'$ $\forall T > 0$ $\exists C = C(\omega, T)$ that $\forall\ 0 \le t_1 \le t_2 \le T$ $\ |X_{t_1} - X_{t_2}| \le C(\omega,T)|t_1 - t_2|^\alpha$ for any $0 \le \alpha < H$. That is, trajectories of X_t a.s. belong to $Lip_\alpha[0,T]$, $\forall T > 0$, $\forall \alpha \in [0, H)$.

Proof. We will try to find the solution X_t of equation (1) in the following form $X_t = h(Y_t, Z_t) = (Y_t - Y_0 + X_0)\mathbf{e}^{c(Z_t - Z_{t_0})}$, where $Z_t = B_t^H$, $Y \in \mathcal{C}^1(\mathbb{R})$, $Y_{t_0} = Y_0$ – $\mathcal{F}_{t_0}$-measurable random value.

Applying the Itô formula ([2]) to the function $h(Y_t, Z_t)$ we obtain

$$\begin{aligned} dX_t = {} & \frac{\partial h}{\partial Z}(Y_t, Z_t)\,dZ_t + \frac{\partial h}{\partial Y}(Y_t, Z_t)Y_t'\,dt + \\ & + \frac{\partial h}{\partial t}(Y_t, Z_t)\,dt + \frac{1}{2}\frac{\partial^2 h}{\partial Z^2}(Y_t, Z_t)\,d\,[Z,Z]_t = \\ & = \frac{\partial h}{\partial Z}(Y_t, Z_t)\,dZ_t + Y_t'\frac{\partial h}{\partial Y}(Y_t, Z_t)\,dt, \end{aligned}$$

since according to [2], $[Z,Z]_t = 0$.

Comparing (1) and the last equality we obtain an ordinary differential equation for Y :

$$\begin{cases} Y_t' = (c_1(t))^{-1}\,(b((Y_t - Y_0 + X_0)\,c_1(t), t)) =: f(t, Y_t) \\ Y_{t_0} = Y_0, \end{cases} \tag{2}$$

where $c_1(t) = \mathbf{e}^{c(Z_t - Z_{t_0})}$.

Further we will assume that $\omega \in \Omega$ is fixed. Put $L_1(T) := \max\limits_{t\in[0,T]} \frac{1}{c_1(t)} > 0$, $\ L_2(T) := \max\limits_{t\in[0,T]} c_1(t) > 0$, $\ D_1 := C_T\,L_1(T)$, $\ D_2 = C_T$.

Then under $|Y - Y_0| \le \beta$; $\ |t - t_0| \le \alpha$:

$$M := \max_{t\in[0,T]} |f(t,y)| \le (L_1(T) + (\beta + |x_0|))\,C_T \le D_1 + D_2(\beta + |x_0|)$$

and according to the Picard theorem there exists solution of (2) on $[t_0, t_0 + l^{(0)}]$, where

$$l^{(0)} = \min\left(\alpha, \frac{\beta}{M}\right) \ge \min\left(\alpha, \frac{\beta}{D_1 + D_2(\beta + |x_0|)}\right) =: l_0,$$

and therefore solution of (2) exists on the smaller segment $[t_0, t_0 + l_0]$.

Now we can estimate the solution of (1) in the point $t_0 + l_0$ in such a way:

$$h_{t_0+l_0} \le |Y - Y_0 + X_0|\,L_2(T) \le (\beta + |X_0|)\,L_2(T).$$

Note that Y_t is a continuous differentiable function on $[t_0, t_0 + l_0]$, and therefore it follows from view of h that for each $\omega \in \Omega$ it is a Hölder function on $[t_0, t_0 + l_0]$, since by Itô formula

$$\mathbf{e}^{c\left(Z_t - Z_{t_0}\right)} = c\int_{t_0}^{t} \mathbf{e}^{c\left(Z_s - Z_{t_0}\right)}\,dZ_s,$$

$Z \in Lip_\alpha[0,T]$ $\forall T > 0$, $\forall\, 0 < \alpha < H$, and then according to the theorem 22 ([5]), $\int\limits_{t_0}^{t} \mathbf{e}^{c\left(Z_s - Z_{t_0}\right)}\, dZ_s \in Lip_\alpha[0,T]$ $\forall T > 0$, $\forall\, 0 < \alpha < H$. Thus, trajectories of X_t a.s. belong to $Lip_\alpha[t_0, t_0 + l_0]$ $\forall\, 0 < \alpha < H$.

Further we will extend the solution h for $t \geqslant t_0 + l_0$. The value of solution h in the point $t_0 + l_0$ will be new initial value $X_0^{(1)} \le (\beta + |X_0|)\, L_2(T)$.

Now, under $|Y(t) - Y(t_0 + l_0)| \le \beta_1$, $|t - (t_0 + l_0)| \le \alpha_1$, the solution of (2) exists on segment $[t_0 + l_0, t_0 + l_0 + l_1]$, where

$$l_1 = \min\left(\alpha_1,\ \frac{\beta_1}{D_1 + D_2(\beta_1 + (\beta + |X_0|)L_2(T))}\right)$$

Further on the n-th step of such solution extension procedure we have:

$$l_n = \min\left(\alpha_n,\ \frac{\beta_n}{D_1 + D_2(\beta_n + \sum\limits_{k=0}^{n-1} \beta_{n-1-k}\, L_2^{k+1}(T) + |X_0| L_2(T))}\right)$$

$\alpha_0 := \alpha$, $\beta_0 := \beta$ and on $[t_0 + \sum\limits_{i=0}^{n-1} l_i,\ t_0 + \sum\limits_{i=0}^{n} l_i]$ there exists a solution of (2).

Now let us investigate the properties of
$z_n =: \frac{\beta_n}{D_1 + D_2(\beta_n + \sum\limits_{k=0}^{n-1} b_{n-1-k}\, L_2^{k+1}(T) + |X_0| L_2(T))}$, $n \ge 0$:

a) under $|X_0| \le 1$ we can chose $\beta_k \equiv 1$, $k = \overline{0,n}$ and then

$$\frac{1}{D_1 + 2D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}} \le z_n \le \frac{1}{D_1 + D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}}.$$

b)under $|X_0| > 1$ we can chose $\beta_k \equiv |X_0|$, $k = \overline{0,n}$ and then

$$\frac{1}{D_1 + 2D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}} \le z_n = \frac{1}{\frac{D_1}{|x_0|} + D_2(1 + 2\sum\limits_{k=1}^{n} L_2^k(T))} \le \frac{1}{D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}}.$$

For both cases a) and b) we put $\alpha_n = \frac{1}{D_1 + 2D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}}$, $n \ge 0$, that is $l_n = \min(a_n, z_n) = a_n \Rightarrow \sum\limits_{n \ge 0} l_n = S$, and then

1) under $L_2(T) \le 1$ the series $\sum\limits_{n \ge 0} z_n$ is divergent and therefore the series $\sum\limits_{i \ge 0} l_i$ is also divergent, and thus there exists a finite quantity N of steps in our solution extension procedure, such that $[t_0, T] \subset [t_0, t_0 + \sum\limits_{i=0}^{N} l_i]$, that is on the $[t_0, T]$ there exists a solution of (2).

2) under $L_2(T) > 1$ the series $\sum\limits_{n \ge 0} z_n$ is convergent and moreover

$$\sum_{n \ge 0} z_n \ge \sum_{n \ge 0} \frac{1}{D_1 + 2D_2 \frac{L_2(T)^{n+1} - 1}{L_2(T) - 1}} =: S$$

In fact we proved that on segment $[t_0, t_0 + \frac{1}{2}S]$ there exists a bounded solution h of (1). Now we can consider a finite value of h in the point $t_0 + \frac{1}{2}S$ as a new initial value of h and for it we can prove that solution h exists on $[t_0 + \frac{1}{2}S, t_0 + S]$, because the step $\frac{1}{2}S$ of solution extension doesn't depend on initial value. Therefore we can extend solution h to whole segment $[t_0, T]$. Since T is arbitrary we obtain the solution (1) on $[0, \infty)$. Uniqueness follows from theorem 7.1.1.([2]). In this theorem it is established that any two of solutions coincide on the common interval of definition. Since the solution extension procedure don't violate the properties of trajectories of a solution, then by mathematical induction we can obtain that the trajectories of a solution a.s. belong to $Lip_\alpha[0, T]\ \forall T > 0,\ \forall\, 0 < \alpha < H$.

3. Differentiability of stochastic integrals with random integrand and fBm as integrator.

Put $I(t) := \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} \int_0^s \alpha(u,\omega)\, dB_u^H\, ds$, where $\alpha = \alpha(u,\omega) : \mathbb{R} \times \mathbf{\Omega} \to \mathbb{R}$ is random Hölder-continuous function on $[0, t]$ with index $\beta > \frac{1}{2}$. It should be noted that according to Lemma 3 from [6], for each $\omega \in \mathbf{\Omega}'$, $\mathbb{P}(\mathbf{\Omega}') = 1$ there exists such integral, and moreover there exists an integral $J(t) := \int_0^t \alpha(u,\omega) \int_u^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\, ds\, dB_u^H$, and equality $I(t) = J(t), t > 0$ a.s. holds.

Lemma 1. *For all $t > 0$ $I(t)$ admits the following representation*

$$I(t) = t^{2-2H} \int_0^t \delta_s\, ds,$$

where function

$$\delta_s := s^{2H-3} \int_0^s u^{\frac{3}{2}-H}(s-u)^{\frac{1}{2}-H} \alpha(u,\omega)\, dB_u^H$$

a.s. belongs to $\mathcal{L}_1([0,t])$, that is $\int_0^t |\delta_s|\, ds < \infty$ a.s., $t \geq 0$.

Proof. Throughout the proof we will assume that $\omega \in \mathbf{\Omega}'$, $\mathbb{P}(\omega \in \mathbf{\Omega}') = 1$ is fixed, and therefore we will omit an argument ω. Retype equality $I(t) = J(t)$ in the following way

$$I(t) = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} \int_0^s \alpha(u)\, dB_u^H\, ds = J(t) =$$

$$= \int_0^t \int_u^t \alpha(u) s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\, ds\, dB_u^H = t^{2-2H} \int_0^t \int_{\frac{u}{t}}^1 \alpha(u) s^{\frac{1}{2}-H}(1-s)^{\frac{1}{2}-H}\, ds\, dB_u^H =$$

$$= t^{2-2H} \int_0^t \int_u^t \alpha(u) \frac{u}{s^2} \left(\frac{u}{s}\right)^{\frac{1}{2}-H} \left(1 - \frac{u}{s}\right)^{\frac{1}{2}-H}\, ds\, dB_u^H =$$

$$= t^{2-2H} \int_0^t \int_u^t s^{2H-3}(s-u)^{\frac{1}{2}-H} u^{\frac{3}{2}-H} \alpha(u)\, ds\, dB_u^H =: t^{2-2H} M(t)$$

Now, let us consider a function

$$N(t) := \int_0^t s^{2H-3} \int_0^s (s-u)^{\frac{1}{2}-H} u^{\frac{3}{2}-H} \alpha(u)\, dB_u^H\, ds.$$

We will use the following results in order to prove an existence of function $N(t)$

1) according to Lemma 2.1 from [1] Hölder-continuous function $f : [0, T] \to \mathbb{R}$ with index $\beta \in (0, 1)$ satisfies the following inequality under $\gamma \in (-\beta, -\beta + 1)$, $T > 0$ and $f(0) = 0$

$$\int_0^t (t-u)^\gamma\, df(u) =$$

$$= \lim_{\varepsilon \to 0} \left(\varepsilon^\gamma \left(f(t-\varepsilon) - f(t) \right) + t^\gamma f(t) + \gamma \int_0^{t-\varepsilon} \left(f(u) - f(t) \right) (t-u)^{\gamma-1}\, du \right); \qquad (3)$$

2) according to Lemma 1 from [6] for Hölder continuous functions $f \in \mathcal{H}^\alpha_{[a,b]}$, $g \in \mathcal{H}^\beta_{[a,b]}$, $\alpha + \beta > 1$, holds

$$\left| \int_a^b f(t)\, dg(t) \right| \le C \|f\|_{\mathcal{H}^\alpha_{[a,b]}} \|g\|_{\mathcal{H}^\beta_{[a,b]}} \max \left\{ (b-a)^{1+\varepsilon}, (b-a)^\beta \right\}. \qquad (4)$$

Using (3) and (4) under $t > s_2 > s_1 > 0$ we can obtain such estimation

$$\left| \int_{s_1}^{s_2} \alpha(z)(s_2 - z)^{\frac{1}{2}-H}\, dB_z^H \right| = \left| \lim_{\varepsilon \to 0} \left[-\varepsilon^{\frac{1}{2}-H} \int_{s_2-\varepsilon}^{s_2} \alpha(v)\, dB_v^H + \right.\right.$$

$$\left.\left. +(s_2 - s_1)^{\frac{1}{2}-H} \int_{s_1}^{s_2} \alpha(z)\, dB_z^H + \left(H - \frac{1}{2} \right) \int_{s_1}^{s_2-\varepsilon} (s_2 - z)^{-\frac{1}{2}-H} \int_z^{s_2} \alpha(v)\, dB_v^H\, dz \right] \right| \le$$

$$\le \lim_{\varepsilon \to 0} C \|\alpha\|_{\mathcal{H}^\beta_{[0,t]}} \|B^H\|_{\mathcal{H}^H_{[0,t]}} \left\{ \varepsilon^{\frac{1}{2}} + (s_2 - s_1)^{\frac{1}{2}} + \left(H - \frac{1}{2} \right) \int_{s_1}^{s_2-\varepsilon} (s_2 - z)^{-\frac{1}{2}}\, dz \right\} \le$$

$$\le K_1 (s_2 - s_1)^{\frac{1}{2}}, \qquad (5)$$

where $K_1 := 2CH \|\alpha\|_{\mathcal{H}^\beta_{[0,t]}} \|B^H\|_{\mathcal{H}^H_{[0,t]}}$

Further we have

$$\int_{s_1}^{s_2} (s_2 - u)^{\frac{1}{2}-H} u^{\frac{3}{2}-H} \alpha(u)\, dB_u^H = \int_{s_1}^{s_2} u^{\frac{3}{2}-H} d \left[\int_0^u (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H \right] =$$

$$= s_2^{\frac{3}{2}-H} \int_0^{s_2} (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H - s_1^{\frac{3}{2}-H} \int_0^{s_1} (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H -$$

$$-\left(\frac{3}{2} - H\right) \int_{s_1}^{s_2} u^{\frac{1}{2}-H} \int_0^u (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H\, du =: L(s_1, s_2).$$

$$|L(s_1, s_2)| \le \left| s_2^{\frac{3}{2}-H} - s_1^{\frac{3}{2}-H} \right| \left| \int_0^{s_2} (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H \right| + s_1^{\frac{3}{2}-H} \times$$

$$\times \left| \int_{s_1}^{s_2} (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H \right| + \left(\frac{3}{2} - H\right) \int_{s_1}^{s_2} u^{\frac{1}{2}-H} \left| \int_0^u (s_2 - z)^{\frac{1}{2}-H} \alpha(z)\, dB_z^H \right| du \le$$

$$\le K_1 \left| s_2^{\frac{3}{2}-H} - s_1^{\frac{3}{2}-H} \right| s_2^{\frac{1}{2}} + K_1 s_1^{\frac{3}{2}-H} (s_2 - s_1)^{\frac{1}{2}} + (3 - 2H) K_1 \int_{s_1}^{s_2} u^{\frac{1}{2}-H} s_2^{\frac{1}{2}}\, du =$$

$$\le K_1 s_2^{\frac{3}{2}-H} (s_2 - s_1)^{\frac{1}{2}} + 3K_1 (s_2 - s_1)^{\frac{3}{2}-H} s_2^{\frac{1}{2}}. \tag{6}$$

Thus $|L(0, s)| \le 4K_1 s^{2-H}$ and

$$|N(t)| \le \int_0^s s^{2H-3} |L(s)|\, ds \le \frac{4K_1}{H} t^H < \infty.$$

Further we consider a function

$$N_\varepsilon(t) := \int_0^t s^{2H-3} \mathbb{I}\{s \in [\varepsilon, t]\} \int_0^{s-\varepsilon} u^{\frac{3}{2}-H} (s - u)^{\frac{1}{2}-H} \alpha(u)\, dB_u^H\, ds.$$

It is obvious that $\forall \varepsilon > 0$ function

$$\varphi_\varepsilon(s, u) = \mathbb{I}\{u \le s - \varepsilon\} \mathbb{I}\{s \in [\varepsilon, t]\} s^{2H-3} u^{\frac{3}{2}-H} (s - u)^{\frac{1}{2}-H} \alpha(u)$$

1) is piecewise Hölder-continuous w.r.t. argument u on segment [0,t] with index $\beta_1 > \frac{1}{2}$ ($u = s - \varepsilon$ is a point of Hölder discontinuity);

2) $\int_0^t \varphi_\varepsilon(s, u)\, dB_u^H = s^{2H-3} \mathbb{I}\{s \in [\varepsilon, t]\} \int_0^{s-\varepsilon} u^{\frac{3}{2}-H} (s - u)^{\frac{1}{2}-H} \alpha(u)\, dB_u^H$ is Riemann integrable function on $[0, t]$.

Therefore, $\varphi_\varepsilon(s, u)$ satisfies conditions of the stochastic Fubini theorem ([6]), that is there exist the following integrals

$$N_\varepsilon(t) = \int_0^t s^{2H-3} \mathbb{I}\{s \in [\varepsilon, t]\} \int_0^{s-\varepsilon} u^{\frac{3}{2}-H} (s - u)^{\frac{1}{2}-H} \alpha(u)\, dB_u^H\, ds$$

$$M_\varepsilon(t) := \int_0^{t-\varepsilon} u^{\frac{3}{2}-H} \alpha(u) \int_{u+\varepsilon}^t s^{2H-3} (s - u)^{\frac{1}{2}-H}\, ds\, dB_u^H$$

and they are coincide under each $\varepsilon > 0$.

Using inequality (6), for the first integral we obtain

$$|N(t) - N_\varepsilon(t)| \le \left| \int\limits_\varepsilon^t s^{2H-3} \int\limits_{s-\varepsilon}^s u^{\frac{3}{2}-H}(s-u)^{\frac{1}{2}-H}\alpha(u)\, dB_u^H\, ds \right| +$$

$$+ \left| \int\limits_0^\varepsilon s^{2H-3} \int\limits_0^s u^{\frac{3}{2}-H}(s-u)^{\frac{1}{2}-H}\alpha(u)\, dB_u^H\, ds \right| \le$$

$$\le \int\limits_\varepsilon^t s^{2H-3} \left[K_1 s^{\frac{3}{2}-H}\varepsilon^{\frac{1}{2}} + 3K_1 s^{\frac{1}{2}}\varepsilon^{\frac{3}{2}-H} \right] ds + 4K_1 \int\limits_0^\varepsilon s^{2H-3}s^{2-H}\, ds =$$

$$= \frac{K_1}{H-\frac{1}{2}}\varepsilon^{\frac{1}{2}}(t^{H-\frac{1}{2}} - \varepsilon^{H-\frac{1}{2}}) + \frac{3K_1}{|2H-\frac{3}{2}|}\left(\varepsilon^{\frac{3}{2}-H}t^{2H-\frac{3}{2}} - \varepsilon^H\right) + \frac{4K_1}{H}\varepsilon^H \to 0, \quad \varepsilon \to 0.$$

Using integral transformation (ii) of Lemma 2.2 ([1]), for the second integral we obtain

$$|M(t) - M_\varepsilon(t)| \le \left| \int\limits_0^{t-\varepsilon} \alpha(u)u^{\frac{3}{2}-H} \int\limits_u^{u+\varepsilon} s^{2H-3}(s-u)^{\frac{1}{2}-H}\, ds\, dB_u^H \right| +$$

$$+ \left| \int\limits_{t-\varepsilon}^t \alpha(u)u^{\frac{3}{2}-H} \int\limits_u^t s^{2H-3}(s-u)^{\frac{1}{2}-H}\, ds\, dB_u^H \right| =$$

$$= \left| \int\limits_0^{t-\varepsilon} \alpha(u)u^{\frac{3}{2}-H} \int\limits_0^{\frac{\varepsilon}{u+\varepsilon}} s^{\frac{1}{2}-H}(1-s)^{\frac{1}{2}-H}\, ds\, dB_u^H \right| +$$

$$+ \left| \int\limits_{t-\varepsilon}^t \alpha(u)u^{\frac{3}{2}-H} \int\limits_0^{1-\frac{u}{t}} s^{\frac{1}{2}-H}(1-s)^{\frac{1}{2}-H}\, ds\, dB_u^H \right| =: A_1(\varepsilon) + A_2(\varepsilon).$$

Further according to stochastic Fubini theorem from [6] we have

$$A_1(\varepsilon) = \int\limits_0^{\frac{\varepsilon}{t}} s^{\frac{1}{2}-H}(1-s)^{\frac{1}{2}-H} \int\limits_0^{t-\varepsilon} \alpha(u)\, dB_u^H\, ds + \int\limits_{\frac{\varepsilon}{t}}^1 s^{\frac{1}{2}-H}(1-s)^{\frac{1}{2}-H} \int\limits_0^{\varepsilon\frac{1-s}{s}} \alpha(u)\, dB_u^H\, ds$$

$$A_2(\varepsilon) = \int\limits_0^{\frac{\varepsilon}{t}} s^{\frac{1}{2}-H}(1-s)^{\frac{1}{2}-H} \int\limits_{t-\varepsilon}^{t(1-s)} \alpha(u)\, dB_u^H\, ds.$$

Now we have

$$|A_1(\varepsilon)| \le \frac{K_1}{2H}t^{2H-\frac{3}{2}}\left(1-\frac{\varepsilon}{t}\right)^{\frac{1}{2}-H}\varepsilon^{\frac{3}{2}-H} + \frac{K_1}{2H}\varepsilon^H \int\limits_{\frac{\varepsilon}{t}}^1 (1-s)^{\frac{1}{2}}s^{\frac{1}{2}-2H}\, ds \le$$

$$\leq \frac{K_1}{2H} t^{2H-\frac{3}{2}} \left(1-\frac{\varepsilon}{t}\right)^{\frac{1}{2}-H} \varepsilon^{\frac{3}{2}-H} + \frac{K_1}{2H} t^{\frac{1}{2}} \varepsilon^{H-\frac{1}{2}} B\left(\frac{3}{2}, 2-2H\right) \to 0, \ \ \varepsilon \to 0.$$

$$|A_2(\varepsilon)| \leq \frac{K_1}{2H} \int\limits_0^{\frac{\varepsilon}{t}} s^{\frac{1}{2}-H} (1-s)^{\frac{1}{2}-H} (\varepsilon - ts)^H \, ds \leq$$

$$\leq \frac{K_1}{2H} t^H \left(1-\frac{\varepsilon}{t}\right)^{\frac{1}{2}-H} \int\limits_0^{\frac{\varepsilon}{t}} s^{\frac{1}{2}-H} \left(\frac{\varepsilon}{t} - s\right)^H \, ds \leq$$

$$\leq \frac{K_1}{2H} t^{H-\frac{3}{2}} \left(1-\frac{\varepsilon}{t}\right)^{\frac{1}{2}-H} \varepsilon^{\frac{3}{2}} B\left(H+1, \frac{3}{2}-H\right) \to 0, \ \ \varepsilon \to 0.$$

Hence, under $\varepsilon \to 0$ we obtain an equality $N(t) = M(t)$, and proof is over.

4. Measure transformation for semilinear SDE representing the "pure fractional Brownian" stock price model.

Let us consider SDE (1) and suppose that $t_0 = 0$, function $b(t,x)$ satisfies conditions of Theorem 1 and can be represented as $b(t,x) = e(t,x)x$, where $e \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})$. Further we will use the following notes: $B = \mathrm{B}(H+1/2, 3/2-H)$, $\mathrm{B}(\cdot,\cdot)$ is beta function, $B_1 = \mathrm{B}(H-1/2, 3/2-H)$, $C_1 = (2HB)^{-1}$, $C_H = \left(\frac{2H\,\Gamma(3/2-H)}{\Gamma(H+1/2)\,\Gamma(2-2H)}\right)^{1/2}$, $C_2 = \frac{C_H}{2H(2-2H)^{1/2}}$, $C_0 = \frac{C_1}{C_2}$. Also, let kernel $K(t,s) = C_1 s^{1/2-H}(t-s)^{1/2-H} \mathbb{I}\{s \in (0,t)\}$.

We will prove the following auxiliary results which will be useful under measure transforming in (1).

Put $\mu(u) := \frac{r - e(u, X_u)}{c}$, $r \in \mathbb{R}$.

Lemma 2. *Let function $e(t,x)$ satisfies conditions:*

1) $\forall T > 0 \ \ \exists C_T > 0, \ \ \forall u \leq T, \ \forall x \in \mathbb{R}$

$$\begin{aligned} &|e(u,x)| \leq C_T(1+|x|), \ |e'_t(u,x)| \leq C_T(1+|x|), \\ &|e'_x(u,x)| \leq C_T(1+|x|), \ |e''_{xx}(u,x)| \leq C_T(1+|x|); \end{aligned} \tag{7}$$

2)

$$\forall T > 0 \ \exists L_T > 0 \ \forall t \in [0,T], \forall x_1, x_2 \in \mathbb{R}$$
$$\sup_{t \in [0,T]} |e'_x(t,x_1) - e'_x(t,x_2)| \leq L_T \, |x_1 - x_2|. \tag{8}$$

Then

1) $\int\limits_0^t K(t,s)|\mu(s)| \, ds < \infty$ *a.s.,*

(9)

2) there exists a representation $\int\limits_0^t K(t,s)\mu(s)\, ds = \int\limits_0^t \varphi_s \, ds$, *where* $\int\limits_0^t |\varphi_s| \, ds < \infty$ *a.s.*

Proof. It is obvious that

$$|I(t)| := \left| \int\limits_0^t C_1 K(t,s) \frac{r - e(s, X_s)}{c} \, ds \right| =$$

$$= \left| \frac{r}{c} C_1 t^{2-2H} B_1 - \frac{C_1}{c} \int\limits_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} e(s, X_s)\, ds \right| < \infty,$$

and therefore statement 1) of our lemma holds.

Using Itô formula ([2]) we have

$$e(s, X_s) = e(0,0) + \int\limits_0^s e'_t(u, X_u)\, du + \int\limits_0^s e'_x(u, X_u)\, dX_u + \frac{1}{2} \int\limits_0^s e''_{xx}(u, X_u)\, d[X]_u =$$

$$= e(0,0) + \int\limits_0^s e'_t(u, X_u)\, du + \int\limits_0^s e'_x(u, X_u) e(u, X_u) X_u\, du + c \int\limits_0^s e'_x(u, X_u) X_u\, dB_u^H.$$

Hence

$$I(t) = (r - e(0,0)) \frac{C_1 B_1}{c} t^{2-2H} - \frac{C_1}{c} \int\limits_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} \int\limits_0^s e'_t(u, X_u)\, du\, ds -$$

$$-\frac{C_1}{c} \int\limits_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} \int\limits_0^s e'_x(u, X_u) e(u, X_u) X_u\, du\, ds - C_1 \int\limits_0^t s^{\frac{1}{2}-H} \times$$

$$\times (t-s)^{\frac{1}{2}-H} \int\limits_0^s e'_x(u, X_u) X_u\, dB_u^H\, ds.$$

Taking in account growth conditions (7) for functions e, e'_t, e'_x, e''_{xx}, we satisfy oneself that the following functions

$$s^{\frac{1}{2}-H} \int\limits_0^s e'_t(u, X_u)\, du = \int\limits_0^s u^{\frac{1}{2}-H} e'_t(u, X_u)\, du + \left(\frac{1}{2} - H\right) \int\limits_0^s u^{-\frac{1}{2}-H} \int\limits_0^u e'_t(v, X_v)\, dv\, du$$

and

$$s^{\frac{1}{2}-H} \int\limits_0^s e'_x(u, X_u) e(u, X_u) X_u\, du = \int\limits_0^s u^{-\frac{1}{2}-H} e'_x(u, X_u) e(u, X_u) X_u\, du +$$

$$+ \left(\frac{1}{2} - H\right) \int\limits_0^s u^{-\frac{1}{2}-H} \int\limits_0^u e'_x(v, X_v) e(v, X_v) X_v\, dv\, du$$

belong to $\mathcal{AC}[0,T]$. Hence, according to Lemma 2.2 ([7]) almost everywhere under Lebesgue measure there exist fractional derivatives of order $H - \frac{1}{2}$ of those functions

$$\frac{d}{dt} \int\limits_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} \int\limits_0^s e'_t(u, X_u)\, du\, ds = \int\limits_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} e'_t(s, X_s)\, ds +$$

$$+ \left(\frac{1}{2} - H\right) \int\limits_0^t s^{-\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} \int\limits_0^s e'_t(u, X_u)\, du\, ds, \tag{10}$$

and

$$\frac{d}{dt}\int\limits_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\int\limits_0^s e'_x(u,X_u)e(u,X_u)X_u\,du\,ds=\int\limits_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}e'_x(s,X_s)\times$$

$$\times e(s,X_s)X_s\,ds+\left(\frac{1}{2}-H\right)\int\limits_0^t s^{-\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\int\limits_0^s e'_x(u,X_u)e(u,X_u)X_u\,du\,ds. \quad (11)$$

Using condition (8) we satisfy oneself that $e'_x(u,X_u)X_u$ is Hölder-continuous function on $[0,T]$ w.r.t. argument u with any index $\beta\in\left(\frac{1}{2},H\right)$. Hence, according to Lemma 1 a.s. there exists an equality

$$\int\limits_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\int\limits_0^s e'_x(u,X_u)X_u\,dB_u^H\,ds=$$

$$=t^{2-2H}\int\limits_0^t s^{2H-3}\int\limits_0^s u^{\frac{3}{2}-H}(s-u)^{\frac{1}{2}-H}e'_x(u,X_u)X_u\,dB_u^H\,ds. \quad (12)$$

Now, using (10)-(12) we can find the function φ_t.

$$\frac{d}{dt}I(t)=\varphi_t=(2-2H)(r-e(0,0))\frac{C_1B_1}{c}t^{1-2H}-\frac{C_1}{c}\left(\int\limits_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}(e'_t(s,X_s)+\right.$$

$$\left.+e'_x(s,X_s)e(s,X_s)X_s)\,ds+\int\limits_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\int\limits_0^s (e'_t(u,X_u)+e'_x(u,X_u)e(u,X_u)X_u)\,du\,ds\right)-$$

$$-(2-2H)C_1t^{1-2H}\int\limits_0^t s^{2H-3}\int\limits_0^s u^{\frac{3}{2}-H}(s-u)^{\frac{1}{2}-H}e'_x(u,X_u)X_u\,dB_u^H\,ds-$$

$$-C_1t^{-1}\int\limits_0^t u^{\frac{3}{2}-H}(t-u)^{\frac{1}{2}-H}e'_x(u,X_u)X_u\,dB_u^H. \quad (13)$$

It follows from (13) that φ_s is a.s. continuous w.r.t. argument s function. Thus statement 2) holds.

Remark 1. *Since φ_s is a.s. continuous w.r.t. argument s function, $\forall t>0$ a.s. there exists an integral $\int\limits_0^t s^{2H-1}\varphi_s^2\,ds$, and therefore the random process $Z_t:=\frac{1}{C_2}\int\limits_0^t s^{H-\frac{1}{2}}\varphi_s\,dW_s$ is square integrable martingale with the quadratic characteristic $\langle Z\rangle_t=\frac{1}{C_2^2}\int\limits_0^t s^{2H-1}\varphi_s^2\,ds$.*

Now in equation (1) we change probability measure $\mathbb{P}$ for another measure $\mathbb{Q}$, such that $\mathbb{Q}_T\ll\mathbb{P}_T$, where $\mathbb{Q}_T=\mathbb{Q}|_{\mathcal{F}_T}$, and such that the drift will be equal to $rX_t\,dt$ (equation (1) will be linear).

Remark 2. *If measure* $\mathbb{Q}$ *such that* $\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = \exp\left\{Z_T - \frac{1}{2}\langle Z\rangle_T\right\}$, *then the process* $\widehat{B}_t^H = B_t^H - \int\limits_0^t \mu(u)\,du$ *is fBm w.r.t. measure* $\mathbb{Q}$ *(see theorem 4 from [8], and also [9], [10]).*

Theorem 2. *Let conditions of Lemma 2 hold and moreover the following condition holds*

$$\mathbb{E}\exp\left\{Z_t - \frac{1}{2}\langle Z\rangle_t\right\} = 1.$$

Then equation (1) admits the following representation w.r.t. measure $\mathbb{Q}$

$$dX_t = rX_t\,dt + cX_t\,d\widehat{B}_t^H$$

Proof. The proof of theorem follows from Lemma 2 and Remarks 1–2.

5. Fractional Burger's equation and equilibrium of financial market.

Definition 2. *Financial market described by the equation (1) is in equilibrium on* $[0, T]$ *if both the kernel* $\varphi_t t^{H-1/2}$ *and likelihood ratio* $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ *are the functions of* t *and* W_t, *and don't depend on the path of* W_s, $s < t$.

This definition generalizes the usual definition of equilibrium ([11], [12]) in which path's independence of $\frac{dQ}{dP}\big|_{\mathcal{F}_t}$ is declared, and the kernel equals simply $e(t, W_t)$. It should be noted that financial market described by the equation (1) admits arbitrage, but it can be in equilibrium.

Definition 3. *We say that function* $f(t, x)$ *satisfies fractional Burger's equation with index* H, *if the function* $g(t, x) := t^{H-1/2} f(t, x)$ *satisfies ordinary Burger's equation*

$$-g(s, x)g_2'(s, x) = g_1'(s, x) + \frac{1}{2}g_{22}''(s, x)$$

It is obvious that fractional Burger's equation with index H has a following form

$$-s^{H-1/2}p(s, x)p_2'(s, x) = (H - 1/2)s^{-1}p(s, x) + p_1'(s, x) + \frac{1}{2}p_{22}''(s, x),\ s > 0,\ x \in \mathbb{R}$$

Theorem 3. *Let financial market described by (1) is in equilibrium and moreover conditions (7), (8) hold. Then* φ_t *satisfies the fractional Burger's equation with index* H.

Proof. Let

$$\int\limits_0^t \varphi_s s^{H-1/2}\,dW_s - \frac{1}{2}\int\limits_0^t \varphi_s^2 s^{2H-1}\,ds = G(t, W_t),$$

where g, $G \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})$. Then

$$\int\limits_0^t g(s,\, W_s)\,dW_s - \frac{1}{2}\int\limits_0^t g^2(s,\, W_s)\,ds = G(t, W_t),\ t \in [0, T].$$

According to Itô formula

$$G(t, W_t) = \int\limits_0^t \left[G_1'(s, W_s) + \frac{1}{2}G_{22}''(s, W_s)\right]\,ds + \int\limits_0^t G_2'(s, W_s)\,dW_s.$$

Hence

$$\begin{array}{l} g(s,x) = G_2'(s,x), \\ -\frac{1}{2}g^2(s,x) = G_1'(s,x) + \frac{1}{2}G_{22}''(s,x) \end{array} \quad (14)$$

Further $g_2'(s,x) = G_{22}''(s,x)$,

$$-\frac{1}{2}g^2(s,x) = G_1'(s,x) + \frac{1}{2}g_2'(s,x). \quad (15)$$

Differentiating (14) on s and then (15) on x, we obtain:

$$g_1'(s,x) = G_{12}''(s,x)$$

$$-g(s,x)g_2'(s,x) = G_{12}''(s,x) + \frac{1}{2}g_{22}''(s,x)$$

or

$$-g(s,x)g_2'(s,x) = g_1'(s,x) + \frac{1}{2}g_{22}''(s,x).$$

Proof is over.

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