

ON ESTIMATION OF NET PREMIUM IN COLLECTIVE LIFE INSURANCE

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Abstract

Nonparametric estimates of net premiums in collective models of insurance are proposed. The asymptotic normality and the mean square convergence of the proposed estimates are proved. The main parts of asymptotic mean square errors of net premiums estimates are found. Simulation results show that the nonparametric estimates are just as good in practice.

Keywords: nonparametric estimate, collective life insurance, joint-life status, last-survivor status, net premium, survival function.

1. Introduction

The concept of status is the useful abstraction in the collective life insurance according to [1]. Consider m members of ages x_1, x_2, \dots, x_m who desire to buy insurance policy. Let us denote the future lifetime of k -th individual by $T(x_k) = X - x_k$. Let us put in a correspondence a status U with its future lifetime $T(U)$ and a set of numbers $T(x_1), T(x_2), \dots, T(x_m)$. Joint-life status and last-survivor status are the most widespread.

Joint-life status is denoted by $U := x_1 : x_2 : \dots : x_m$ and is considered as failed upon the first death, i.e. $T(U) = \min(T(x_1), T(x_2), \dots, T(x_m))$.

It is evident that

$$\mathbf{P}\{T(U) > t\} = \mathbf{P}\{\min(T(x_1), T(x_2), \dots, T(x_m)) > t\} = \mathbf{P}\{T(x_1) > t, T(x_2) > t, \dots, T(x_m) > t\},$$

so when the deaths are independent we have

$$\mathbf{P}\{T(U) > t\} = \prod_{i=1}^m \mathbf{P}\{T(x_i) > t\} .$$

Last-survivor status is denoted by $U := \overline{x_1 : x_2 : \dots : x_m}$ and fails upon the last death, and exists as long as at least one member of a group is alive, i.e.

$T(U) = \max(T(x_1), T(x_2), \dots, T(x_m))$. Similarly,

$$\mathbf{P}\{T(U) \leq t\} = \mathbf{P}\{\max(T(x_1), T(x_2), \dots, T(x_m)) \leq t\} = \mathbf{P}\{T(x_1) \leq t, T(x_2) \leq t, \dots, T(x_m) \leq t\} ,$$

and in the case of independent deaths we have

$$\mathbf{P}\{\max(T(U) \leq t\} = \prod_{i=1}^m \mathbf{P}\{T(x_i) \leq t\} .$$

2. Functionals of Net Premiums

Consider the case of two members: the joint-life status $U := x_1 : x_2$ with $T(U) = \min(T(x_1), T(x_2))$ and the last-survivor status $U := \overline{x_1 : x_2}$ with $T(U) = \max(T(x_1), T(x_2))$. Analogously to the case of individual whole life insurance [2] the net premiums can be written as

$$\overline{A}_{x_1 : x_2} = \int_0^{\infty} \exp(-\delta t) f_{x_1 : x_2}(t) dt ,$$

$$\overline{A}_{\overline{x_1 : x_2}} = \int_0^{\infty} \exp(-\delta t) f_{\overline{x_1 : x_2}}(t) dt ,$$

where δ is the instantaneous interest rate and the according probability density functions of these statistics are

$$f_{x_1 : x_2}(t) = (\mathbf{P}\{\min(T(x_1), T(x_2)) \leq t\})'_t = f_{x_1}(t) S_{x_2}(t) + f_{x_2}(t) S_{x_1}(t) ,$$

$$f_{\overline{x_1 : x_2}}(t) = (\mathbf{P}\{\max(T(x_1), T(x_2)) \leq t\})'_t = f_{x_1}(t) F_{x_2}(t) + f_{x_2}(t) F_{x_1}(t) ,$$

where $F_x(t)$, $f_x(t)$ and $S_x(t) = 1 - F_x(t)$ are the cumulative distribution function, the probability density function and the survival function for random variable $T(x)$ accordingly.

From here

$$\overline{A}_{x_1 : x_2} = \int_0^{\infty} \exp(-\delta t) [f_{x_1}(t) S_{x_2}(t) + f_{x_2}(t) S_{x_1}(t)] dt , \quad (1)$$

$$\overline{A}_{\overline{x_1 : x_2}} = \int_0^{\infty} \exp(-\delta t) [f_{x_1}(t) F_{x_2}(t) + f_{x_2}(t) F_{x_1}(t)] dt . \quad (2)$$

Using (1) and (2) we can see that net premiums $\bar{A}_{x_1:x_2}$ and $\bar{A}_{\overline{x_1:x_2}}$ less than the sum of individual net premiums \bar{A}_{x_1} and \bar{A}_{x_2} in view of $S_{x_1}(t) \leq 1$, $S_{x_2}(t) \leq 1$, $F_{x_1}(t) \leq 1$ and $F_{x_2}(t) \leq 1$. Note that we can use formulae (1) and (2) in parametric estimation [2].

For example, de Moivre's model has the following characteristics:

$$F_x(t) = I_t(\omega - x, \infty) + \frac{t I_t(0, \omega - x)}{\omega - x},$$

$$f_x(t) = \frac{I_t(0, \omega - x)}{\omega - x},$$

$$S_x(t) = I_t(-\infty, \omega - x) - \frac{t I_t(0, \omega - x)}{\omega - x},$$

where

$$I_t(a, b) = \begin{cases} 1, & t \in (a, b) \\ 0, & t \notin (a, b) \end{cases},$$

and functionals (1) and (2) are written as

$$\begin{aligned} \bar{A}_{x_1:x_2} &= \frac{1 - \exp(-\delta \min(\omega - x_1, \omega - x_2))}{\delta(\omega - x_1)} + \frac{1 - \exp(-\delta \min(\omega - x_1, \omega - x_2))}{\delta(\omega - x_2)} + \\ &+ \frac{2[(\delta \min(\omega - x_1, \omega - x_2) + 1) \exp(-\delta \min(\omega - x_1, \omega - x_2)) - 1]}{\delta^2(\omega - x_1)(\omega - x_2)}, \end{aligned}$$

$$\begin{aligned} \bar{A}_{\overline{x_1:x_2}} &= \frac{\exp(-\delta(\omega - x_1)) - \exp(-\delta \max(\omega - x_1, \omega - x_2))}{\delta(\omega - x_1)} + \\ &+ \frac{\exp(-\delta(\omega - x_2)) - \exp(-\delta \max(\omega - x_1, \omega - x_2))}{\delta(\omega - x_2)} + \\ &+ \frac{2 - [2\delta \min(\omega - x_1, \omega - x_2) + 2] \exp(-\delta \min(\omega - x_1, \omega - x_2))}{\delta(\omega - x_1)(\omega - x_2)}, \end{aligned}$$

where ω is the limiting age. If the parameter ω is unknown we can use the following estimate:

$$\hat{\omega} = \max(X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n),$$

where $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ is the sample of moments of deaths of pairs of individuals.

Functionals (1) and (2) for the Gompertz's, Makeham's and Weibull's models are estimated by computed methods, and unknown parameters are found by method of moments.

The tables of net premiums for different ages are designed. It is important note that Makeham's model characterizes the mortality process the most adequately.

Let us write the net premium functionals (1) and (2) in another manner

$$\bar{A}_{x_1:x_2} = \frac{1}{S(x_1, x_2)} \int_0^{\infty} \exp(-\delta t) d\mathbf{P}\{\min(X - x_1, Y - x_2) \leq t\} = \frac{\Phi(x_1, x_2, \delta)}{S(x_1, x_2)},$$

$$\bar{A}_{\overline{x_1:x_2}} = \frac{1}{\bar{S}(x_1, x_2)} \int_0^{\infty} \exp(-\delta t) d\mathbf{P}\{\max(X - x_1, Y - x_2) \leq t\} = \frac{\bar{\Phi}(x_1, x_2, \delta)}{\bar{S}(x_1, x_2)},$$

where $S(x_1, x_2) = \mathbf{P}\{\min(X - x_1, Y - x_2) > 0\}$ and $\bar{S}(x_1, x_2) = \mathbf{P}\{\max(X - x_1, Y - x_2) > 0\}$ are the survival functions of the joint-life status and the last-survivor status accordingly.

3. The Estimates of Net Premiums and Their Properties

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be two-dimensional sample of independent and identically distributed random variables. We estimate the distribution functions $\mathbf{P}\{\min(X - x_1, Y - x_2) \leq t\}$ and $\mathbf{P}\{\max(X - x_1, Y - x_2) \leq t\}$, the survival functions $S(x_1, x_2)$ and $\bar{S}(x_1, x_2)$ by the following unbiased nonparametric estimates

$$\frac{1}{n} \sum_{i=1}^n I(\min(X_i - x_1, Y_i - x_2) \leq t), \quad \frac{1}{n} \sum_{i=1}^n I(\max(X_i - x_1, Y_i - x_2) \leq t),$$

$$\frac{1}{n} \sum_{i=1}^n I(\min(X_i - x_1, Y_i - x_2) > 0) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n I(\max(X_i - x_1, Y_i - x_2) > 0),$$

where $I(A)$ is an indicator of the set A . Following reasonings for a single client model (see [3] and [4]), we obtain:

$$\begin{aligned} \hat{\bar{A}}_{x_1:x_2} &= \frac{1}{S_n(x_1, x_2)} \int_0^{\infty} \exp(-\delta t) d\left(\frac{1}{n} \sum_{i=1}^n I(\min(X_i - x_1, Y_i - x_2) \leq t)\right) = \\ &= \frac{1}{S_n(x_1, x_2)} \int_0^{\infty} \exp(-\delta t) \bar{\delta}(t - \min(X_i - x_1, Y_i - x_2)) dt = \\ &= \frac{\sum_{i=1}^n \exp(-\delta \min(X_i - x_1, Y_i - x_2))}{n S_n(x_1, x_2)} I(\min(X_i - x_1, Y_i - x_2) > 0) = \frac{\Phi_n(x_1, x_2, \delta)}{S_n(x_1, x_2)}, \end{aligned} \quad (3)$$

$$\begin{aligned}
\hat{A}_{x_1:x_2} &= \frac{1}{\bar{S}_n(x_1, x_2)} \int_0^\infty \exp(-\delta t) d\left(\frac{1}{n} \sum_{i=1}^n I(\max(X_i - x_1, Y_i - x_2) \leq t)\right) = \\
&= \frac{1}{\bar{S}_n(x_1, x_2)} \int_0^\infty \exp(-\delta t) \bar{\delta}(t - \max(X_i - x_1, Y_i - x_2)) dt = \\
&= \frac{\sum_{i=1}^n \exp(-\delta \max(X_i - x_1, Y_i - x_2))}{n \bar{S}_n(x_1, x_2)} I(\max(X_i - x_1, Y_i - x_2) > 0) = \frac{\bar{\Phi}_n(x_1, x_2, \delta)}{\bar{S}_n(x_1, x_2)}. \quad (4)
\end{aligned}$$

Here $\bar{\delta}(x)$ is Dirac delta-function.

Let: $t_n = (t_{1n}, \dots, t_{sn})$ be a s -dimensional statistic, $t_{jn} = t_{jn}(x) = t_{jn}(x; X_1, \dots, X_n)$, $j = \overline{1, s}$; $M_\nu(t_{jn}) = \mathbf{E}(t_{jn} - \varphi_j)^\nu$ be the moment of the order ν of the deviation $t_{jn}(x) - \varphi_j(x)$ at a point x . It is clear that $M_1(t_{jn})$ and $M_2(t_{jn})$ are bias and mean square error (MSE) of the estimate t_{jn} correspondingly.

Let the function $H(\varphi) : R^s \rightarrow R^1$, and $\varphi = \varphi(x) = (\varphi_1(x), \dots, \varphi_s(x))$ be s -dimensional bounded function.

Definition. Function $H(\cdot) \in N_{\nu, s}(\varphi)$ if $H(z) : R^s \rightarrow R^1$ and function $\varphi = \varphi(x)$ at a point x has an ε -neighborhood $\sigma = \{z : |z_i - \varphi_i| < \varepsilon, i = \overline{1, s}\}$, in which $H(z)$ and its derivatives $\frac{\partial H(z)}{\partial z_j}$ up to the order ν are continuous and bounded.

Let us denote by $\Rightarrow N_s(\mu, \sigma)$ the symbol of weak convergence of sequence of random variables to the s -dimensional normal random variable with mean $\mu = (\mu_1, \dots, \mu_s)$ and symmetric covariance matrix $\sigma = \|\sigma_{ij}\|$, $0 < \sigma_{jj} = \sigma_{jj}(x) < \infty$.

Let $H_j(\varphi)$ will denote a partial derivative $\frac{\partial H(z)}{\partial z_j} \Big|_{z=\varphi}$, $j = \overline{1, s}$,

$\nabla H(\varphi) = (H_1(\varphi), \dots, H_s(\varphi))$; the symbol T will denote the transpose. To prove the asymptotic normality of the net premiums estimates $\hat{A}_{x_1:x_2}$ and $\hat{A}_{x_1:x_2}$ we shall use the following result.

Theorem 1 [5]. If random vector $d_n(t_n - \varphi) \Rightarrow N_s(\mu, \sigma)$ for some number sequence $d_n \uparrow \infty$, the function $H(\cdot) \in N_{1, s}(\varphi)$, $\nabla H(\varphi) \neq 0$, then the random variable

$$d_n(H(t_n) - H(\varphi)) \Rightarrow N_1\{\nabla H(\varphi)\mu^T, \nabla H(\varphi)\sigma\nabla H(\varphi)^T\}.$$

Theorem 2. Let $S(x_1, x_2)$ be the continuous survival function of the joint-life status and $S(x_1, x_2) \neq 0$. Then

$$\sqrt{n}[\hat{A}_{x_1:x_2} - \bar{A}_{x_1:x_2}] \Rightarrow N_1\{0, Q_{x_1:x_2}\},$$

where

$$Q_{x_1:x_2} = \frac{\Phi(x_1, x_2, 2\delta)S(x_1, x_2) - \Phi^2(x_1, x_2, \delta)}{nS^3(x_1, x_2)}.$$

Proof. In notations of Theorem 3 we have: $s = 2$, $\varphi = (\varphi_1, \varphi_2)$, $H(\varphi) = \Phi_1/\varphi_2$, $\varphi_1 = \Phi(x_1, x_2, \delta)$, $\varphi_2 = S(x_1, x_2)$, $t_n = (t_{1n}, t_{2n}) = (\Phi_n(x_1, x_2, \delta), S_n(x_1, x_2))$, $d_n = \sqrt{n}$, $\nabla H(\varphi) = (H_1, H_2)$, $H_1 = 1/S(x_1, x_2)$, $H_2 = -\frac{\Phi(x_1, x_2, \delta)}{S^2(x_1, x_2)}$.

It is clear that in view of $S(x_1, x_2) \neq 0$ the function $H(\cdot) \in N_{1,2}(\varphi)$. Since

$$\mathbf{E}(S_n(x_1, x_2)) = S(x_1, x_2), \quad \mathbf{E}(\Phi_n(x_1, x_2, \delta)) = \Phi(x_1, x_2, \delta), \quad \mathbf{D}(S_n(x_1, x_2)) = \frac{1}{n}(S(x)(1-S(x))),$$

$$\mathbf{D}(\Phi_n(x_1, x_2, \delta)) = \frac{1}{n}(\Phi(x_1, x_2, 2\delta) - \Phi^2(x_1, x_2, \delta)),$$

$$\text{cov}(\Phi_n(x_1, x_2, \delta), S_n(x_1, x_2)) = \frac{1}{n}(\Phi(x_1, x_2, \delta))(1-S(x_1, x_2)), \text{ it follows that } \mu_1 = \mu_2 = 0,$$

$$\sigma_{11} = \Phi(x_1, x_2, 2\delta) - \Phi^2(x_1, x_2, \delta), \quad \sigma_{12} = \sigma_{21} = \Phi(x_1, x_2, \delta)(1-S(x_1, x_2)),$$

$$\sigma_{22} = S(x_1, x_2)(1-S(x_1, x_2)).$$

Using the two-dimensional central limit theorem, we have

$$\sqrt{n}(t_{1n} - \varphi_1, t_{2n} - \varphi_2) \Rightarrow N_2\{(0,0), \sigma\}.$$

Now the statement of Theorem 2 follows from Theorem 1.

Theorem 3. Let $\bar{S}(x_1, x_2)$ be the continuous survival function of the last-survivor status and $\bar{S}(x_1, x_2) \neq 0$. Then

$$\sqrt{n}[\hat{\bar{A}}_{x_1:x_2} - \bar{\bar{A}}_{x_1:x_2}] \Rightarrow N_1\{0, Q_{x_1:x_2}\},$$

where

$$Q_{x_1:x_2} = \frac{\bar{\Phi}(x_1, x_2, 2\delta)\bar{S}(x_1, x_2) - \bar{\Phi}^2(x_1, x_2, \delta)}{n\bar{S}^3(x_1, x_2)}.$$

The proof of Theorem 3 is similar to proof of Theorem 2.

Now it is interesting to find the main parts of asymptotic MSE of the estimates (3) and (4) and their almost sure convergence (see Theorems 5 and 6).

Let $\|t_n\| = \sqrt{t_{1n}^2 + t_{2n}^2 + \dots + t_{sn}^2}$ be the Euclidean norm of t_n , N and N^+ be sets of integers, and even integers accordingly.

We shall use the following result.

Theorem 4 [6]. Let: 1) $H(z)$, $\{H(t_n)\} \in N_{2,s}(t)$, 2) $\mathbf{E}\|t_n - t\|^i = O(d_n^{-i/2})$, $i = \overline{1, \infty}$.

Then for any $k \in N$

$$\left| \mathbf{E}[H(t_n) - H(t)]^k - \mathbf{E}[\nabla H(t)(t_n - t)^T]^k \right| = O(d_n^{-k+1/2}).$$

Theorem 5. If the continuous survival function of the joint-life status $S(x_1, x_2) \neq 0$, then the estimate of net premium $\hat{A}_{x_1:x_2}$ is asymptotic unbiased, i.e.

$$|\mathbf{E}(\hat{A}_{x_1:x_2}) - \bar{A}_{x_1:x_2}| = O(n^{-1}),$$

MSE is equal to

$$u^2(\hat{A}_{x_1:x_2}) = \frac{\Phi(x_1, x_2, 2\delta)S(x_1, x_2) - \Phi^2(x_1, x_2, \delta)}{n S^3(x_1, x_2)} + O(n^{-3/2}), \quad (5)$$

and

$$\mathbf{P}\left\{\lim_{n \rightarrow \infty} |\hat{A}_{x_1:x_2} - \bar{A}_{x_1:x_2}| = 0\right\} = 1.$$

Proof. We will use the base notations introduced in the proof of Theorem 2. For $k = 1$, $d_n = n$, $\mathbf{E}\|\Phi_n(x_1, x_2, \delta) - \Phi(x_1, x_2, \delta)\|^i = O(n^{-i/2})$,

$\mathbf{E}\|S_n(x_1, x_2) - S(x_1, x_2)\|^i = O(n^{-i/2})$, $i = \overline{1, \infty}$ (cf. [7, p.255]), and according to Theorem 4 we obtain $|\mathbf{E}(\hat{A}_{x_1:x_2}) - \bar{A}_{x_1:x_2}| = O(n^{-1})$.

Analogously, taking $k = 2$, we can show the formula (5) is fulfilled. Since

$M_4(\hat{A}_{x_1:x_2}) = O(n^{-2})$ according to Theorem 4 under $k = 4$, almost sure convergence of $\hat{A}_{x_1:x_2}$ to $\bar{A}_{x_1:x_2}$ follows the Borel-Cantelli lemma.

Theorem 6. If the continuous survival function of the last-survivor status $\bar{S}(x_1, x_2) \neq 0$, then the estimate of net premium $\hat{A}_{x_1:x_2}$ is asymptotic unbiased, i.e.

$$|\mathbf{E}(\hat{A}_{x_1, x_2}) - \bar{A}_{x_1, x_2}| = O(n^{-1}),$$

and MSE is equal to

$$u^2(\hat{A}_{x_1, x_2}) = \frac{\bar{\Phi}(x_1, x_2, 2\delta)\bar{S}(x_1, x_2) - \bar{\Phi}^2(x_1, x_2, \delta)}{n\bar{S}^3(x_1, x_2)} + O(n^{-3/2}).$$

4. Modifications of the Estimates (3) and (4)

Note that estimate \hat{A}_{x_1, x_2} has the shortcoming, because sometimes $S_n(x_1, x_2) = 0$.

This shortcoming is overcome by use of truncated modifications \hat{A}_{x_1, x_2} or piecewise-smoothed approximations \tilde{A}_{x_1, x_2} [4]:

$$\hat{A}_{x_1, x_2} = \begin{cases} \hat{A}_{x_1, x_2}, & \text{if } \hat{A}_{x_1, x_2} \leq Cn^\gamma, \\ Cn^\gamma, & \text{otherwise} \end{cases}, \quad C, \gamma > 0, \quad (6)$$

$$\tilde{A}_{x_1, x_2} = \frac{\hat{A}_{x_1, x_2}}{(1 + \delta_n \hat{A}_{x_1, x_2}^\tau)^\rho}, \quad \tau > 0, \rho > 0, \rho\tau \geq 1, \delta_n = O(n^{-1}). \quad (7)$$

Analogously, the modifications of estimation (4) are constructed.

Denote as in [6] $\tilde{\Phi}(z, \alpha) = \frac{H(z)}{(1 + \alpha |H(z)|^\tau)^\rho}$, where $\tau > 0, \rho > 0, \rho\tau \geq 1, \alpha > 0$.

Introduce for (τ, k) and $m \in N$ the set

$$T(m) = \{(\tau, k) : \tau \geq \tau(m) = 2k/(m - k + 1), m \geq m_0 = [3, k = 1; 2k, k \geq 2, k \in N]\},$$

We shall use the following result.

Theorem 7 [6]. Let: 1) $H(\cdot) \in N_{2,s}(\varphi)$, 2) $E_l \|t_n\| = O(q_n^{-1/2})$ for some $m \geq 3, m \in N$, 3) $\alpha = \delta_n = Cq_n^{-1}, 0 < C < \infty, q_n \uparrow \infty$, 4) $H(\varphi) \neq 0$ or $\tau \in N^+$. Then for any $(\tau, k) \in T(m)$

$$\left| \mathbf{E}[\tilde{\Phi}(t_n, \alpha) - H(\varphi)]^k - \mathbf{E}[\nabla H(\varphi)(t_n - \varphi)^T]^k \right| = O\left(\frac{1}{q_n^{(k+1)/2}}\right).$$

Theorem 8. If the continuous survival function of the joint-life status $S(x_1, x_2) \neq 0$, $\delta_n = n^{-1}$, $0 < C < \infty$ or $\tau \in N^+$, then as $n \rightarrow \infty$

$$\left| \mathbf{E} \left[\tilde{\bar{A}}_{x_1:x_2} - \bar{A}_{x_1:x_2} \right]^2 - \frac{\Phi(x_1, x_2, 2\delta)S(x_1, x_2) - \Phi^2(x_1, x_2, \delta)}{n S^3(x_1, x_2)} \right| = O\left(\frac{1}{n^{3/2}}\right),$$

$$\mathbf{P} \left\{ \lim_{n \rightarrow \infty} \left| \tilde{\bar{A}}_{x_1:x_2} - \bar{A}_{x_1:x_2} \right| = 0 \right\} = 1.$$

Proof. We will use the base notations introduced in the proof of Theorem 4. So, according to Theorem 7, $k = 2$, $q_n = n$. Taking $l = l_0 = 4$, we get

$$M_4 \|\Phi_n(x_1, x_2, \delta), S_n(x_1, x_2)\| \leq 2 \left[M_4(\Phi_n(x_1, x_2, \delta)) + M_4(S_n(x_1, x_2)) \right] = O(n^{-2}).$$

Now, if $\tau \geq \tau_0 = 4$, then all conditions of Theorem 7 for $\tilde{\bar{A}}_{x_1:x_2}$ hold. The first statement of Theorem 8 is proved. Analogously to Theorem 5 the second statement is proved.

Theorem 9 (main part of MSE of truncated modification). Let $S(x_1, x_2)$ be continuous survival function of the joint-life status, $S(x_1, x_2) \neq 0$, and $0 < \gamma < 1$ in the formula (6). Then the statements of Theorem 5 for the truncated modification $\tilde{\bar{A}}_{x_1:x_2}$ hold.

The proof of Theorem 9 is based on using the results of [5].

Remark 1. Under proper conditions the estimates (6) and (7) are asymptotically normally distributed also (cf. [5]).

The analogues of Theorems 7, 8 and 9 for the last-survivor status are proved too.

5. The Case of m Clients

In the case of m clients the functionals of net premiums can be written as

$$\bar{A}_{x_1:\dots:x_m} = \frac{\int_0^\infty \exp(-\delta t) d\mathbf{P}\{\min(X_1 - x_1, \dots, X_m - x_m) \leq t\}}{S(x_1, \dots, x_m)},$$

$$\bar{A}_{x_1:\dots:x_m} = \frac{\int_0^\infty \exp(-\delta t) d\mathbf{P}\{\max(X_1 - x_1, \dots, X_m - x_m) \leq t\}}{\bar{S}(x_1, \dots, x_m)}.$$

Let $(X_{11}, \dots, X_{1m}), \dots, (X_{n1}, \dots, X_{nm})$ be the m -dimensional sample. Then in the manner similar to (3) and (4), the estimates of the net premium are

$$\hat{A}_{x_1, \dots, x_m} = \frac{\sum_{i=1}^n \exp(-\delta \min(X_{i1} - x_1, \dots, X_{im} - x_m)) I(\min(X_{i1} - x_1, \dots, X_{im} - x_m) \leq t)}{n S_n(x_1, \dots, x_m)},$$

$$\hat{A}_{x_1, \dots, x_m} = \frac{\sum_{i=1}^n \exp(-\delta \max(X_{i1} - x_1, \dots, X_{im} - x_m)) I(\max(X_{i1} - x_1, \dots, X_{im} - x_m) \leq t)}{n \bar{S}_n(x_1, \dots, x_m)}.$$

These estimates are asymptotically unbiased, i.e. $|\mathbf{E}(\hat{A}_{x_1, \dots, x_m}) - \bar{A}_{x_1, \dots, x_m}| = O(n^{-1})$,

$|\mathbf{E}(\hat{A}_{x_1, \dots, x_m}) - \bar{A}_{x_1, \dots, x_m}| = O(n^{-1})$, their MSE are equal to

$$u^2(\hat{A}_{x_1, \dots, x_m}) = \frac{\Phi(x_1, \dots, x_m, 2\delta) S(x_1, \dots, x_m) - \Phi^2(x_1, \dots, x_m, \delta)}{n S^3(x_1, \dots, x_m)} + O(n^{-3/2}),$$

$$u^2(\hat{A}_{x_1, \dots, x_m}) = \frac{\bar{\Phi}(x_1, \dots, x_m, 2\delta) \bar{S}(x_1, \dots, x_m) - \bar{\Phi}^2(x_1, \dots, x_m, \delta)}{n \bar{S}^3(x_1, \dots, x_m)} + O(n^{-3/2}).$$

It is easily to obtain results that are analogous to the above-mentioned results for the net premiums estimates for models of p -year term life insurance, p -year pure endowment and r -year deferred insurance.

6. Simulation Results

The nonparametric estimates show adaptability if the distribution is changed and exceed parametric estimates, oriented on the best result only for its own distributions. Often the MSE of nonparametric estimates are less than the MSE of nonparametric estimates in 2-3 times. The main modeling results obtained using data from Makeham and de Moivre distributions.

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