

Pricing Formulae for Barrier Caps

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Abstract: in this paper, we present pricing formulae for barrier caps. To lower the cost of vanilla caps we fix a barrier in such a way that when hit, the protection is totally or partially suppressed. Quasi explicit formulae are given. The term structure of interest rates is given by a standard one factor model. A numerical analysis is performed which illustrates the suggested approach.

Résumé : dans cet article nous proposons des formules d'évaluation pour des caps à barrière. Dès que le taux de référence franchit une barrière la protection du cap est partiellement ou totalement supprimée. Des formules quasi explicites sont données et une analyse numérique illustre la méthode.

1 Introduction

Caps and floors are versatile tools to manage interest rate risk. However they are well known to be expensive. To lower their costs a first and immediate solution is to trade collars obtained by buying a cap and selling a floor or vice-versa. Here we consider another approach : the possibility to use a barrier, a solution which proves to be very successful in stocks and foreign exchange markets. In this context we define and study three kinds of exotic caps we call point barrier cap, continuous barrier cap and restricted continuous barrier cap. Using a Heath Jarrow Morton framework we obtain quasi-explicit formulae to price them. For the present study the underlying price of our options is a one dimensionnal Markov process which legitimates the use of Fortet's equation and paves the way to a fast and accurate approximations of the equilibrium prices. The paper is organized as follows : section 2 gives the setting of the economic environment considered especially the term structure of interest rates, section 3 is devoted to the pricing methodology and section 4 presents a numerical analysis. Technical proofs can be found in a appendix.

2 Setting and yield curve model

The market is complete and frictionless. The uncertainty is modelled by a filtered space $(\Omega, \{F_t\}_{0 \leq t \leq T}, \Pi)$ where Ω is the usual fundamental space, Π is the historical probability measure and F_t is the σ -field collecting all the information up to time t , generated by a brownian motion. Trading takes place continuously and the bond's price follows a diffusion process :

$$\frac{dP(s, t)}{P(s, t)} = \alpha(s, t)ds - \sigma_P(s, t)dZ(s)$$

where $\sigma_P(., t)$ is a deterministic function and $Z(.)$ a standard Brownian motion.

$P(s, t)$ denotes the price at time s of the riskfree zero-coupon bond maturing at t .

Its yield to maturity is defined by :

$$Y(s, t) = \frac{-1}{t-s} \ln(P(s, t))$$

We recall that the instantaneous riskfree interest rate $r(\cdot)$ verifies at all times:

$$r(s) = \lim_{t \searrow s} Y(s, t)$$

For a fixed s and variable t , $Y(s, t)$ defines the yield curve at time s .

$$\rho(s, t) = \frac{1}{t-s} (\exp((t-s)Y(s, t)) - 1)$$

is the proportional rate prevailing at s for lending or borrowing up to t .

The forward rate prevailing at s for the future period $[t_1, t_2]$, $t_2 \geq t_1 \geq s$, is defined by :

$$\begin{aligned} \phi(s; t_1, t_2) &= \frac{(t_2 - s)Y(s, t_2) - (t_1 - s)Y(s, t_1)}{t_2 - t_1} \\ &= \frac{1}{t_2 - t_1} \ln \left(\frac{P(s, t_1)}{P(s, t_2)} \right) \end{aligned}$$

We write the zero-coupon bond in the s -forward-neutral universe defined by its probability measure Q_s :

$$P(s, t) = \frac{P(0, t)}{P(0, s)} \exp \left(\begin{array}{l} - \int_0^s (\sigma_P(u, t) - \sigma_P(u, s)) dZ_s(u) \\ - \frac{1}{2} \int_0^s (\sigma_P(u, t) - \sigma_P(u, s))^2 du \end{array} \right)$$

where $Z_s(\cdot)$ is a Q_s -Brownian motion. So :

$$\begin{aligned} Y(s, t) &= \phi(0; s, t) + \frac{1}{t-s} \int_0^s (\sigma_P(u, t) - \sigma_P(u, s)) dZ_s(u) \\ &\quad + \frac{1}{2(t-s)} \int_0^s (\sigma_P(u, t) - \sigma_P(u, s))^2 du \end{aligned}$$

or, setting $\theta = t - s$:

$$\begin{aligned} Y(s, s + \theta) &= \phi(0; s, s + \theta) + \frac{1}{\theta} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s)) dZ_s(u) \\ &\quad + \frac{1}{2\theta} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s))^2 du \end{aligned}$$

Usually, there is a lag, θ , corresponding to the duration of the reference floating rate, between the payoff setting at time t , and the payment date

$t + \theta$. The martingale pricing method requires the computation of the expected discounted payoff, known in t under the $t + \theta$ - forward probability measure associated with the payment date. Suppose we have to pay in $t + \theta$ a deterministic function of $(Y(s, s + \theta))_{0 \leq s \leq t}$. For this case it is useful to define the $Q_{t+\theta}$ - Brownian motion $Z_{t+\theta}(\cdot)$ through:

$$dZ_{t+\theta}(u) = dZ_s(u) + (\sigma_P(u, t + \theta) - \sigma_P(u, s)) du$$

then we obtain :

$$\begin{aligned} Y(s, s + \theta) &= \phi(0; s, s + \theta) \\ &+ \frac{1}{\theta} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s)) dZ_{t+\theta}(u) \\ &+ \frac{1}{\theta} \int_0^s \begin{pmatrix} (\sigma_P(u, s + \theta) - \sigma_P(u, s)) \\ (\sigma_P(u, s) - \sigma_P(u, t + \theta)) \end{pmatrix} du \\ &+ \frac{1}{2\theta} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s))^2 du \end{aligned}$$

In this form, $Y(s, s + \theta)$ is expressed in the $t + \theta$ -forward-neutral universe and this gives a useful key to value deferred options on interest rates.

$(Y(s, s + \theta))_{0 \leq s \leq t}$ is Gaussian with mean $M^Y(s)$, variance, $V^Y(s)$, and auto-covariance $C^Y(s, v)$. In this setting :

$$\begin{aligned} M^Y(s) &= \phi(0; s, s + \theta) + \frac{1}{\theta} \int_0^s \begin{pmatrix} (\sigma_P(u, s + \theta) - \sigma_P(u, s)) \\ (\sigma_P(u, s) - \sigma_P(u, t + \theta)) \end{pmatrix} du \\ &+ \frac{1}{2\theta} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s))^2 du \end{aligned}$$

$$V^Y(s) = \frac{1}{\theta^2} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s))^2 du$$

$$C^Y(s, v) = \frac{1}{\theta^2} \int_0^s (\sigma_P(u, s + \theta) - \sigma_P(u, s)) (\sigma_P(u, v + \theta) - \sigma_P(u, v)) du$$

These moments can be given explicitly in two familiar cases, linear and exponential, which respectively correspond to the Ho and Lee [1986] model and to the Vasicek [1977] model . We give in appendix 2 the required formulae.

$(Y(s, s + \theta))_{0 \leq s \leq t}$ being a gaussian process, $Y(v, v + \theta)$ given $(Y(s, s + \theta) = a)$, is normal with mean:

$$\widehat{M}^Y(s, v) = M^Y(v) + \frac{C^Y(s, v)}{V^Y(s)} (a - M^Y(s))$$

and variance:

$$\widehat{V}^Y(s, v) = V^Y(v) - \frac{(C^Y(s, v))^2}{V^Y(s)}$$

3 Pricing methodology

Caps are financial contracts designed to provide insurance against the rise of an underlying floating-rate note above a predetermined level during a chosen period $[T_0, T]$, T_0 and T being respectively the time when insurance takes place and when it stops. A cap can be seen as a sum of caplets. Indeed, the interval $[T_0, T]$ is subdivided into m subintervals $[t_i, t_{i+1}]$, $0 \leq i \leq m$, of length $\theta = \frac{T-T_0}{m}$, in such a way that $t_0 = T_0$, $t_m = T$. Each time t_i , $1 \leq i \leq m-1$, is a reset date and a payment date, and $t_i + \theta = t_{i+1}$. A caplet is a contract which pays at t_{i+1} the amount $M\theta(\rho(t_i, t_i + \theta) - t_g)_+$, where M is a notional normalized to one in the subsequent developments. $\rho(t_i, t_i + \theta)$ is a θ proportional rate. This rate is known at t_i , for this reason the caplet is said to be settled in arrears. t_g is a strike rate or a guaranteed rate which is predetermined. A cap on $[T_0, T]$ is the sum of these caplets. In a gaussian HJM setting, pricing formulae are well known and not reproduced here. El Karoui and alii [1992], Brace and Musiela [1994] gave valuations of standard caps in this setting. We introduce here modified caps which are path dependent: they are linked to a barrier which can activate or suppress the insurance leading to knocked in or knocked out caps.

3.1 Point barrier cap

Caps are known to be expensive. To reduce the cost of an ordinary cap, we modify the payoff in such a way that the insurance against a rise in the floating rate is deactivated when the rate at t_i is greater than the level α , so the owner's caplet will receive at t_{i+1}

$$\theta(\rho(t_i, t_{i+1}) - t_g)_+ I_{(\rho(t_i, t_{i+1}) < \alpha)}$$

By doing so, one can take advantage of the floating rate decrease while being protected against the rise of it, if the floating rate at time t_i is strictly inferior to the predefined barrier α . This caplet can be called a knock out point barrier caplet. Let us denote $Q_{t_{i+1}}$ the t_{i+1} -forward neutral probability. The price at time 0 is :

$$P(0, t_{i+1}) \theta E_{Q_{t_{i+1}}} \left((\rho(t_i, t_{i+1}) - t_g)_+ I_{(\rho(t_i, t_{i+1}) < \alpha)} \right)$$

or using the yield:

$$P(0, t_{i+1}) \theta \left(\begin{array}{l} \left(\frac{1}{\theta} + t_g \right) \left(\begin{array}{l} E_{Q_{t_{i+1}}} \left(I_{(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta\alpha)}{\theta})} \right) \\ - E_{Q_{t_{i+1}}} \left(I_{(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta})} \right) \end{array} \right) \\ + \frac{1}{\theta} \left(\begin{array}{l} E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta})} \right) \\ - E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta\alpha)}{\theta})} \right) \end{array} \right) \end{array} \right)$$

In the t_{i+1} -forward-neutral universe, $Y(t_i, t_i + \theta)$ can be expressed through the $Q_{t_{i+1}}$ -Brownian motion $Z_{t_{i+1}}$:

$$\begin{aligned} Y(t_i, t_i + \theta) &= \phi(0; t_i, t_i + \theta) + \frac{1}{\theta} \int_0^{t_i} (\sigma_P(u, t_i + \theta) - \sigma_P(u, t_i)) dZ_{t_{i+1}}(u) \\ &\quad - \frac{1}{2\theta} \int_0^{t_i} (\sigma_P(u, t_i + \theta) - \sigma_P(u, t_i))^2 du \end{aligned}$$

Let us denote $N(\cdot)$ the standard normal cumulative distribution function. We have then the following proposition.

The point barrier knock out caplet's price is :

$$kocaplet(0, t_i, t_g, \alpha) = P(0, t_{i+1}) ((1 + \theta t_g) (E_1 - E_2) + (E_3 - E_4))$$

where:

$$\begin{aligned} E_1 &= N \left(\frac{M^Y(t_i) - \frac{\ln(1+\theta\alpha)}{\theta}}{\sqrt{V^Y(t_i)}} \right) \\ E_2 &= N \left(\frac{M^Y(t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{V^Y(t_i)}} \right) \end{aligned}$$

$$\begin{aligned}
E_3 &= \exp\left(\frac{\theta^2 V^Y(t_i)}{2} + \theta M^Y(t_i)\right) N\left(\frac{\theta V^Y(t_i) + M^Y(t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{V^Y(t_i)}}\right) \\
E_4 &= \exp\left(\frac{\theta^2 V^Y(t_i)}{2} + \theta M^Y(t_i)\right) N\left(\frac{\theta V^Y(t_i) + M^Y(t_i) - \frac{\ln(1+\theta\alpha)}{\theta}}{\sqrt{V^Y(t_i)}}\right)
\end{aligned}$$

The new cap is obtained by adding these caplets. Being cheaper than a standard cap, this knock out cap does not offer a perfect protection and if the floating rate moves beyond α no insurance is provided any longer. To avoid this situation we can consider a new modified cap whose caplets have payoffs at t_{i+1} :

$$\theta(\rho(t_i, t_{i+1}) - t_g)_+ I_{(\rho(t_i, t_{i+1}) < \alpha)} + \theta(\rho(t_i, t_{i+1}) - k) I_{(\rho(t_i, t_{i+1}) \geq \alpha)}$$

The second term can be viewed as a new caplet which is a knock in caplet. It is activated when the rate is above or equal to the barrier level. Its valuation is easy and gives the price:

$$kicaplet(0, t_i, k, \alpha) = P(0, t_{i+1}) (E_1 - (1 + \theta k)E_2)$$

where :

$$\begin{aligned}
E_1 &= \exp\left(\frac{\theta^2 V^Y(t_i)}{2} + \theta M^Y(t_i)\right) N\left(\frac{\theta V^Y(t_i) + M^Y(t_i) - \frac{\ln(1+\theta\alpha)}{\theta}}{\sqrt{V^Y(t_i)}}\right) \\
E_2 &= N\left(\frac{M^Y(t_i) - \frac{\ln(1+\theta\alpha)}{\theta}}{\sqrt{V^Y(t_i)}}\right)
\end{aligned}$$

The price of this modified cap is therefore:

$$Cap(0, t, t_g, k, \alpha) = \sum_{i=1}^{m-1} kicaplet(0, t_i, t_g, \alpha) + kicaplet(0, t_i, k, \alpha)$$

This cap gives a simple example of an exotic cap. We suggest in the following subsections other exotic caps where the barrier is continuous.

3.2 Continuous barrier cap

Another way, for the borrower, to cap this reference floating rate is to subscribe at time 0 to the following more restrictive contract whose payoff at expiry time t_{i+1} is :

$$\theta (\rho(t_i, t_{i+1}) - t_g)_+ I_{(0 \leq t \leq t_i, \max \rho(t, t+\theta) < \alpha)}$$

By doing so, he can take advantage of the floating rate decrease while being protected against the rise of it as long as the barrier α is not breached during the time interval $[0, t_i]$. The effect of the barrier is to lower the price and thus to make this product more attractive.

The price at time 0 is :

$$P(0, t_{i+1}) \theta E_{Q_{t_{i+1}}} \left((\rho(t_i, t_{i+1}) - t_g)_+ I_{(0 \leq t \leq t_i, \max \rho(t, t+\theta) < \alpha)} \right)$$

Let γ be the first hitting time of $(\rho(t, t+\theta))_{t \geq 0}$ to α . This first passage time is the same as the one of the $(Y(t, t+\theta))_{t \geq 0}$ process to $\frac{\ln(\alpha\theta+1)}{\theta}$. The price then becomes :

$$\begin{aligned} & P(0, t_{i+1}) \theta E_{Q_{t_{i+1}}} \left((\rho(t_i, t_{i+1}) - t_g)_+ I_{(\gamma > t_i)} \right) \\ = & P(0, t_{i+1}) \theta \left(E_{Q_{t_{i+1}}} \left(\begin{array}{c} \rho(t_i, t_{i+1}) I_{(\rho(t_i, t_{i+1}) > t_g)} I_{(\gamma > t_i)} \\ -t_g I_{(\rho(t_i, t_{i+1}) > t_g)} I_{(\gamma > t_i)} \end{array} \right) \right) \\ = & P(0, t_{i+1}) \theta \left(\begin{array}{c} -t_g E_{Q_{t_{i+1}}} \left(I_{(\rho(t_i, t_{i+1}) > t_g)} \right) \\ + E_{Q_{t_{i+1}}} \left(\rho(t_i, t_{i+1}) I_{(\rho(t_i, t_{i+1}) > t_g)} \right) \\ + t_g E_{Q_{t_{i+1}}} \left(I_{(\rho(t_i, t_{i+1}) > t_g)} I_{(\gamma \leq t_i)} \right) \\ - E_{Q_{t_{i+1}}} \left(\rho(t_i, t_{i+1}) I_{(\rho(t_i, t_{i+1}) > t_g)} I_{(\gamma \leq t_i)} \right) \end{array} \right) \end{aligned}$$

or using the yield:

$$P(0, t_{i+1}) \theta \begin{pmatrix} -\left(\frac{1}{\theta} + t_g\right) E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} \right) \\ \frac{1}{\theta} E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} \right) \\ + \left(\frac{1}{\theta} + t_g\right) E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} I_{(\gamma \leq t_i)} \right) \\ -\frac{1}{\theta} E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} I_{(\gamma \leq t_i)} \right) \end{pmatrix}$$

Only the two last expectations need a particular treatment. We compute them with a discrete approximation of the probability density function of γ obtained via an implicit equation used by Fortet [1943] (See Appendix 1). For simplicity, we take $T_0 = 0$. The time interval $[0, T]$ is divided into m subintervals $[0, t_{i+1}]$, and for each of these subintervals the time step is equal to $\frac{\theta}{k}$, so the procedure to obtain the γ density is done on $[0, t_m]$ but only use on the interval $[0, t_i]$ for any caplet on $[t_i, t_{i+1}]$. We have then the following proposition.

The continuous barrier caplet's price is :

$$P(0, t_{i+1}) ((1 + \theta t_g) (E_3 - E_1) + E_2 - E_4)$$

where:

$$\begin{aligned} E_1 &= N \left(\frac{M^Y(t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{V^Y(t_i)}} \right) \\ E_2 &= \exp \left(\frac{\theta^2 V^Y(t_i)}{2} + \theta M^Y(t_i) \right) N \left(\frac{\theta V^Y(t_i) + M^Y(t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{V^Y(t_i)}} \right) \\ E_3 &= \frac{\theta}{k} \sum_{j=1}^{i*k-1} N \left(\frac{\widehat{M}^Y \left(j \frac{\theta}{k}, t_i \right) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y \left(j \frac{\theta}{k}, t_i \right)}} \right) p_j \end{aligned}$$

$$E_4 = \frac{\theta}{k} \sum_{j=1}^{i*k-1} \exp \left(\frac{\theta^2 \widehat{V}^Y \left(j \frac{\theta}{k}, t_i \right)}{2} + \theta \widehat{M}^Y \left(j \frac{\theta}{k}, t_i \right) \right) \\ N \left(\frac{\theta \widehat{V}^Y \left(j \frac{\theta}{k}, t_i \right) + \widehat{M}^Y \left(j \frac{\theta}{k}, t_i \right) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y \left(j \frac{\theta}{k}, t_i \right)}} \right) p_j$$

We only give the proof for the two last expectations.

$$E_3 = E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) > \frac{\log(1+\theta t_g)}{\theta} \right)} I_{(\gamma \leq t_i)} \right) \\ = \int_0^{t_i} E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) > \frac{\log(1+\theta t_g)}{\theta} \right)} \mid (\gamma = \tau) \right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau$$

The event $(\gamma = \tau)$ can be written as :

$$(\gamma = \tau) = \left(Y(\tau, \tau + \theta) = \frac{\log(1 + \theta \alpha)}{\theta} \right) \cap \left(Y(s, s + \theta) < \frac{\log(1 + \theta \alpha)}{\theta}, \forall s < \tau \right)$$

Since $(Y(t, t + \theta))_{t \geq 0}$ is a markovian process as the solution of a stochastic differential equation :

$$E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) > \frac{\log(1+\theta t_g)}{\theta} \right)} \mid (\gamma = \tau) \right) \\ = E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) > \frac{\log(1+\theta t_g)}{\theta} \right)} \mid \left(Y(\tau, \tau + \theta) = \frac{\log(1 + \theta \alpha)}{\theta} \right) \right) \\ = E_{Q_{t_{i+1}}} \left(I_{\left(\frac{Y(t_i, t_{i+1}) - \widehat{M}^Y(\tau, t_i)}{\sqrt{\widehat{V}^Y(\tau, t_i)}} > \frac{\frac{\log(1+\theta t_g)}{\theta} - \widehat{M}^Y(\tau, t_i)}{\sqrt{\widehat{V}^Y(\tau, t_i)}} \right)} \mid \left(Y(\tau, \tau + \theta) = \frac{\log(1 + \theta \alpha)}{\theta} \right) \right) \\ = 1 - N \left(\frac{\frac{\log(1+\theta t_g)}{\theta} - \widehat{M}^Y(\tau, t_i)}{\sqrt{\widehat{V}^Y(\tau, t_i)}} \right) \\ = N \left(\frac{\widehat{M}^Y(\tau, t_i) - \frac{\log(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(\tau, t_i)}} \right)$$

We finally obtain :

$$\begin{aligned} E_3 &= \int_0^{t_i} N \left(\frac{\widehat{M}^Y(\tau, t_i) - \frac{\log(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(\tau, t_i)}} \right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \\ &\simeq \frac{\theta}{k} \sum_{j=1}^{i*k-1} N \left(\frac{\widehat{M}^Y(j\frac{\theta}{k}, t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(j\frac{\theta}{k}, t_i)}} \right) p_j \end{aligned}$$

Similarly, we have :

$$\begin{aligned} E_4 &= E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} I_{(\gamma \leq t_i)} \right) \\ &= \int_0^{t_i} E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} \mid (\gamma = \tau) \right) \\ &\quad Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \end{aligned}$$

but :

$$\begin{aligned} &E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) > \frac{\ln(1+\theta t_g)}{\theta}\right)} \mid (\gamma = \tau) \right) \\ &= \exp \left(\frac{\theta^2 \widehat{V}^Y(\tau, t_i)}{2} + \theta \widehat{M}^Y(\tau, t_i) \right) N \left(\frac{\theta \widehat{V}^Y(\tau, t_i) + \widehat{M}^Y(\tau, t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(\tau, t_i)}} \right) \end{aligned}$$

that leads to :

$$\begin{aligned} E_4 &\simeq \frac{\theta}{k} \sum_{j=1}^{i*k-1} \exp \left(\frac{\theta^2 \widehat{V}^Y(j\frac{\theta}{k}, t_i)}{2} + \theta \widehat{M}^Y(j\frac{\theta}{k}, t_i) \right) \\ &\quad N \left(\frac{\theta \widehat{V}^Y(j\frac{\theta}{k}, t_i) + \widehat{M}^Y(j\frac{\theta}{k}, t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(j\frac{\theta}{k}, t_i)}} \right) p_j \end{aligned}$$

3.3 Restricted continuous barrier cap

We can also define the following contract halfway between the first and the second product presented above. Its payoff at expiry time t_{i+1} is :

$$\theta (\rho(t_i, t_{i+1}) - t_g)_+ I_{(t_{i-1} \leq t \leq t_i, \max_{\rho(t, t+\theta) < \alpha})}$$

By subscribing this contract, one can take advantage of the floating rate decrease while being protected against the rise of it as long as the barrier α is not breached during the time interval $[t_{i-1}, t_i]$. The barrier lowers the price to a lesser degree than when the barrier is activated during $[0, t_i]$, but on the other hand the probability for the option to be extinguished is lower. Indeed, when the barrier is breached during $[t_{i-1}, t_i]$, only the caplet on $[t_i, t_{i+1}]$ vanishes.

The price at time 0 is :

$$\begin{aligned} & P(0, t_{i+1}) \theta E_{Q_{t_{i+1}}} \left((\rho(t_i, t_{i+1}) - t_g)_+ I_{(t_{i-1} \leq t \leq t_i, \max_{\rho(t, t+\theta)} < \alpha)} \right) \\ = & P(0, t_{i+1}) \theta E_{Q_{t_{i+1}}} \left(\begin{array}{c} (\rho(t_i, t_{i+1}) - t_g)_+ \\ - (\rho(t_i, t_{i+1}) - t_g)_+ I_{(t_{i-1} \leq t \leq t_i, \max_{\rho(t, t+\theta)} \geq \alpha)} \end{array} \right) \end{aligned}$$

Let us define γ as :

$$\gamma \triangleq \inf_{s \geq t_{i-1}} \{s : \rho(s, s + \theta) \geq \alpha\}$$

or equivalently :

$$\gamma \triangleq \inf_{s \geq t_{i-1}} \left\{ s : Y(s, s + \theta) \geq \frac{\log(1 + \theta\alpha)}{\theta} \right\}$$

since :

$$\begin{aligned} (t_{i-1} \leq t \leq t_i, \max_{\rho(t, t + \theta)} \geq \alpha) &= (\gamma \leq t_i) \\ &= (\gamma = t_{i-1}) \cup (t_{i-1} < \gamma \leq t_i) \end{aligned}$$

the price can be rewritten as :

$$P(0, t_{i+1}) \theta E_{Q_{t_{i+1}}} \left(\begin{array}{c} (\rho(t_i, t_{i+1}) - t_g)_+ \\ - (\rho(t_i, t_{i+1}) - t_g)_+ I_{(\gamma = t_{i-1})} \\ - (\rho(t_i, t_{i+1}) - t_g)_+ I_{(t_{i-1} < \gamma \leq t_i)} \end{array} \right)$$

and with the identity :

$$(\gamma = t_{i-1}) = (\rho(t_{i-1}, t_i) \geq \alpha)$$

we obtain :

$$P(0, t_{i+1}) \theta \begin{pmatrix} E_{Q_{t_{i+1}}} \left(\rho(t_i, t_{i+1}) I_{(\rho(t_i, t_{i+1}) \geq t_g)} I_{(\rho(t_{i-1}, t_i) < \alpha)} \right) \\ -t_g E_{Q_{t_{i+1}}} \left(I_{(\rho(t_i, t_{i+1}) \geq t_g)} I_{(\rho(t_{i-1}, t_i) < \alpha)} \right) \\ -E_{Q_{t_{i+1}}} \left(\rho(t_i, t_{i+1}) I_{(\rho(t_i, t_{i+1}) \geq t_g)} I_{(t_{i-1} < \gamma \leq t_i)} \right) \\ +t_g E_{Q_{t_{i+1}}} \left(I_{(\rho(t_i, t_{i+1}) \geq t_g)} I_{(t_{i-1} < \gamma \leq t_i)} \right) \end{pmatrix}$$

or using the yield:

$$P(0, t_{i+1}) \begin{pmatrix} E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) \geq \frac{\ln(1+\theta t_g)}{\theta} \right)} I_{\left(Y(t_{i-1}, t_i) < \frac{\ln(1+\theta \alpha)}{\theta} \right)} \right) \\ - (1 + \theta t_g) E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) \geq \frac{\ln(1+\theta t_g)}{\theta} \right)} I_{\left(Y(t_{i-1}, t_i) < \frac{\ln(1+\theta \alpha)}{\theta} \right)} \right) \\ - E_{Q_{t_{i+1}}} \left(\exp(\theta Y(t_i, t_{i+1})) I_{\left(Y(t_i, t_{i+1}) \geq \frac{\ln(1+\theta t_g)}{\theta} \right)} I_{(t_{i-1} < \gamma \leq t_i)} \right) \\ + (1 + \theta t_g) E_{Q_{t_{i+1}}} \left(I_{\left(Y(t_i, t_{i+1}) \geq \frac{\ln(1+\theta t_g)}{\theta} \right)} I_{(t_{i-1} < \gamma \leq t_i)} \right) \end{pmatrix}$$

We easily obtain the first two terms in closed-form. For the two last expectations, we must extract an approximation of the γ density. As we know the density at time t_{i-1} because $Q_{t_{i+1}}(\gamma = t_{i-1}) = Q_{t_{i+1}}(\rho(t_{i-1}, t_i) \geq \alpha)$, we just need to write an implicit equation on $[t_{i-1}, t_i]$ similar to that already seen in the previous section. Since :

$$\left(Y(t_{i-1}, t_i) < \frac{\log(1 + \theta \alpha)}{\theta}, Y(t, t + \theta) \geq \frac{\log(1 + \theta \alpha)}{\theta} \right) \subset (t_{i-1} < \gamma \leq t)$$

then :

$$\begin{aligned} & Q_{t_{i+1}} \left(Y(t_{i-1}, t_i) < \frac{\log(1 + \theta \alpha)}{\theta}, Y(t, t + \theta) \geq \frac{\log(1 + \theta \alpha)}{\theta} \right) \\ = & Q_{t_{i+1}} \left(Y(t_{i-1}, t_i) < \frac{\log(1 + \theta \alpha)}{\theta}, Y(t, t + \theta) \geq \frac{\log(1 + \theta \alpha)}{\theta}, t_{i-1} < \gamma \leq t \right) \end{aligned}$$

but :

$$(t_{i-1} < \gamma \leq t) \subset (t_{i-1} < \gamma) \subset \left(Y(t_{i-1}, t_i) < \frac{\log(1 + \theta\alpha)}{\theta} \right)$$

then :

$$\begin{aligned} & Q_{t_{i+1}} \left(Y(t_{i-1}, t_i) < \frac{\log(1 + \theta\alpha)}{\theta}, Y(t, t + \theta) \geq \frac{\log(1 + \theta\alpha)}{\theta} \right) \\ = & Q_{t_{i+1}} \left(Y(t, t + \theta) \geq \frac{\log(1 + \theta\alpha)}{\theta}, t_{i-1} < \gamma \leq t \right) \\ = & \int_{t_{i-1}}^t Q_{t_{i+1}} \left(Y(t, t + \theta) \geq \frac{\log(1 + \theta\alpha)}{\theta} \mid \gamma = s \right) Q_{t_{i+1}}(\gamma \in [s, s + ds]) ds \end{aligned}$$

Finally, using the Markov property, the implicit equation is given by :

$$\begin{aligned} & Q_{t_{i+1}} \left(Y(t_{i-1}, t_i) < \frac{\log(1 + \theta\alpha)}{\theta}, Y(t, t + \theta) \geq \frac{\log(1 + \theta\alpha)}{\theta} \right) \\ = & \int_{t_{i-1}}^t Q_{t_{i+1}} \left(Y(t, t + \theta) \geq \frac{\log(1 + \theta\alpha)}{\theta} \mid Y(s, s + \theta) = \frac{\log(1 + \theta\alpha)}{\theta} \right) \\ & Q_{t_{i+1}}(\gamma \in [s, s + ds]) ds \end{aligned}$$

Let us denote $NN(., ., c)$ the two dimensional standard normal cumulative distribution function with correlation c . We have then the following proposition.

The restricted continuous barrier caplet's price is :

$$P(0, t_{i+1})(E_1 - E_3 + (1 + \theta t_g)(E_4 - E_2))$$

where:

$$E_1 = \left(\begin{array}{c} N \left(\frac{\frac{\log(1+\theta\alpha)}{\theta} - M^Y(t_{i-1})}{\sqrt{V^Y(t_{i-1})}} - \frac{C^Y(t_{i-1}, t_i)\theta}{\sqrt{V^Y(t_{i-1})}} \right) \\ - NN \left(\frac{\frac{\log(1+\theta t_g)}{\theta} - M^Y(t_i) - \theta V^Y(t_i)}{\sqrt{V^Y(t_i)}}, \frac{\frac{\log(1+\theta\alpha)}{\theta} - M^Y(t_{i-1})}{\sqrt{V^Y(t_{i-1})}} - \frac{C^Y(t_{i-1}, t_i)\theta}{\sqrt{V^Y(t_{i-1})}}, \right. \\ \left. \frac{C^Y(t_{i-1}, t_i)}{\sqrt{V^Y(t_{i-1})V^Y(t_i)}} \right) \end{array} \right) \exp \left(\theta M^Y(t_i) + \frac{\theta^2 V^Y(t_i)}{2} \right)$$

$$\begin{aligned}
E_2 &= N \left(\frac{\frac{\log(1+\theta\alpha)}{\theta} - M^Y(t_{i-1})}{\sqrt{V^Y(t_{i-1})}} \right) \\
&\quad - NN \left(\frac{\frac{\log(1+\theta t_g)}{\theta} - M^Y(t_i)}{\sqrt{V^Y(t_i)}}, \frac{\frac{\log(1+\theta\alpha)}{\theta} - M^Y(t_{i-1})}{\sqrt{V^Y(t_{i-1})}}, \frac{C^Y(t_{i-1}, t_i)}{\sqrt{V^Y(t_{i-1}) V^Y(t_i)}} \right) \\
E_3 &= \frac{\theta}{k} \sum_{j=1}^{k-1} \exp \left(\frac{\theta^2 \widehat{V}^Y(t_{i-1} + j\frac{\theta}{k}, t_i)}{2} + \theta \widehat{M}^Y(t_{i-1} + j\frac{\theta}{k}, t_i) \right) \\
&\quad N \left(\frac{\theta \widehat{V}^Y(t_{i-1} + j\frac{\theta}{k}, t_i) + \widehat{M}^Y(t_{i-1} + j\frac{\theta}{k}, t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(t_{i-1} + j\frac{\theta}{k}, t_i)}} \right) p_j \\
E_4 &= \frac{\theta}{k} \sum_{j=1}^{k-1} N \left(\frac{\widehat{M}^Y(t_{i-1} + j\frac{\theta}{k}, t_i) - \frac{\ln(1+\theta t_g)}{\theta}}{\sqrt{\widehat{V}^Y(t_{i-1} + j\frac{\theta}{k}, t_i)}} \right) p_j
\end{aligned}$$

4 Numerical Analysis

The formulae given in this paper are sensitive to many parameters, so we limit ourselves to a rapid study of the impact on the price of both the barrier and the volatility structure of interest rate. The notional amount M and the maturity T are taken to be equal to one. The initial term structure of interest rate, which is ascendant, is given through the following yield curve :

$$Y(0, T) = 0.07356 - (0.04) \exp(- (1.7) T)$$

corresponding to a 6.85% one year-maturity proportional rate.

We take :

$$\begin{aligned}
\sigma^P &= 0.033333 \\
a &= 1
\end{aligned}$$

for the volatility parameters in the exponential case, and:

$$\sigma^P = 0.0210070$$

in the linear case. With these values the linear volatility and the exponential one start from the same value at time 0.

We define an exotic cap as the sum of 9 exotic caplets uniformly distributed on the interval $[0, 1]$. More precisely, the interval $[0, 1]$ is subdivided into 10 subintervals $[t_i, t_{i+1}]$ of length $\frac{1}{10}$. The prevailing rate of the first exotic caplet is known at time t_1 but the payment takes place at time t_2 , and the prevailing rate of the last exotic caplet is known at time t_9 and the payment date is $t_{10} = 1$.

We recall that the pay-off of the knock out point barrier caplet, the knock-out continuous barrier caplet and the knock-out restricted continuous barrier caplet are given respectively by:

$$\theta (\rho (t_i, t_{i+1}) - t_g)_+ I_{(\rho(t_i, t_{i+1}) < \alpha)}$$

$$\theta (\rho (t_i, t_{i+1}) - t_g)_+ I_{(0 \leq t \leq t_i \max(\rho(t, t+\theta) < \alpha)}$$

$$\theta (\rho (t_i, t_{i+1}) - t_g)_+ I_{(t_{i-1} \leq t \leq t_i \max(\rho(t, t+\theta) < \alpha)}$$

In the following tables, t_g is fixed to 7% . The method proposed in this paper is used with $n = 600$, on the interval $[0, 1]$ for the continuous barrier caplet; and with $n = 60$, on each of the subintervals $[t_i, t_{i+1}]$, for the restricted barrier caplet . In the following tables, *Proba.*, *Point.*, *Restr.* and *Cont.* respectively denote the approximations of $Q_T (\gamma \leq 1)$, the point barrier cap's price, the restricted barrier cap's price and the continuous barrier cap's price.

Table 1: Exotic caps with linear volatility structure of interest rate and guaranteed rate

α %	<i>Proba.</i>	<i>Point.</i>	<i>Restr.</i>	<i>Cont.</i>
6.86	1.000			
7	0.990	0.000	0.000	0.000
9	0.623	0.039	0.028	0.023
9.5	0.501	0.053	0.042	0.037
10	0.387	0.066	0.056	0.051
10.5	0.287	0.076	0.068	0.064
11	0.205	0.084	0.078	0.075
11.5	0.140	0.090	0.086	0.084
12	0.093	0.094	0.091	0.090
12.5	0.059	0.097	0.095	0.094
13	0.036	0.099	0.097	0.097
14	0.012	0.100	0.100	0.100
15	0.003	0.101	0.101	0.101
40	0.000	0.101	0.101	0.101

Table 2: Exotic caps with exponential volatility structure of interest rate and guaranteed rate

α	<i>Proba.</i>	<i>Point.</i>	<i>Restr.</i>	<i>Cont.</i>
6.86	1.000			
7	0.997	0.000	0.000	0.000
8.5	0.805	0.030	0.017	0.013
9	0.658	0.047	0.033	0.027
9.5	0.492	0.062	0.049	0.043
10	0.333	0.074	0.064	0.059
10.5	0.204	0.082	0.076	0.072
11	0.113	0.087	0.083	0.081
11.5	0.056	0.090	0.088	0.087
12	0.025	0.091	0.090	0.090
12.5	0.010	0.092	0.091	0.091
13	0.004	0.092	0.092	0.092
14	0.000	0.092	0.092	0.092
15	0.000	0.092	0.092	0.092
40	0.000	0.092	0.092	0.092

In the first row of these two tables, there are no values for the different exotic caps simply because, the pricing formulae are only defined for a barrier level α superior or equal to the guaranteed rate t_g . In fact, α is taken to be equal to 6.86 % in order to verify the accuracy of the discrete approximation of the γ density. Indeed, the one-year proportional rate starting from 6.85 %

, it is quite logical that the probability for this rate to breach α , is very close to 1. Inversely, in the last row we take a large value for α , and we fortunately observe a probability equal to 0. In the same spirit, in the second row, α is taken to be equal to 7 %, that permits us to consider again the validity of the approximations, knowing that for this value, the exotic cap's prices are, by definition, equal to zero.

The barrier's influence on the price is explicit. As a matter of fact, when α goes from 9 % to 15 %, the price varies respectively about 96 %, 181 % and 243 % for the point barrier, the restricted barrier and the continuous barrier. For any fixed value of α , we can also compare the impact on the price of the interest rate's modelisation through the volatility structure.

5 Conclusion

In this paper we suggest a method to value barrier options when the underlying asset is an interest rate, the barrier being constant or more generally deterministic. The pricing formulae are given in quasi-explicit form. They only involve parameters which can easily be estimated or recovered from the market. Because our model is parametrized both in its structural parameters: initial yield curve and volatility structure and in its intrinsic parameters: barrier levels, guaranteed rate, one advantage of this approach is its flexibility. Furthermore, a numerical analysis shows that the recursive computations needed to price these options are fairly quick, so our results can be implemented for practical purposes and can be useful for trading desks.

Appendix 1: Fortet equation and approximation of the first passage time to a barrier

Let

$$f(x, t | a, s)$$

and

$$F(x, t | a, s)$$

be respectively the density at x , of the conditional law of $\ln(S_t)$ given $(\ln(S_s) = a)$, and the cumulative distribution function at x of $\ln(S_t)$ given $(\ln(S_s) = a)$.

Let :

$$Q_T(\gamma \in [t, t + dt])$$

denote the first passage time γ of $(\ln(S_t))_{0 \leq t \leq T}$ to $\ln(H)$ density at t , under Q_T .

Due to Fortet we have the following equation :

$$f(x, t | \ln(S_0), 0) = \int_0^t f(x, t | \ln(H), \tau) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau$$

with $\ln(S_0) < \ln(H)$ and $x \geq \ln(H)$.

One integrates this equation with respect to x from $\ln(H)$ to $+\infty$:

$$1 - F(\ln(H), t | \ln(S_0), 0) = \int_0^t (1 - F(\ln(H), t | \ln(H), \tau)) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \quad (*)$$

We know that $\ln(S_t)$ is gaussian with mean $M(t)$ and variance $V(t)$, and that the conditional law of $\ln(S_t)$ given $(\ln(S_\tau) = \ln(H))$ is normal with mean $\widehat{M}(\tau, t)$ and variance $\widehat{V}(\tau, t)$, hence :

$$\begin{aligned} F(\ln(H), t | \ln(S_0), 0) &= P(\ln(S_t) \leq \ln(H)) \\ &= P\left(\frac{\ln(S_t) - M(t)}{\sqrt{V(t)}} \leq \frac{\ln(H) - M(t)}{\sqrt{V(t)}}\right) \\ &= N\left(\frac{\ln(H) - M(t)}{\sqrt{V(t)}}\right) \end{aligned}$$

Where $N(\cdot)$ is the standard normal cumulative distribution function.

Similarly :

$$\begin{aligned} &F(\ln(H), t | \ln(K), \tau) \\ &= P(\ln(S_t) \leq \ln(H) | (\ln(S_\tau) = \ln(H))) \\ &= P\left(\frac{\ln(S_t) - \widehat{M}(\tau, t)}{\sqrt{\widehat{V}(\tau, t)}} \leq \frac{\ln(H) - \widehat{M}(\tau, t)}{\sqrt{\widehat{V}(\tau, t)}} | (\ln(S_\tau) = \ln(H))\right) \\ &= N\left(\frac{\ln(H) - \widehat{M}(\tau, t)}{\sqrt{\widehat{V}(\tau, t)}}\right) \end{aligned}$$

equation (*) writes :

$$1 - N\left(\frac{\ln(H) - M(t)}{\sqrt{V(t)}}\right) = \int_0^t \left(1 - N\left(\frac{\ln(H) - \widehat{M}(\tau, t)}{\sqrt{\widehat{V}(\tau, t)}}\right)\right)$$

$$Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau$$

$$\Leftrightarrow N\left(\frac{M(t) - \ln(H)}{\sqrt{V(t)}}\right) = \int_0^t N\left(\frac{\widehat{M}(\tau, t) - \ln(H)}{\sqrt{\widehat{V}(\tau, t)}}\right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \quad (**)$$

In order to recursively determine $(Q_T(\gamma \in [\tau, \tau + d\tau]))_{0 \leq \tau \leq T}$, first we divide the interval $[0, T]$ into n subperiods of length $h = \frac{T}{n}$ and then rewrite (**) for $[0, h]$:

$$N\left(\frac{M(h) - \ln(H)}{\sqrt{V(h)}}\right) = \int_0^h N\left(\frac{\widehat{M}(\tau, h) - \ln(H)}{\sqrt{\widehat{V}(\tau, h)}}\right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau$$

Using the following result :

$$\lim_{\tau \nearrow h} N\left(\frac{\widehat{M}(\tau, h) - \ln(H)}{\sqrt{\widehat{V}(\tau, h)}}\right) = \frac{1}{2}$$

we get :

$$\int_0^h N\left(\frac{\widehat{M}(\tau, h) - \ln(H)}{\sqrt{\widehat{V}(\tau, h)}}\right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \simeq h Q_T(\gamma \in [h, h + dh]) \frac{1}{2}$$

so :

$$Q_T(\gamma \in [h, h + dh]) \simeq p_1 = N\left(\frac{M(h) - \ln(H)}{\sqrt{V(h)}}\right) \frac{2}{h}$$

The same argument is used on $[0, 2h]$:

$$\begin{aligned} & N\left(\frac{M(2h) - \ln(H)}{\sqrt{V(2h)}}\right) \\ &= \int_0^{2h} N\left(\frac{\widehat{M}(\tau, 2h) - \ln(H)}{\sqrt{\widehat{V}(\tau, 2h)}}\right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \\ &= \int_0^h N\left(\frac{\widehat{M}(\tau, 2h) - \ln(H)}{\sqrt{\widehat{V}(\tau, 2h)}}\right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \\ &\quad + \int_h^{2h} N\left(\frac{\widehat{M}(\tau, 2h) - \ln(H)}{\sqrt{\widehat{V}(\tau, 2h)}}\right) Q_T(\gamma \in [\tau, \tau + d\tau]) d\tau \end{aligned}$$

The first integral is computed with the approximation p_1 , the second one is managed the same way, leading to :

$$N\left(\frac{M(2h) - \ln(H)}{\sqrt{V(2h)}}\right) \simeq hp_1 N\left(\frac{\widehat{M}(h, 2h) - \ln(H)}{\sqrt{\widehat{V}(h, 2h)}}\right) + hQ_T(\gamma \in [2h, 2h + dh]) \frac{1}{2}$$

so :

$$Q_T(\gamma \in [2h, 2h + dh]) \simeq p_2 = \frac{2}{h} N\left(\frac{M(2h) - \ln(H)}{\sqrt{V(2h)}}\right) - 2p_1 N\left(\frac{\widehat{M}(h, 2h) - \ln(H)}{\sqrt{\widehat{V}(h, 2h)}}\right)$$

and more generally for $2 \leq j \leq n$:

$$Q_T(\gamma \in [jh, jh + dh]) \simeq p_j = \frac{2}{h} N\left(\frac{M(jh) - \ln(H)}{\sqrt{V(jh)}}\right) - 2 \sum_{i=1}^{j-1} p_i N\left(\frac{\widehat{M}(ih, jh) - \ln(H)}{\sqrt{\widehat{V}(ih, jh)}}\right)$$

Appendix 2: First and second moments of $Y(t, t + \theta)$ for different volatility structure

With a linear volatility structure :

$$M_{lin}^Y(t) = \phi(0; t, t + \theta) + \left(\frac{\theta\sigma_P^2}{2} - \sigma_P^2(T + \theta)\right)t + \sigma_P^2 t^2$$

$$V_{lin}^Y(t) = \sigma_P^2 t$$

$$C_{lin}^Y(v, t) = \sigma_P^2 v$$

With an exponential volatility structure :

$$M_{exp}^Y(t) = \phi(0; t, t + \theta) + \frac{\theta}{2} V^Y \exp(t)$$

$$\begin{aligned}
& -\frac{\sigma_P^2}{2\theta a^3} (1 - \exp(-a\theta)) (\exp(-at) - \exp(-a(T + \theta))) \\
& (\exp(at) - \exp(-at)) \\
V_{\text{exp}}^Y(t) &= \frac{\sigma_P^2}{2\theta^2 a^3} (1 - \exp(-a\theta))^2 (1 - \exp(-2at)) \\
C_{\text{exp}}^Y(v, t) &= \frac{\sigma_P^2}{2\theta^2 a^3} (1 - \exp(-a\theta))^2 (\exp(-a(t - v)) - \exp(-a(t + v)))
\end{aligned}$$

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