Hedging Strategies and Insurance Securitization

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Abstract

The application of financial theory to insurance pricing seems to be extremely popular in the recent actuarial literature. This is undoubtedly due to the increasing interaction between financial and insurance markets, caused by the emergence of products such as segregated funds, insurance securities, catastrophe risk bonds, etc., which in turn cause the proliferation of new markets and products, in accordance to Merton's financial innovation spiral. This paper reviews some recent advances in the construction of hedging strategies in incomplete markets, essential to price insurance contracts in a financial framework and to provide alternatives to equity capital for managing risks.

1 Introduction

The insurance market has undoubtedly benefited from the developments of modern finance, not only by using financial derivatives to hedge insurers' financial risks over their assets, but also by providing actuaries with new tools for product design. Equity-linked insurance contracts, segregated funds, catastrophe bonds, some defined-benefit pension schemes and the securitization of insurance contracts are just examples of this market evolution, which demands for adequate pricing, hedging and reserving methodologies. From the financial point of view, perfect replication of such products is in general not possible, and therefore we are in the context of incomplete markets. In this framework, the valuation of new assets can be done in two ways: pricing according to the general equilibrium theory for incomplete markets or pricing via no arbitrage arguments. The first typically requires very detailed assumptions to guarantee appropriate results; the second, although less demanding in terms of assumptions, requires the selection of an adequate equivalent martingale measure to value the contingent claims by taking expectations. Since only the maximum price over all equivalent martingale measures can ensure a successful hedge with probability one, appropriate measures of the risk associated to a particular price become necessary. Quantile hedging, a dynamic version of VaR, is a technique of construction of hedging strategies that allows, among other things, to maximize the probability of success, or alternatively, to minimize the cost given a probability of success. This paper briefly reviews the problem of pricing non-redundant securities, gives an introduction to quantile hedging and some comments related to securitization of insurance risk, providing some background for the presentation to follow.

2 Arbitrage Pricing

We start this presentation by assuming that the evolution of the discounted price of some underlying asset over a time interval $[0, T]$ can be represented by an adapted stochastic process $S = \{S_t\}_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. A contingent claim is a contract whose payoff X at time T is an $\mathcal{F}_T$-measurable random variable and the determination of a fair price for such contingent claim is one of the main topics of modern finance. One possible approach, formally introduced by Harrison and Kreps (1979), assumes the absence of arbitrage opportunities to guarantee the existence of a measure Q, equivalent to P, under which S becomes a martingale, implying that claims can be priced by taking expectations under this measure. For a more precise statement, valid under very general circumstances, see Delbaen and Schachermayer (1994). Unfortunately, with rare exceptions, the set $\mathcal{Q}$ of equivalent martingale measures is infinite and, in consequence, the prices consistent with the no arbitrage assumption generate an interval of the form

$$\left[\inf_{Q\in\mathcal{Q}} E_Q[X], \quad \sup_{Q\in\mathcal{Q}} E_Q[X]\right] \tag{1}$$

For any price outside this interval, we can construct a portfolio involving the option and the risky asset that gives a riskless profit. But unfortunately there is no obvious way to choose an appropriate $Q \in \mathcal{Q}$; moreover, the interval is in general much wider than the corresponding to the bid-ask spread and for most models is just the trivial one (where the minimum price is the discounted payoff given that S_T increased at the risk-free rate and the maximum is simply S_0). See Eberlein and Jacod (1997) for details. The literature has therefore focused on defining additional criteria to choose an equivalent martingale measure. Some examples are given below.

First notice that pricing the contingent claim as $\sup_{Q \in \mathcal{Q}} E_Q[X]$ guarantees a hedging portfolio with value at time T of at least X, although at a great cost. This approach is therefore called superhedge. See El Karoui and Quenez (1995) and Karatzas and Shreve (1998), chapter 5 for details. This method is a first step in the construction of quantile hedging, which will be discussed in the next section.

Davis (1997) in Dempster and Pliska (eds.) shows that a risk averse agent will select the equivalent martingale measure that maximizes his/her expected utility at maturity. In insurance securitization, similar equilibrium arguments are presented for example in Meister (1995). Föllmer and Sondermann (1986) propose the use of strategies which minimize the variance of the hedge error, prove that the solution is unique and show that this strategy is mean-self-financing. Föllmer and Schweizer (1990) study a more general case and obtain a minimal (relative entropy) martingale measure.

A different approach consists in reducing the set $\mathcal{Q}$. This is done for example in Bizid, Jouini and Koehl (1999) and Jouini and Napp (1999) using market completion and a representative agent framework. In the context of reinsurance, Delbaen and Haezendonck (1989) restrict the set $\mathcal{Q}$ in order to preserve the compound Poisson property under Q. This method is generalized by Meister (1995) and applied to CAT-futures. Also relevant to our discussion is the popular Esscher pricing method, introduced by Gerber and Shiu (1994) and generalized by Bühlmann et al. (1996), this method has been applied to both financial and insurance problems in the literature, reducing cumbersome formulae to elegant and simpler expressions. The excellent survey by Embrechts (1996) shows evidence that this actuarial/financial technique has much more to offer in the near future.

We already mentioned that there is no obvious criteria to choose a particular Q. Thus, there is a need for an indicator of the risk inherent to a particular pricing rule. Quantile hedging can be considered as a method to obtain the probability of a successful hedge given the price paid for the contingent claim, and such probability is, at least a first approximation to a suitable measure of risk in a dynamic setting.

The following section discusses quantile hedging based on Föllmer and Leukert (1999), hereafter [FL].

3 Quantile hedging

As it is standard in mathematical finance, a hedging portfolio is defined by an initial amount V_0 and a strategy ξ (a predictable process), such that its value at time t is given by

$$\begin{aligned} V_t &= V_0 + \int_0^t \xi_u dS_u \quad \forall t \in [0,T], \quad P-\text{a.s.} \\ V_t &\geq 0 \quad \forall t \in [0,T], \quad P-\text{a.s.} \end{aligned} \tag{2}$$

The portfolio in (2) is an admissible, self-financing strategy, since there are no inflows nor outflows and is always positive. If such portfolio is used to replicate the contingent claim X, then V_0 becomes the initial capital that the investor is willing to allocate to hedge this product. In a complete market perfect replication is possible, so the fair value of the contingent claim (at least from the point of view of the writer) is the smallest V_0 such that $V_T = X$,

$P-$a.s. In the incomplete case we are interested in obtaining the (maximal) probability of a successful hedge (the event $V_T \geq X$) , given a choice of Q (and thus of V_0). This optimization is analogous to the problem of statistical tests of hypotheses, thus, it is natural to define a critical function ("success ratio") φ. This function takes values in $[0,1]$ and depends on the choice of portfolio, as defined by (2). The optimal strategy will correspond to the superhedge associated to a modified claim of the form $X\varphi$. Let $\mathcal{R}$ denote the class of all functions φ, and fix $\widetilde{V}_0$ below the cost of superhedge. Then, according to [FL, theorem 4.9], there exists a function $\widetilde{\varphi} \in \mathcal{R}$ such that

$$E_P\left[\widetilde{\varphi}\right] = \max_{\varphi \in \mathcal{R}} E_P\left[\varphi\right] \tag{3}$$

under the constraints

$$E_Q\left[X\varphi\right] \leq \widetilde{V}_0 \quad \forall Q \in \mathcal{Q}. \tag{4}$$

That is, we are interested in strategies that maximize (under P measure) the probability of a successful hedge, keeping the initial cost (calculated under Q) below a certain level. It is (almost) directly to show that this is the case of a statistical test of a compound null hypothesis against a simple alternative, therefore, the solution can be interpreted as a uniformly most powerful size $\frac{\widetilde{V}_0}{E_Q[X]}$ test. See [FL, (4.16)-(4.17)].

Given the modified claim $\widetilde{X} = X\widetilde{\varphi}$, the optimal strategy $\widetilde{\xi}$ can be obtained from the Itô decomposition of $\widetilde{X}$, if $\widetilde{X}$ is attainable, or by the optional decomposition of $\widetilde{X}$, if $\widetilde{X}$ is not attainable. The optimal form of $\widetilde{\varphi}$ is given in [FL, section 4.2]: There exists a measure $\widetilde{Q} \in \mathcal{Q}$ such that

$$\widetilde{\varphi} = \begin{cases} 1 & \text{if } \frac{dP}{d\widetilde{Q}} > \lambda X \\ 0 & \text{if } \frac{dP}{d\widetilde{Q}} < \lambda X \end{cases} \tag{5}$$

where λ is a constant and

$$E_{\widetilde{Q}}\left[X\widetilde{\varphi}\right] = \widetilde{V}_0. \tag{6}$$

Therefore, the modified claim can be generated by a trading strategy based on the optional decomposition theorem (see Kramkov (1996)): Let $\widehat{X}$ denote a right continuous version of the process

$$\widehat{X}_t = ess. \sup_{Q \in \mathcal{Q}} E_Q\left[X \left| \mathcal{F}_t \right.\right], \tag{7}$$

then, the strategy $\widetilde{\xi}$ satisfies a decomposition

$$\widehat{X}_t = \widehat{X}_0 + \int_0^t \widetilde{\xi}_u dS_u - O_t, \tag{8}$$

where the process O_t is an optional, nondecreasing process. Notice that the process $\widehat{X}$ is a supermartingale with respect to all $\widetilde{Q} \in \mathcal{Q}$.

Examples of the applications of the quantile hedging approach in a financial context are: Sekine (1999), who analyses the robustness of the quantile hedging problems in the Black Scholes model, with unknown constant drift of the risky asset. Lotz (1999) discusses the applicability of quantile hedging in the context of credit risk models both in the complete and the incomplete cases. Krutchenko and Melnikov (2000) derive optimal strategies associated to a jump-diffusion process (restricted to a complete market). Further developments are given in Föllmer and Leukert (2000).

4 Some comments regarding insurance securitization

The economics of insurance markets have always occupied a particular place in economic theory, due to theoretical and technical difficulties in modelling economic agent's behavior under uncertainty. In finance, market imperfections do not have the same impact than in the insurance market and thus, arbitrage pricing theory provides results that are a reasonable approximation to real financial markets. (Re-)insurance markets do not observe the basic assumptions of contingent claim valuation, since, as Albrecht (1992) reminds us, "Insurance companies exhibit economies of scale with respect to risk taking", and thus, contingent claims do not have unique prices. See also Cummins (1990) p. 142.

Even if we assume completeness in the insurance market, there is a conceptual problem when comparing an actuarial premium

$$E_P[X] + \theta \tag{9}$$

(with θ chosen according to a premium calculation principle) against the no arbitrage price

$$E_Q[X]. \tag{10}$$

The actuarial premium is simply the price for the risk transfer and thus, the insurer faces the random loss

$$X - E_P[X] - \theta \tag{11}$$

whereas the no arbitrage price (10) is truly the *deterministic equivalent to* *the contingent claim* X: the amount that can be invested in an admissible, self-financing strategy, providing a *riskless* replication of X:

$$E_Q\left[X\left|\mathcal{F}_t\right.\right] = E_Q\left[X\right] + \int_0^t \xi_u dS_u, \quad \forall t \in [0,T] \qquad P - \text{a.s.} \tag{12}$$

And this explanation also clarifies why under the no arbitrage assumption the prices satisfy strict additivity while the insurance premiums do not.

These basic differences between the insurance and the financial markets have been amply discussed in the literature. For example, Sondermann (1988) analyses the problem of risk neutral valuation in "dynamic reinsurance policies", which require certain liquidity conditions of a reinsurance market. Venter (1991) analyses the implications of no arbitrage in the context of premium calculation principles. More detailed conditions are given by Delbaen and Haezendonck (1989), who consider a liquid insurance market where contracts can be traded very frequently and with perfect divisibility.

Although the insurance market does not exhibit these properties, more appropriate for financial models in continuous time, this pioneering work opened the door to appropriate methods in the field of insurance securitization, as illustrated by the SoA monograph of the 1995 Bowles symposium. Developments that are particularly related to our discussion are Embrechts and Meister (1995) and Meister (1995) who extend some results from Delbaen and Haezendonck (1989) to other models relevant to actuarial science, such as the mixed Poisson and doubly stochastic Poisson process. Christensen (2000) proposes a model for the underlying loss index which recognizes the heavy tailed behavior of the claims and obtains prices based on the Esscher method.

Some specific problems in insurance securitization will be discussed in the conference, as an illustration of the applicability of quantile hedging.

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