EFFICIENT ASSET LIABILITY PORTFOLIOS USING MEAN-ERC AND MEAN-VARIANCE ANALYSIS

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Abstract.

In the context of asset and liability management, we propose a portfolio selection model based on the expected return of the assets and the economic risk capital (ERC) associated to the asset liability portfolio, for short called mean-ERC asset liability portfolio selection. Mean-ERC efficiency in asset and liability management is closely related and compared to mean-variance efficiency in asset management. Distinguished but similar results are obtained for an economy without and with a riskless asset. An illustration for the important situation of a life insurance business is presented and discussed.

Keywords : ALM, portfolio selection, economic risk capital, value-at-risk, expected shortfall, life insurance

1. Introduction.

Modern asset liability management takes into account both the characteristics of the assets and liabilities to make investment decisions. Early and more recent asset liability portfolio models include Wise(1984a/b), Wilkie(1985), Sharpe and Tint(1990), Elton and Gruber(1992), Leibowitz et al.(1992), Keel and Müller(1995), etc. In the present paper we propose a portfolio selection model based on the expected return of the assets and the economic risk capital (ERC) associated to the asset liability portfolio, for short called mean-ERC asset liability portfolio selection. The considered mean-ERC efficiency criterion is related and compared to the usual mean-variance criterion in asset management. Our method follows closely Alexander and Baptista(2000) and generalizes their results to an asset liability context.

The paper is organized as follows. In Section 2 we introduce our asset liability portfolio model and study its economic risk capital (ERC) using both the value-at-risk and the expected shortfall approach. To examine and point out diversification effects between assets and liabilities, we look separately at the ERC of the liabilities, called liability risk capital (LRC), and at the ERC of the assets, called asset risk capital (ARC). Formulas are derived under a multivariate normal assumption. One has the subadditive property $ERC \leq LRC + ARC$, which shows that simultaneous asset and liability management has a diversification advantage compared to a separate asset management and liability management. A simple and practical additive allocation of ERC on the asset and liability components is obtained applying the covariance principle. Following Ballmann and Hürlimann(2001), we present ERC formulas for the important situation of a life insurance business, which is our main application. In Section 3 mean-ERC asset liability portfolio selection is studied for an economy without riskless asset. Using the well-known mean-variance analytical results by Merton(1972), we derive the mean-ERC boundary, the mean-ERC efficient frontier, and the minimum ERC portfolio, which exists only under a restriction depending on the confidence level under which ERC is calculated. Furthermore, we characterize mean-ERC efficiency and present an interesting characterization of the mean-variance efficient frontier in terms of a minimum ERC portfolio at a prescribed confidence level. Section 4 adapts the results of Section 3 for an economy with a riskless asset. Finally, Section 5 presents and discusses an illustration for a life insurance business.

2. ERC for an asset liability portfolio model.

Modern asset liability management takes into account both the characteristics of the assets and liabilities to make investment decisions. Early and more recent asset liability portfolio models include Wise(1984a/b), Wilkie(1985), Sharpe and Tint(1990), Elton and Gruber(1992), Leibowitz et al.(1992), Keel and Müller(1995), etc. In the present Section, we first introduce our asset liability portfolio model, and then define the economic risk capital of the portfolio.

2.1. An asset liability portfolio model.

At an initial time $t=0$ the assets are valued according to market prices with an initial value A_0 and are supposed to be tradable on the financial market. After one period at time $t=T$ the assets with terminal random value A_T meet the liabilities with random value L_T. The liabilities, which are supposed to be non-tradable are valued according to some specific accounting rules. It is assumed that the certainty equivalent of L_T at time $t=0$ is equal to $H[L_T]$, the value obtained from a so-called pricing principle. For example, if one applies the standard deviation principle one has $H[L_T]=m_{L_T}+n_{L_T}\cdot s_{L_T}$ with $m_{L_T}=E[L_T]$ the expected value of the liabilities, $s_{L_T}=\sqrt{Var[L_T]}$ the standard deviation of the liabilities, and n_{L_T} the loading factor (or risk-reward ratio). The assets are invested in n risky securities with rates of return R_i over the one-period $[0,T]$, $i=1,...,n$. The investment strategy consists of a portfolio choice $\boldsymbol{w}=(w_1,...,w_n)$ with $\sum_{i=1}^{n} w_i=1$, where w_i is the proportion of wealth invested in security i. The vector of expected returns is denoted by $\boldsymbol{m}=(m_1,...,m_n)$ with $m_i=E[R_i]$, and the covariance matrix is $\boldsymbol{S}=(s_{ij})$ with $s_{ij}=Cov[R_i,R_j]$, $i,j=1,...,n$. The rate of return of a portfolio $\boldsymbol{w}$ is denoted by $R_w=\sum_{i=1}^{n} w_i R_i$ with mean $m_w=m\cdot w^T$ and variance $s_w^2=w\cdot\Sigma\cdot w^T$. To summarize, the asset and liability portfolio is described by the set of quantities

$$PF_w=\{A_0,L_T,H[L_T],m_{L_T},s_{L_T},m,\Sigma,w\}. \tag{2.1}$$

2.2. The notion of economic risk capital.

Depending on the portfolio choice $\mathbf{w}$, the initial surplus is $S_0 = A_0 - H[L_T]$, and after one period the final surplus is $S_T = A_0 \cdot (1+R_w) - L_T$. In order to associate an economic risk capital to the asset liability portfolio, one considers the random loss over the period defined as the negative increase in surplus at time T, that is

$$V_T = S_0 - S_T = (L_T - H[L_T]) - A_0 \cdot R_w . \tag{2.2}$$

To be able to cover any loss with a high probability, the portfolio manager borrows at time $t=0$ the amount ERC_0, called economic risk capital. At time T, interest on this is due at the rate m_R over the period. To guarantee with certainty the value of the borrowed capital at time T, the amount ERC_0 is invested at the riskless rate $m_f < m_R$. The value of the economic risk capital at time T is $ERC_T = ERC_0 \cdot (1 + m_f - m_R)$.

There exist several risk management principles applied to determine ERC_T. Two simple methods are the value-at-risk and the expected shortfall approach (e.g. Arztner et al.(1997), Arztner(1999), Embrechts(1995), Wirch(1999)). According to the value-at-risk method one identifies ERC_T with the *value-at-risk* at the confidence level a of the loss setting

$$ERC_T = VaR_a[V_T] := Q_{V_T}(a), \tag{2.3}$$

where $Q_X(u) = \inf\{x | F_X(x) \ge u\}$ denotes a quantile function of the random variable X and $F_X(x) = \Pr(X \le x)$ ist the distribution function of X. This quantile represents the maximum possible loss, which is not exceeded with the (high) confidence level a. According to the expected shortfall method one identifies ERC_T with the *risk-adjusted capital* at the confidence level a of the loss setting

$$ERC_T = RaC_a[V_T] := E[V_T | V_T > VaR_a[V_T]]. \tag{2.4}$$

This value represents the conditional loss given the loss exceeds its value-at-risk. One has

$$RaC_a[V_T] = Q_{V_T}(a) + m_{V_T}[Q_{V_T}(a)] = Q_{V_T}(a) + \frac{1}{e} \cdot p_{V_T}[Q_{V_T}(a)], \tag{2.5}$$

where $m_X(x) := E[X - x | X > x]$ denotes the mean excess function of X, $p_X(x) = (1 - F_X(x)) \cdot m_X(x)$ is the stop-loss transform, and $e = 1 - a$ is interpreted as loss probability (also called loss tolerance level). Mathematically, VaR and RaC, which have been defined as functions of random variables, may be viewed as functionals defined on the space of probability distributions associated with these random variables. By abuse of language, we will use the terminology functionals when appropriate.

It is important to observe that both ERC functionals satisfy two important risk-preference criteria in the economics of insurance (see Denuit et al.(1999) for a recent review). They are consistent with the risk preferences of profit-seeking decision makers respectively profit-seeking risk averse decision makers. Indeed, one has the following stochastic and stop-loss order relations (e.g. Hürlimann(2001)) :

$$X \le_{st} Y \quad \Leftrightarrow \quad VaR_\alpha[X] \le VaR_\alpha[Y], \text{ for all } \alpha \in [0,1], \tag{2.6}$$
$$X \le_{sl} Y \quad \Leftrightarrow \quad RaC_\alpha[X] \le RaC_\alpha[Y], \text{ for all } \alpha \in [0,1]. \tag{2.7}$$

However, except in a world of elliptical linear portfolio losses (Embrechts et al.(1998), Fundamental Theorem of Risk Management), the VaR functional has several shortcomings. It is not subadditive and not scalar multiplicative, and it cannot discriminate between risk-averse and risk-taking portfolios (examples 1 to 3 in Wirch(1999)). In contrast to this, the RaC functional, which is subadditive and scalar multiplicative, is a coherent risk measure in the sense of Arztner et al.(1997) and appears thus more suitable in general.

2.3. <u>ERC formulas</u>.

To determine ERC_T, let us use the *normalized loss* V_w of the portfolio PF_w defined by

$$V_w = \frac{V_T}{A_0} = (R_L - H[R_L]) - R_w, \tag{2.8}$$

where $R_L = \frac{L_T}{A_0}$ represents the liability rate per unit of the initial investment amount, and the calculation principle $H[\cdot]$ has the positive homogeneous property such that $H[R_L] = \frac{H[L_T]}{A_0}$. Denote by μ_L and σ_L the mean and standard deviation of R_L. Without specification of the multivariate distribution of $(R_L, R_1, ..., R_n)$ the quantities (2.3) and (2.4) cannot be calculated. For simplicity, let us assume a multivariate normal distribution, and let ρ_w denote the correlation coefficient between R_L and R_w. More generally, it is reasonable to assume that $(R_1, ..., R_n)$ has a multivariate normal, t or elliptical distribution, such that R_w has still a normal, t or elliptical distribution, and to specify a bivariate distribution for (R_L, R_w), where R_L is not necessarily normally distributed, for example applying copula functions. To examine and point out diversification effects between assets and liabilities, it is of interest to look separetely at the economic risk capital of the normalized liability loss $V^L = R_L - H[R_L]$, called *liability risk capital* and abbreviated LRC, and at the economic risk capital of the normalized asset loss $V_w^A = -R_w$, called *asset risk capital* and abbreviated ARC.

<u>Proposition 2.1</u>. The value-at-risk economic risk capitals associated to the normalized losses of an asset liability portfolio PF_w under a multivariate normal distribution are given by

$$VaR_\alpha[V_w] = -\mu_w - \nu_L \sigma_L + \Phi^{-1}(\alpha)\sqrt{(\sigma_w - \rho_w \sigma_L)^2 + (1 - \rho_w^2)\sigma_L^2} \tag{2.9}$$
$$VaR_\alpha[V^L] = (\Phi^{-1}(\alpha) - \nu_L)\sigma_L \tag{2.10}$$
$$VaR_\alpha[V_w^A] = \Phi^{-1}(\alpha)\sigma_w - \mu_w \tag{2.11}$$

Proof. This follows from the definition of $VaR_\alpha[\cdot]$ noting that $Q_X(\alpha) = \mu_X + \Phi^{-1}(\alpha)\sigma_X$ for any normally distributed random variable X. ◊

Proposition 2.2. The expected shortfall economic risk capitals associated to the normalized losses of an asset liability portfolio PF_w under a multivariate normal distribution are

$$RaC_\alpha[V_w] = -\mu_w - \nu_L \sigma_L + \frac{1}{1-\alpha}\varphi\left[\Phi^{-1}(\alpha)\right]\sqrt{(\sigma_w - \rho_w \sigma_L)^2 + \left(1-\rho_w^2\right)\sigma_L^2} \tag{2.12}$$

$$RaC_\alpha\left[V^L\right] = \left(\frac{1}{1-\alpha}\varphi\left[\Phi^{-1}(\alpha)\right] - \nu_L\right)\sigma_L \tag{2.13}$$

$$RaC_\alpha\left[V_w^A\right] = \frac{1}{1-\alpha}\varphi\left[\Phi^{-1}(\alpha)\right]\sigma_w - \mu_w \tag{2.14}$$

Proof. This follows from the definition of $RaC_\alpha[\cdot]$ noting that $\pi_X[Q_X(\alpha)] = \sigma_X \varphi\left[\Phi^{-1}(\alpha)\right] - (1-\alpha)([Q_X(\alpha)] - \mu_X)$ for any normally distributed random variable *X*. ◊

The algebraic structure of the ERC formulas is identical in both approaches, the difference lying in the factors $\Phi^{-1}(\alpha)$ respectively $\frac{1}{1-\alpha}\varphi\left[\Phi^{-1}(\alpha)\right]$, denoted α^* for a common use in the following. The ERC quantities are increasing functions of α^*, hence also of the confidence level α. The effect of the liability risk on ERC is threefold depending on the factor loading ν_L, the standard deviation σ_L of the liability rate R_L, and the correlation between R_L and R_w. Comparing ERC of V_w with the economic risk capitals LRC and ARC of the additive components V^L and V_w^A reveals the subadditive property

$$ERC \le LRC + ARC, \tag{2.15}$$

which follows from the inequality $(\sigma_w - \rho_w \sigma_L)^2 + \left(1-\rho_w^2\right)\sigma_L^2 \le (\sigma_w + \sigma_L)^2$ valid for all $\rho_w \ge -1$. Performing asset and liability management simultaneously has a diversification advantage compared to a separate asset management and liability management.

If strict inequality $ERC < LRC + ARC$ holds, then a participant of a global asset liability market can sell LRC and ARC of the components V^L and V_w^A separately and buy back ERC of V_w, making a riskless profit of amount $LRC + ARC - ERC > 0$ on economic risk capital. To avoid such arbitrage opportunities, one looks for "fair" allocations of ERC on the asset liability components satisfying the additive property $ERC = LRC + ARC$. A simple and practical solution is a covariance principle such that

$$LRC = E\left[V_w^L\right] + \frac{Cov\left[V_w^L, V_w\right]}{Var[V_w]}\left(ERC - E[V_w]\right), \tag{2.16}$$

$$ARC = E\left[V_w^A\right] + \frac{Cov\left[V_w^A, V_w\right]}{Var[V_w]}\left(ERC - E[V_w]\right). \tag{2.17}$$

2.4. ERC for life insurance.

An important and main example to which the considered asset liability portfolio model applies is life insurance, including the special instance of pensions funds. In this situation

ERC modelling has been considered in details by Ballmann and Hürlimann(2001). Recall that, refraining from the cost process, the random loss over a period $[t, t+1)$ of a life business at time $t+1$ can be represented as

$$V_{t+1} = (DK_t + p_t) \cdot (i_t^L - I_{t+1}^w) + (S_{t+1} - p_t^R (1 + i_t^L)), \tag{2.18}$$

where DK_t are the mathematical reserves at time t, p_t, p_t^R are the net and risk premiums due at time t, i_t^L is the guaranteed technical interest rate, S_{t+1} are the aggregate claims at time $t+1$ (stochastic sum of the difference between individual benefits and mathematical reserves over all claims in the period $[t, t+1)$), and $I_{t+1}^w = I_{t+1} \cdot w^T$ is the rate of return over the period $[t, t+1)$ of an asset portfolio with weights $\boldsymbol{w}$ and vector of returns $I_{t+1} = (I_{t+1}^1, ..., I_{t+1}^n)$. Comparing (2.18) with (2.2) it appears adequate to define the asset liability portfolio (2.1) as follows :

$$\begin{aligned} &PF_w = \{DK_t + p_t, S_{t+1}, H[S_{t+1}] = p_t^R (1 + i_t^L), m_{S_{t11}}, s_{S_{t+1}}, m - i_t^L \cdot e, \Sigma, w\}, \\ &e = (1, ..., 1), \quad s_{ij} = Cov[I_{t+1}^i, I_{t+1}^j] \end{aligned} \tag{2.19}$$

It is assumed that the rate of return I_{t+1}^w is *independent* of the aggregate claims S_{t+1}, hence $Cov[R_w, R_L] = 0$ and $r_w = 0$. The assumption of normally distributed aggregate claims in life insurance seems justified for practical ERC calculations, as shown in Ballmann and Hürlimann(2001), Section 6. The value-at-risk and expected shortfall economic risk capitals associated to the normalized losses of PF_w are calculated according to the Propositions 2.1 and 2.2, and allocated to the liability and asset components using (2.16) and (2.17). In the terminology of Ballmann and Hürlimann(2001), the liability risk capital LRC corresponds to the insurance risk capital IRC and the asset risk capital ARC corresponds to the market risk capital MRC. Measured in units of the invested capital $DK_t + p_t$, the economic risk capital of a life insurance business is given by the formula

$$ERC = a^* \sqrt{s_w^2 + s_L^2} - m_w - n_L s_L + r_L \tag{2.20}$$

where one sets $a^* = \Phi^{-1}(a)$ (value-at-risk method) respectively $a^* = \frac{1}{1-a} j\left[\Phi^{-1}(a)\right]$ (expected shortfall method) and

$$s_L = \frac{s_{S_{t+1}}}{DK_t + p_t}, \tag{2.21}$$

$$n_L = \frac{p_t^R (1 + i_t^L) - m_{S_{t+1}}}{s_{S_{t+1}}}, \tag{2.22}$$

$$r_L = i_t^L. \tag{2.23}$$

3. Mean-ERC versus mean-variance analysis without riskless asset.

By fixed liability structure, the economic risk capital to the confidence level a associated to the normalized loss of an asset liability portfolio PF_w, under a multivariate normal distribution and the assumption $r_w = 0$ made in life insurance, is described by the function

$$C(a,w) = a^* \sqrt{s_w^2 + s_L^2} - m_w - n_L o_L , \tag{3.1}$$

Note that the shift of r_L in (2.20) due to the guaranteed technical interest rate in life insurance is not taken into account here. One makes the assumption $a \in \left(\frac{1}{2},1\right)$, which implies that $a^* > 0$.

3.1. Mean-ERC boundary and mean-ERC efficient frontier.

We consider the notions of mean-ERC boundary and mean-ERC efficient frontier and relate them with the similar notions in mean-variance analysis. We closely follow Alexander and Baptista(2000) and generalize their results to the asset liability context. The set $W = \left\{ w \in R^n \,\middle|\, \sum w_i = 1 \right\}$ describes all portfolio choices.

Definitions 3.1. A portfolio $\overline{w} \in W$ belongs to the *mean-ERC boundary* at the confidence level a if and only if, for some $\overline{m}$, $\overline{w}$ solves the problem

$$\min_{w \in W} C(a,w) \quad \text{subject to} \quad m_w = \overline{m} . \tag{3.2}$$

A portfolio $\overline{w} \in W$ belongs to the *mean-variance boundary* if and only if, for some $\overline{m}$, $\overline{w}$ solves the problem $\min_{w \in W} s_w^2$ subject to $m_w = \overline{m}$.

Property 3.1. Since $a^* > 0$ a portfolio belongs to the mean-ERC boundary if and only if it belongs to the mean-variance boundary.

From Merton(1972) or Huang and Litzenberger(1988), Section 3.11, one knows that portfolio w belongs to the mean-variance boundary if and only if

$$\frac{s_w^2}{\frac{1}{C}} - \frac{\left(m_w - \frac{A}{C}\right)^2}{\frac{D}{C^2}} = 1 , \tag{3.3}$$

where the constants are defined by

$$\begin{aligned} A &= e\Sigma^{-1}m^T, \quad B = m\Sigma^{-1}m^T, \\ C &= e\Sigma^{-1}e^T, \quad D = BC - A^2, \end{aligned} \tag{3.4}$$

where $e=(1,\ldots,1)$ denotes the n-dimensional unit vector, and B, C, D are positive. In the standard deviation mean (s,m)-space the equation (3.3) represents a hyperbola with center $\left(0,\frac{A}{C}\right)$ and asymptotes $m=\frac{A}{C}\pm\sqrt{\frac{D}{C}}\cdot s$, where the point $m_s=\left(\sqrt{\frac{1}{C}},\frac{A}{C}\right)$ defines the *minimum variance portfolio*. Using Property 3.1 the mean-ERC boundary equation is obtained from (3.3) using (3.1). Solving (3.1) for s_w one obtains the *necessary condition* on s_L (assumed in the following) :

$$s_w^2=\left(\frac{m_w+n_L s_L+C(a,w)}{a^*}\right)^2-s_L^2>0\ . \qquad (3.5)$$

It follows that a portfolio w belongs to the mean-ERC boundary if and only if

$$\frac{\left(\frac{m_w+n_L s_L+C(a,w)}{a^*}\right)^2-s_L^2}{\frac{1}{C}}-\frac{\left(m_w-\frac{A}{C}\right)^2}{\frac{D}{C^2}}=1\ . \qquad (3.6)$$

In contrast to the above, the geometric representation in the ERC-mean (C,m)-space of the equation (3.6) is not a conic section, but a simple transformation of it. Solving (3.6) for m_w one obtains the hyperbola like curve

$$m_w=\frac{A}{C}+\frac{\frac{D}{C}[C(a,w)+E]\pm a^*\sqrt{\frac{D}{C}}\sqrt{[C(a,w)+E]^2-\left(a^{*2}-\frac{D}{C}\right)F}}{a^{*2}-\frac{D}{C}}\ , \qquad (3.6')$$

$$E=\frac{A}{C}+n_L s_L,\quad F=\frac{1}{C}+s_L^2.$$

As seen later in Proposition 3.1, the minimum ERC portfolio exists exactly when $a^*>\sqrt{\frac{D}{C}}$ and, as will be shown, the economic properties of (3.6') are similar to those of the mean-variance boundary. A detailed discussion of the geometric structure of (3.6'), including the limiting cases $a^*\to 0$ and $a^*\to\infty$ is left to the reader (see Alexander and Baptista(2000) in the asset only case, and Section 5 for an example).

Definitions 3.2. A portfolio $w\in W$ belongs to the *mean-ERC efficient frontier* at the confidence level a if and only if no portfolio $v\in W$ exists such that $m_v\ge m_w$ and $C(a,v)\le C(a,w)$, where at least one of the inequalities is strict. A portfolio $w\in W$ belongs to the *mean-variance efficient frontier* if and only if no portfolio $v\in W$ exists such that $m_v\ge m_w$ and $s_v\le s_w$, where at least one of the inequalities is strict.

3.2. The minimum ERC portfolio.

As a portfolio w moves along the mean-ERC efficient frontier, one observes two effects : the *standard deviation effect*, which acts through the term proportional to a^*, and the *mean effect*, which acts through the term $m_w + n_L o_L$. If the confidence level is not large enough so that the standard deviation effect prevails the mean effect, then the minimum ERC problem has no solution. In the following the existence problem is settled, and the minimum ERC portfolio with its minimum ERC are expressed in analytical closed-form. One requires a preliminary result.

Lemma 3.1. If the miminum ERC portfolio at the confidence level a exists, then it is mean-variance efficient.

Proof. Suppose the minimum ERC portfolio exists and denote it m_a. If m_a is not mean-variance efficient, then there exists $v \in W$ such that $m_v \geq m_{m_a}$, $s_v \leq s_{m_a}$, with at least one strict inequality. One obtains from (3.1) that $C(a,v) < C(a,m_a)$, which contradicts the fact that m_a is the minimum ERC portfolio. ◊

Proposition 3.1. The minimum ERC portfolio at the confidence level a exists if and only if $a^* > \sqrt{\frac{D}{C}}$. If the condition $a^* > \sqrt{\frac{D}{C}}$ is fulfilled, then the minimum ERC portfolio $m_a \in W$ is given by

$$m_a = g + h\left[\frac{A}{C} + \sqrt{\frac{D}{C}\left(s_{m_a}^2 - \frac{1}{C}\right)}\right] \tag{3.7}$$

where g and h are the vectors defined by

$$g = \frac{1}{D}\left[B\left(\Sigma^{-1}e^T\right) - A\left(\Sigma^{-1}m^T\right)\right], \quad h = \frac{1}{D}\left[C\left(\Sigma^{-1}m^T\right) - A\left(\Sigma^{-1}e^T\right)\right], \tag{3.8}$$

and

$$s_{m_a} = \sqrt{\frac{a^{*2} + Ds_L^2}{Ca^{*2} - D}}\,. \tag{3.9}$$

Furthermore, the minimum ERC is given by

$$C(a,m_a) = a^*\sqrt{s_{m_a}^2 + s_L^2} - \left[\frac{A}{C} + \sqrt{\frac{D}{C}\left(s_{m_a}^2 - \frac{1}{C}\right)}\right] - n_L o_L\,. \tag{3.10}$$

Remark 3.1. In the asset only case, that is if $s_L = 0$, the formulas (3.7) and (3.10) are identical to the formulas (11) and (12) in Alexander and Baptista(2000).

It is useful to express the necessary and sufficient condition $a^* > \sqrt{\frac{D}{C}}$ in terms of the confidence level.

Lemma 3.2. The condition $a^* > \sqrt{\frac{D}{C}}$ is equivalent to the following conditions :

VaR case : $a > \Phi\left(\sqrt{\frac{D}{C}}\right)$

RaC case : $a > 1 - \Phi\left(M^{-1}\left(\sqrt{\frac{D}{C}}\right)\right)$, where M^{-1} denotes the inverse function of the Mill's ratio $M(x) = \frac{j(x)}{\Phi(x)}$.

Proof. The VaR case is trivial. In the RaC case one has $a^* = \frac{1}{1-a} \cdot j\left[\Phi^{-1}(1-a)\right] > \sqrt{\frac{D}{C}}$. Setting $b = \Phi^{-1}(1-a)$ one has $M(b) > \sqrt{\frac{D}{C}}$, which is equivalent to $b < M^{-1}\left(\sqrt{\frac{D}{C}}\right)$ because $M(x)$ is strictly decreasing. ◊

Proof of Proposition 3.1. First, we show that $a^* > \sqrt{\frac{D}{C}}$ is a necessary condition for the existence of m_a. From (3.3) one has

$$m_w = \frac{A}{C} + \sqrt{\frac{D}{C}\left(s_w^2 - \frac{1}{C}\right)} \tag{3.11}$$

for all $w \in W$ on the mean-variance efficient frontier. One has to solve $\min_{w \in W} C(a, w)$. As noted in Section 3.1, the standard deviation of the minimum variance portfolio is given by $s_{m_s} = \sqrt{C^{-1}}$. Using Lemma 3.1, (3.1) and (3.11), one has to solve

$$\min_{s \in \left[\sqrt{C^{-1}}, \infty\right)} a^* \sqrt{s^2 + s_L^2} - \left[\frac{A}{C} + \sqrt{\frac{D}{C}\left(s^2 - \frac{1}{C}\right)}\right] - n_L o_L \tag{3.12}$$

to determine the ERC of the minimum ERC portfolio. A necessary condition for $s = s_{m_s}$ to solve (3.12) is

$$\frac{\partial}{\partial s}\left\{a^* \sqrt{s^2 + s_L^2} - \left[\frac{A}{C} + \sqrt{\frac{D}{C}\left(s^2 - \frac{1}{C}\right)}\right] - n_L o_L\right\} = a^* \frac{s}{\sqrt{s^2 + s_L^2}} - \frac{s\sqrt{\frac{D}{C}}}{\sqrt{s^2 - \frac{1}{C}}} = 0, \tag{3.13}$$

hence s_{m_a} is given by (3.9), and necessarily $a^* > \sqrt{\frac{D}{C}}$. To show that this is a sufficient condition for the existence of the minimum, we show that the function in (3.12) is convex. This follows from the calculation

$$\frac{\partial^2}{\partial s^2}\left\{a^*\sqrt{s^2+s_L^2}-\left[\frac{A}{C}+\sqrt{\frac{D}{C}\left(s^2-\frac{1}{C}\right)}\right]-n_L o_L\right\}=\frac{\partial}{\partial s}\left\{a^*\frac{s}{\sqrt{s^2+s_L^2}}-\frac{s\sqrt{\frac{D}{C}}}{\sqrt{s^2-\frac{1}{C}}}\right\}$$

$$=a^*\cdot\frac{s_L^2}{\left[s^2+s_L^2\right]^{3/2}}+\frac{1}{C}\frac{\sqrt{\frac{D}{C}}}{\left[s^2-\frac{1}{C}\right]^{3/2}}>0,\quad s\in\left(\sqrt{C^{-1}},\infty\right) \tag{3.14}$$

By Huang and Litzenberger(1988), pp. 64-65, for any m_w there exists a unique mean-variance boundary portfolio $w \in W$ with expected return m_w such that $w = g + hm_w$. Using (3.11) this implies the relation (3.7). Finally, (3.10) is immediate. ◊

The solutions to the minimization problems (3.1) and (3.2) are always distinct, as stated in the following result.

Corollary 3.1. If it exists, the minimum ERC portfolio at the confidence level $a < 1$ lies above the minimum variance portfolio on the mean-variance efficient frontier.

Proof. From Lemma 3.1 one knows that the minimum ERC portfolio belongs to the mean-variance efficient frontier. As noted in Section 3.1, the mean of the minimum variance portfolio equals $m_{m_s} = \frac{A}{C}$. Using (3.11) one obtains $m_{m_a} = \frac{A}{C} + \sqrt{\frac{D}{C}\left(s_{m_a}^2 - \frac{1}{C}\right)} > m_{m_s}$, which shows the result. ◊

3.3. Characterization of mean-ERC efficiency.

The following result characterizes the mean-ERC efficient frontier.

Proposition 3.2. If $a^* > \sqrt{\frac{D}{C}}$ then a portfolio is mean-ERC efficient at the confidence level a if and only if it belongs to the mean-ERC boundary and it has an expected rate of return greater than or equal to the expected rate of return of the minimum ERC portfolio at that confidence level. If $a^* \le \sqrt{\frac{D}{C}}$ then no mean-ERC efficient portfolio exists at the confidence level a.

Proof. To show the first part of the affirmation, it suffices to verify that $\frac{\partial^2 C(a,w)}{\partial m_w^2} > 0$ for every mean-ERC boundary portfolio $w \in W$. Using (3.1) one has

$$\frac{\partial C}{\partial m_w} = a^* \frac{s_w \frac{\partial s_w}{\partial m_w}}{\sqrt{s_w^2 + s_L^2}} - 1,$$

$$\frac{\partial^2 C}{\partial m_w^2} = a^* \frac{s_w \left(\frac{\partial s_w}{\partial m_w}\right)^2 s_L^2 + s_w \frac{\partial^2 s_w}{\partial m_w^2}\left[s_w^2 + s_L^2\right]}{\left[s_w^2 + s_L^2\right]^{3/2}}. \qquad (3.15)$$

Since w belongs to the mean-variance boundary by property 3.1, one has from (3.3) that

$$s_w = \sqrt{\frac{1}{C} + \frac{\left(m_w - \frac{A}{C}\right)^2}{\frac{D}{C}}}. \qquad (3.16)$$

Since $\frac{\partial^2 s_w}{\partial m_w^2} = \frac{1}{D s_w^2} > 0$ (3.15) implies $\frac{\partial^2 C}{\partial m_w^2} > 0$. The second part of the affirmation follows from the inequality

$$\frac{\partial C}{\partial m_w} = a^* \frac{s_w \frac{\partial s_w}{\partial m_w}}{\sqrt{s_w^2 + s_L^2}} - 1 \le a^* \frac{m_w - \frac{A}{C}}{\sqrt{\frac{D}{C}}\sqrt{\frac{1}{C} + \frac{\left(m_w - \frac{A}{C}\right)^2}{\frac{D}{C}}}} - 1 < 0,$$

which implies that for any $w \in W$ there is a $v \in W$ with both higher expected rate of return and lower ERC, which is equivalent to mean-ERC inefficiency. ◊

Corollary 3.2. The minimum variance portfolio is mean-ERC inefficient at any confidence level $a < 1$.

Proof. This follows from Proposition 3.2 using the inequality $m_{m_a} > m_{m_s}$ shown in the proof of Corollary 3.1. ◊

Similarly to the Corollaries 3 to 5 in Alexander and Baptista(2000) it is possible to show the following assertions :

- The minimum ERC portfolio converges to the minimum variance portfolio as $a \to 1$.
- The set of mean-ERC efficient portfolios is a proper subset of the set of mean-variance efficient portfolios in case $a < 1$, but the former converges to the latter as $a \to 1$.
- The expected rate of return of the minimum ERC portfolio converges to infinity and the set of mean-ERC efficient portfolios converges to the empty set as $a^* \downarrow \sqrt{\frac{D}{C}}$.

The last result has two important consequences. First, asset liability managers that minimize ERC at relatively small confidence levels choose portfolios with large expected rates of return and are thus exposed to high risks as measured by standard deviation. Second, using ERC at a small confidence level reduces the set of mean-ERC efficient portfolios, which may even become empty.

3.4. Characterization of mean-variance efficiency.

The close relationship between mean-ERC and mean-variance analysis can be exploited further to characterize the mean-variance efficient frontier.

Proposition 3.3. A portfolio $w \in W - \{m_s\}$ belongs to the mean-variance efficient frontier if and only if w is the minimum ERC portfolio at the confidence level a determined by

$$a^{*2} = \left[1 + \left(\frac{s_L}{s_{m_a}}\right)^2\right] \cdot \frac{D}{C} \cdot \frac{s_{m_a}^2}{s_{m_a}^2 - \frac{1}{C}}, \tag{3.17}$$

$$s_{m_a} = \sqrt{\frac{1}{C} + \frac{\left(m_w - \frac{A}{C}\right)^2}{\frac{D}{C}}}. \tag{3.18}$$

Proof. The sufficient condition follows immediately from Lemma 3.1. To show the necessary condition, let $w \in W - \{m_s\}$ be arbitrary. Since w is mean-variance efficient one has $w = g + hm_w$, mit g, h as in (3.8). By Proposition 3.1, if there exists s_{m_a} satisfying (3.9) such that

$$m_w = \frac{A}{C} + \sqrt{\frac{D}{C}\left(s_{m_a}^2 - \frac{1}{C}\right)}, \tag{3.19}$$

then w is the minimum ERC portfolio at the confidence level a. From (3.19) one obtains immediately (3.18). Solving (3.9) for a^* shows (3.17). ◊

This result has the following interesting interpretation for strategic asset allocation. Asset liability managers that select their portfolios on the mean-variance efficient frontier without taking into account the liabilities act as if they were choosing a minimum ERC portfolio at a *prescribed confidence level*, which depends on the volatility of the liability rate per unit of invested capital. Moreover, any mean-variance efficient portfolio $w \in W - \{m_s\}$ is mean-ERC inefficient (efficient) at any confidence level lower than (greater than or equal) to the level determined by (3.17).

4. <u>Mean-ERC versus mean-variance analysis with a riskless asset</u>.

Suppose that in the economy there are n risky assets and a riskless one with rate of return $m_f > 0$. Let $W_f = \{w \in R^{n+1} \mid \sum w_i = 1\}$, where w_{n+1} is the proportion of wealth invested in the riskless asset. One says that $\overline{w} \in W_f$ belongs to the *mean-ERC boundary* at the confidence level a if and only if, for some $\overline{m}$, $\overline{w}$ solves the problem

$$\min_{w \in W_f} C(a, w) \quad \text{subject to} \quad m_w = \overline{m}\,. \tag{4.1}$$

Clearly Property 3.1 holds in the extended economy. From Merton(1972) or Huang and Litzenberger(1988), Section 3.11, one knows that $w \in W_f$ belongs to the mean-variance boundary if and only if

$$s_w^2 = \frac{1}{H}\left(m_w - m_f\right)^2, \tag{4.2}$$

where $H = Cm_f^2 - 2Am_f + B > 0$, with A, B, C as in (3.4). Under the necessary condition (3.5) a portfolio $w \in W_f$ belongs to the mean-ERC boundary if and only if

$$C(a, w) = a^* \sqrt{\frac{\left(m_w - m_f\right)^2}{H} + s_L^2} - m_w - n_L s_L\,. \tag{4.3}$$

Solving (4.3) for m_w one obtains in the (C, m)-space

$$m_w = m_f + \frac{H\left[C(a,w)+E\right] \pm a^* \sqrt{H}\sqrt{\left[C(a,w)+E\right]^2 - \left(a^{*2} - H\right)s_L^2}}{a^{*2} - H} \tag{4.3'}$$
$$E = m_f + n_L s_L.$$

This looks like a hyperbola and has a more complicated geometric structure than the mean-variance boundary (4.2), which consists of two lines. The notions of mean-ERC efficiency and mean-variance efficiency are similar to those in Definitions 3.1, and Lemma 3.1 holds in the extended economy. The minimum ERC portfolio is determined as follows. For convenience, the vector w is decomposed in the following as $w = \left(w^r, w^f\right)$, where $w^r = \left(w_1, \ldots, w_n\right)$ and $w^f = w_{n+1}$.

<u>Proposition 4.1</u>. The minimum ERC portfolio at the confidence level a exists if and only if $a^* > \sqrt{H}$. If $a^* > \sqrt{H}$ then the minimum ERC portfolio $m_a = \left(m_a^r, m_a^f\right) \in W_f$ is

$$m_a^r = \Sigma^{-1}\left(m - m_f e\right)^T \frac{s_{m_a}}{\sqrt{H}}, \quad m_a^f = 1 - \sum_{i=1}^{n} (m_a)_i\,, \tag{4.4}$$

$$m_{m_a} = m_f + \frac{H}{\sqrt{a^{*2} - H}} \cdot s_L\,, \quad s_{m_a} = \sqrt{\frac{H}{a^{*2} - H}} \cdot s_L \tag{4.5}$$

and the minimum ERC is given by

$$C(a, m_a) = \sqrt{a^{*2} - H} \cdot s_L - m_f - n_L s_L\,. \tag{4.6}$$

Proof. The minimum variance portfolio m_s has mean $m_{m_s} = m_f$ and standard deviation $s_{m_s} = 0$. Using Lemma 3.1, (3.1), and the fact that the mean-variance efficient frontier consists of the half-line $m_w = m_f + \sqrt{H} \cdot s_w$, one has to solve

$$\min_{s \in [0,\infty)} \left\{ a^* \sqrt{s^2 + s_L^2} - \sqrt{H} s - m_f - n_L s_L \right\}. \tag{4.6}$$

A necessary condition for $s = s_{m_a}$ is $\frac{\partial}{\partial s}\{\ \} = a^* \frac{s}{\sqrt{s^2 + s_L^2}} - \sqrt{H} = 0$, which implies

$$\left(\frac{a^{*2} - H}{H} \right) s^2 = s_L^2, \tag{4.7}$$

hence $a^* > \sqrt{H}$. Since $\frac{\partial^2}{\partial s^2}\{\ \} = a^* \frac{s_L^2}{(s^2 + s_L^2)^{\frac{3}{2}}} > 0$, these conditions are sufficient. Using that $m_w = m_f + \sqrt{H} \cdot s_w$ the formulas (4.5) and (4.6) follow immediately. By Huang and Litzenberger(1988), p.76, for any m_w there exists a unique mean-variance boundary portfolio $w \in W_f$ with expected return m_w such that $w^r = \Sigma^{-1}(m - m_f e)^T \frac{m_w - m_f}{H}$, $w^f = 1 - \sum_{i=1}^{n} w_i$. Using that $m_w = m_f + \sqrt{H} \cdot s_w$ and setting $w = m_a$ one gets (4.4). ◊

Corollary 4.1. If it exists, the minimum ERC portfolio at the confidence level $a < 1$ lies above the minimum variance portfolio on the mean-variance efficient frontier.

Proof. This follows from (4.4) noting that $m_{m_a} > m_f = m_{m_s}$. ◊

The characterization of the mean-ERC efficient frontier is similar to Proposition 3.2.

Proposition 4.2. If $a^* > \sqrt{H}$ then a portfolio is mean-ERC efficient at the confidence level a if and only if it belongs to the mean-ERC boundary and it has an expected rate of return greater than or equal to m_{m_a}. If $a^* \leq \sqrt{H}$ then no mean-ERC efficient portfolio exists at the confidence level a.

Proof. To show the first assertion, it suffices to verify that $\frac{\partial^2 C(a, w)}{\partial m_w^2} > 0$ for every $w \in W_f$. The formula (3.15) still holds. If $m_w \geq m_f$ one has $s_w = \frac{m_w - m_f}{\sqrt{H}}$ because w belongs to the mean-variance boundary. Since $\frac{\partial^2 s_w}{\partial m_w^2} = 0$ (3.15) implies $\frac{\partial^2 C}{\partial m_w^2} > 0$. If $m_w < m_f$ one has $s_w = \frac{m_f - m_w}{\sqrt{H}}$ and $\frac{\partial C}{\partial m_w} = -\frac{a^*}{\sqrt{H}} \frac{s_w}{\sqrt{s_w^2 + s_L^2}} - 1 < 0$, and no portfolio on the mean-variance boundary is mean-ERC efficient. If $a^* \le \sqrt{H}$ and $m_w \ge m_f$ one has $s_w = \frac{m_w - m_f}{\sqrt{H}}$ and $\frac{\partial C}{\partial m_w} = \frac{a^*}{\sqrt{H}} \frac{s_w}{\sqrt{s_w^2 + s_L^2}} - 1 < 0$, which implies mean-ERC inefficiency. ◊

<u>**Corollary 4.2**</u>**.** The minimum variance portfolio is mean-ERC inefficient at any confidence level $a < 1$.

<u>**Proof**</u>**.** This follows from Proposition 4.2 noting that $m_{m_a} > m_f = m_{m_s}$. ◊

It is possible to state in the extended economy assertions similar to those made after Corollary 3.2. This task is left to the reader. As a further result, it appears most interesting to characterize the mean-variance efficient frontier, as done in Proposition 3.3.

<u>**Proposition 4.3**</u>**.** A portfolio $w = (w^r, w^f) \ne (0,1)$ belongs to the mean-variance efficient frontier if and only if w is the minimum ERC portfolio at the confidence level a determined by

$$a^{*2} = H \cdot \left[1 + H \left(\frac{s_L}{m_w - m_f} \right)^2 \right]. \tag{4.9}$$

Proof. By Lemma 3.1 the condition is sufficient. Let now $w \ne (0,1)$. Since w is mean-variance efficient one has necessarily $w^r = \Sigma^{-1}(m - m_f e)^T \frac{m_w - m_f}{H}$. By Proposition 4.1, if there exists s_{m_a} satisfying (4.5) such that

$$m_w = m_f + \sqrt{H} \cdot s_{m_a} = m_f + \frac{H s_L}{\sqrt{a^{*2} - H}}, \tag{4.10}$$

then w is the minimum ERC portfolio at the confidence level a. Solving (4.10) for a^* shows (4.9). ◊

It is important to remark that the interpretations given after Proposition 33 also hold in the extended economy with a riskless asset. Let us conclude by pointing out a difference between the obtained mean-ERC efficient sets in the two economies without and with a riskless asset, which is completely similar to the observation made in Alexander and Baptista(2000). One knows that $H \ge \frac{D}{C}$, and that strict inequality holds if and only if $m_f \ne \frac{A}{C}$. Given a confidence level it is easier to obtain an empty mean-ERC efficient set when there is a riskless asset in the economy. If $H > \frac{D}{C}$, then for any confidence level a such that $a^* \in \left(\sqrt{\frac{D}{C}}, \sqrt{H} \right)$, the mean-ERC efficient set is empty in the economy with a riskless asset even though it is not empty in the economy without riskless asset.

5. Example from life insurance.

Let us illustrate our results at a simple asset liability portfolio model from life insurance with the following characteristics. There are three asset categories consisting of a riskless asset with return $m_f = 0.03$, a bond portfolio with return $m_1 = 0.05$ and volatility $s_1 = 0.1$, and an equity portfolio with return $m_2 = 0.1$ and volatility $s_2 = 0.2$. Let $r_{12} = 0.5$ be the correlation between the bond and equity portfolios. The covariance matrix $\Sigma = (r_{ij} s_i s_j)$ and its inverse $\Omega = \Sigma^{-1}$ are given by

$$\Sigma = 10^{-2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad \Omega = \frac{10^2}{3} \cdot \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}. \tag{5.1}$$

Suppose the life insurance business invests the amount $DK_t + p_t = 50$ Mrd., and the net risk premium at the end of the period reads $p_t^R (1 + i_t^L) = 500$ Mio., where $i_t^L = 0.035$ is the technical interest rate. The aggregate claims have a mean $m_{S_{t+1}} = 375$ Mio. and a standard deviation $s_{S_{t+1}} = 235.875$ Mio. . According to (2.21)-(2.23) the liability characteristics are herewith

$$s_L = 0.00472, \quad n_L = 0.52994, \quad r_L = 0.035. \tag{5.2}$$

We distinguish between mean-ERC analysis without and with a riskless asset, and display the fundamental difference between the two approaches for the management of a life insurance business. We calculate ERC according to the value-at-risk method.

5.1. Mean-ERC analysis without riskless asset.

To construct the mean-ERC boundary, one needs the figures (see (3.4) and (3.6')) :

$$A = 5, \quad B = \frac{1}{3}, \quad C = 100, \quad D = \frac{25}{3},$$

$$E = \frac{A}{C} + n_L s_L - r_L = 0.0175 \quad (\textit{shift of } r_L \textit{ in } (3.6')!),$$

$$F = \frac{1}{C} + s_L^2 = 0.01002 .$$

Let $a = 0.99$ be the confidence level such that $a^* = \Phi^{-1}(a) = 2.32635$. A graph of the mean-ERC boundary is displayed in Figure 5.1.

To obtain the minimum ERC portfolio according to Proposition 3.1, one needs the vectors g and h defined in (3.8) and equal to $g = (2,-1), h = (-18,19)$. One obtains $C_{\min}(a, m_a) = 0.21359$ for the minimum ERC portfolio $m_a = (0.92771, 0.07228)$ with corresponding mean $m_{m_a} = 0.05361$ and standard deviation $s_{m_a} = 0.10078$. In contrast to this the minimum variance portfolio is $m_s = (1,0)$ (100% in the risky bond portfolio) with corresponding mean $m_{m_s} = 0.05$, standard deviation $s_{m_s} = 0.1$, and $C(a, m_s) = 0.21539$. For more general applications, we note by passing that the minimum variance portfolio is the bond portfolio $m_s = (1,0)$ if and only if one has $s_1 = r_{12} s_2$. One notes that m_s is mean-

ERC inefficient, and even more all the mean-variance efficient portfolios with return in the interval $[0.05, 0.05361]$ are mean-ERC inefficient at the confidence level $a = 0.99$ (Proposition 3.2 and Corollary 3.2). For comparison the mean-ERC efficient portfolio with return $m_w = 0.0625$ is given by $w = g + hm_w = (0.75, 0.25)$. Its standard deviation is $s_w = 0.10897$ and requires the economic risk capital $C(a, w) = 0.22375$. In the considered economy with only risky bonds and equities, a reasonable value of the economic risk capital at the confidence level $a = 0.99$ lies in the interval $[0.05361, 0.0625]$. Suppose the chief financial officer of the life insurance business chooses the mean-variance efficient portfolio $w = (0.9, 0.1)$ with corresponding $m_w = 0.05$, $s_w = 0.10149$ and $C(a, w) = 0.21385$. According to Proposition 3.3 w belongs to the mean-variance efficient frontier if and only if w is the minimum ERC portfolio at the confidence level $a = 0.955$ with corresponding minimum economic risk capital $C(a, w) = 0.14954$. This financial decision is acceptable at the usual statistical significance level of 5%. However, a capital requirement at the higher confidence level $a = 0.99$ requires additional 0.06431 per unit of capital when compared to the implied confidence level $a = 0.955$ obtained from mean-variance analysis.

<u>Figure 5.1</u> : mean-ERC boundary without riskless asset

m_w

0.06

0.055

m_a

0.05 m_s

0.045

$C_{\min}(a, w)$ $C_{\min}(a, w) + 0.01$ $C(a, w)$

5.2. <u>Mean-ERC analysis with a riskless asset</u>.

Suppose there is a riskless asset in the economy with return $m_f = 0.03$. To draw the mean-ERC boundary (4.3') one needs the figures (see (4.2) and (4.3')) :

$$H = Cm_f^2 - 2Am_f + B = 0.12333,$$
$$E = m_f + n_L s_L - r_L = -0.0025 \quad (\mathit{shift\ of}\ r_L\ \mathit{in}\ (4.3')!).$$

A graph of the mean-ERC boundary at the confidence level $a = 0.99$ is displayed in Figure 5.2. In contrast to Figure 5.1 the mean-ERC efficient curve looks very much like a straight-line, at least for the higher values of ERC. This is reminiscent of the mean-standard deviation efficient line in classical mean-variance analysis.

Figure 5.2 : mean-ERC boundary with a riskless asset

m_w

0.07

0.06

0.05

0.04

0.03 m_a

0.02

0.01

$C(a, m_a)$ $C(a, w)$

According to Proposition 4.1 the minimum ERC portfolio is $m_a = (0.00068, 0.00342, 0.9959)$ with a mean $m_{m_a} = 0.03025$, standard deviation $s_{m_a} = 0.00072$ and minimum ERC $C_{\min}(a, m_a) = 0.01335$. Suppose the chief financial officer chooses the mean-variance efficient portfolio with return $m_w = 0.055$. The weight vector is $w = (w^r, w^f)$ with $w^r = \Omega(m - m_f e)^T \frac{m_w - m_f}{H} = (0.06757, 0.33784)$, $w^f = 0.59459$. Its standard deviation is $s_w = 0.07119$ and the corresponding economic risk capital is $C(a, m_a) = 0.14348$. According to Proposition 4.3 w is the minimum ERC portfolio at the confidence level $a = 0.63757$ with minimum risk capital $C(a, w) = 0.00261$.

5.3. Economy without versus economy with riskless asset.

It is instructive to note the differences in strategic asset allocation between the economies without and with a riskless asset. Choosing mean-variance efficient portfolios with return $m_w = 0.055$, the economy without riskless asset yields a moderate share in equities of 10%, a high share in bonds of 90%, and a minimum ERC of 14.95% at the relatively high confidence level 95.5%. At the same return, the economy with a riskless asset yields a significant share in equities of 33.78%, a small share in bonds of 6.76% and a considerable share in the riskless asset of 59.46%. The corresponding minimum ERC of 0.00261 is attained at the low confidence level 63.8%, which points out the high riskiness of this asset allocation from the minimum economic risk capital viewpoint. For comparison, the ERC at the confidence level 95.5% is 9.83%. It is not clear, which asset allocation should be preferred. For a *fixed confidence level*, say $a = 0.955$, the asset allocation with the riskless asset yields a smaller economic risk capital and is therefore preferred. However, if the goal is a *minimum economic risk capital*, the strategy with the riskless asset yields the insufficient confidence level 63.8% and the strategy without riskless asset at the acceptable confidence level 95.5% is preferred. Further theoretical and empirical research is required to clarify these conflicting viewpoints.

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