

Using Lattice Rules to Value Low-Dimensional Derivative Contracts

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Abstract

This paper discusses the use of lattice rules to evaluate low and medium dimensional integrals. Lattice rules are based on the use of deterministic sequences rather than random sequences. They are a special type of so-called low discrepancy sequences. We find that as long as the integral is sufficiently regular, lattice rules generally outperform not only basic Monte Carlo but also other types of low discrepancy sequences. We implement a specific lattice rule known as good lattice points and apply this approach to two problems of practical interest. The first application is to the valuation of the certain types of financial options known as lookback options. The second application is to the valuation of the embedded option in certain equity-indexed annuities. In the case of these examples we show that the good lattice points are dramatically more efficient than competing methods.

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1 Introduction

The Monte Carlo method has proven to a very useful tool for numerical analysis, particularly when the number of dimensions ranging from medium to large. Such problems occur in a broad range of applications in science, physics and engineering. In recent years the Monte Carlo method has also become a popular computational device for problems in finance. The finance discipline has become more sophisticated and more quantitative in the last two decades. There are several applications of the Monte Carlo method in finance and we will now briefly describe two of the main ones.

The first application is to the pricing of securities. Boyle, Broadie and Glasserman (1997) provide a survey of the applications of the Monte Carlo methods in this area. The basic valuation paradigm for a derivative security states that the price of the derivative can be obtained by taking the expectation of its future cash flows under a particular probability measure known as the risk-neutral measure. Now if the number of state variables is large, or if the number of underlying assets is large or if the contract involves path dependency, this expectation becomes a multivariate integral. The Monte Carlo method is often the most efficient way of evaluating such high dimensional integrals. The Monte Carlo method is also widely applied in the case of the risk analysis of portfolios. For example, it provides a practical method of obtaining the probability that the value of the portfolio falls below some threshold at the end of some period. Such calculations are needed to compute the so-called Value at Risk of the portfolio. Glasserman, Heidelberger and Shahabuddin (2001) discuss the applications of the Monte Carlo method in computing Value at Risk.

In the last few years new approaches have been developed that outperform standard Monte Carlo in terms of numerical efficiency. It has been found that there can be efficiency gains in using deterministic sequences rather than the random sequences which are a feature of standard Monte Carlo. These deterministic sequences are carefully selected so that they are well dispersed throughout the region of integration. Sequences with this property are known as low discrepancy sequences. These sequences are often more efficient than standard Monte Carlo in evaluating high dimensional integrals if the integrand is sufficiently regular and for many finance applications this is the case. Applications of low discrepancy sequences to finance problems have been discussed by Boyle, Broadie and Glasserman (1997), Caffisch, Morokoff and Owen (1997), Joy, Boyle and Tan (1996), Ninomiya and Tezuka (1996), Tan and Boyle (2000) and Paskov and Traub (1995).

Much of the focus to-date has been on high-dimensional problems since these are more challenging from a computational viewpoint. However, it is also of interest to examine low to medium dimension problems and this is the objective of the present paper. Low to medium sized problems are of practical interest since there are popular contracts whose value depends on a small to medium number of variables. Here are a few examples:

- Options whose payoff depends on the relative performance of two underlying assets. A particular version of this option known as a spread option popular in the energy industry.¹

¹Oil refineries refine crude oil to produce a number of products including heating oil and gasoline. Their revenues depend on the difference between their costs and their output prices. The crack spread which

- Basket options where the payoff depends on the ending values of a number of assets such as different common stocks or stock market indices. The payoff could be based on the average, the maximum or the minimum of the asset prices.
- Path dependent options where the payoff is a function of the asset price at a number of discrete monitoring points along the path. In the case of *Asian* options the payoff is based on the average of these points. In the case of *lookback* options the payoff is based on the largest (or smallest) value recorded at one of these monitoring points. The dimensions of the problem are directly related to the number of discrete points in the path.
- Equity Indexed Annuities (EIA) which have been very popular types of insurance contracts in the United States. These contracts have typically embedded options related to the performance of some equity index such as the S&P 500. Common examples are the high-water-mark (or discrete lookback) options and Asian (or average) options. For the high-water-mark option, the payoff is path-dependent in the sense that it depends on the level of the index fund at each anniversary date of the contract. For the Asian-type feature, the payoff typically depends on the monthly index fund level in the final year of the contract. Since EIA have maturities ranging from 3 to 10 years they can be regarded as a 3 to 10 dimensional problems while the Asian-type is a 12 dimensional problem.

This paper shows that a particular type of low discrepancy sequence known as lattice rules have strong advantages in the case of low to medium dimension problems as long as the integrands are sufficiently regular. We concentrate on a particular type of lattice rule known as good lattice points. The basic idea behind this method can be explained as follows. For all the functions in a certain class,² we can find a useful expression for the integration error. This enables us to find the *worst case function* in the class in the sense that it gives the largest integration error. The good lattice points are obtained by finding those points that minimize the integration error for this worst case function. So the method proceeds by searching for points that minimize the maximum integration error. We demonstrate that good lattice points outperform other methods in terms of numerical efficiency for low dimensional problems. Both Tan (1998) and Ross (1998) consider the method of good lattice points and method of good points, respectively, on pricing low dimensional European-style exotic derivatives. They conclude the superiority of these methods over competitive methods. For much higher dimensions, the numerical works conducted by Lemieux and L'Ecuyer (1998, 1999) indicate that the lattice rules can be as competitive as other low-discrepancy point sets.

The layout of the rest of this paper is as follows. The next section summarizes various approaches that can be used to evaluate multi-dimensional integrals. These include the stan-

denotes the difference between the price of gasoline and the price of crude oil is a measure of their profit margin. Crack spread options are based on this difference and are natural hedging tools for oil refineries. Spread options enable refineries to control their margins and stabilize their profits.

²These functions satisfy specified regularity conditions related to the existence and continuity of partial derivatives. They should be periodic but non-periodic functions can often be transformed into periodic functions in a sense that will be made clear later in the paper.

standard Monte Carlo method as well as approaches based on low discrepancy sequences. We also discuss lattice rules in general and good lattice points (g.l.p.) in particular. Section 3 discusses the pricing of European derivative securities and shows the genesis of the multivariate integrals in this context. The next two sections explore the numerical efficiency of the g.l.p. method against alternative approaches using different examples. Thus Section 4 discusses discretely monitored lookback options while Section 5 examines a particular type of Equity Indexed Annuity. We find that the g.l.p. method generally outperforms alternative approaches in terms of efficiency. We also show that the g.l.p. dramatically outperforms the standard Monte Carlo approach in the computation of the so-called Greek sensitivities. We emphasize that these superior results have been obtained for low to medium dimensional problems where the integrands are smooth. The last section contains a brief summary.

2 Valuing Multiple Integrals using Sampling Methods

We now summarize various methods for evaluating multiple integrals that are all based on some form of sampling. Consider the multiple integral

$$\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x} = \int_0^1 dx^{(1)} \int_0^1 dx^{(2)} \cdots \int_0^1 dx^{(s)} f(x^{(1)}, x^{(2)}, \dots, x^{(s)}) = \theta \quad (1)$$

where $\mathbf{x} = (x^{(1)}, \dots, x^{(s)}) \in [0, 1]^s$, the function f is square-integrable in $[0, 1]^s$ and $|\theta| < \infty$.

For small values of s , numerical integration methods such as Simpson's rule or the trapezoidal rule (see Davis and Rabinowitz (1984)) can be used to approximate the integral (1). These methods, however, suffer from the so-called *curse of dimensionality* and become impractical as s increases beyond 3 or 4. One viable technique for larger values of s is to use *sampling* methods such that the estimator of θ becomes

$$\hat{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \quad (2)$$

where $P_N = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}\} \in [0, 1]^s$ is some point set. Different techniques are available for selecting these point sets and we now examine three of them.

2.1 Monte Carlo Integration

When the integration nodes P_N are N independently and identically distributed random points in $[0, 1]^s$, the above sampling method becomes the standard Monte Carlo integration method and the resulting estimator $\hat{\theta} \equiv \hat{\theta}^{\text{MC}}$ is known as the Monte Carlo estimator. Important properties of $\hat{\theta}^{\text{MC}}$ are as follows:

- It is an unbiased estimator of θ with variance $\frac{\sigma^2}{N}$; i.e.

$$\text{E}[\hat{\theta}^{\text{MC}}] = \theta \quad \text{and} \quad \text{Var}[\hat{\theta}^{\text{MC}}] = \frac{\sigma^2}{N}$$

- The Strong Law of Large Number asserts that $\hat{\theta}^{\text{MC}}$ converges to θ almost surely.
- The Central Limit Theorem guarantees that the distribution of $\hat{\theta}$ converges asymptotically to a normal distribution with mean θ and variance $\frac{\sigma^2}{N}$ as $N \rightarrow \infty$. In other words, the error $|\theta - \hat{\theta}^{\text{MC}}|$ converges probabilistically at a rate of $O(N^{-1/2})$.
- A consequence of above result is that the accuracy of $\hat{\theta}^{\text{MC}}$ can be assessed by constructing the confidence interval. For instance, the 95% confidence interval is usually reported along with the estimated $\hat{\theta}^{\text{MC}}$:

$$\left[\hat{\theta}^{\text{MC}} - 1.96 \frac{\hat{\sigma}(f)}{\sqrt{N}}, \hat{\theta}^{\text{MC}} + 1.96 \frac{\hat{\sigma}(f)}{\sqrt{N}} \right], \quad (3)$$

where $\hat{\sigma}(f)$ can be estimated by

$$\hat{\sigma}(f) \approx \sqrt{\frac{\sum_{n=0}^{N-1} [f(\mathbf{x}_n) - \hat{\theta}^{\text{MC}}]^2}{N-1}}.$$

- The rate of convergence $O(N^{-1/2})$ is independent of s .

2.2 Quasi-Monte Carlo Integration

We can see intuitively that if we pick points that are evenly dispersed through the domain of integration this may lead to a better approximation of the integral than if we use purely random points. It turns out that we can achieve a rate of convergence that is better than $O(N^{-1/2})$ if the point set P_N is chosen judiciously. This is clearly evident when $s = 1$ since the typical quadrature methods such as Simpson's rule yields a much higher convergence rate.

In higher dimensions the so-called quasi-Monte Carlo methods or low discrepancy methods have been developed which use point sets P_N that are more evenly (or uniformly) distributed over $[0, 1]^s$ than random points. The uniformity of a point set is measured by its discrepancy. Point sets with discrepancy that of order $O(N^{-1}(\log N)^{s-1})$ are referred to as low discrepancy point sets. These point sets are designed to have greater uniformity than the random points.

The critical role of discrepancy measure can be gleaned from the Koksma-Hlawka inequality which states that

$$\left| \frac{\sum_{n=0}^{N-1} f(\mathbf{x}_n)}{N} - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq V(f) D_N^*, \quad (4)$$

where the function f is of bounded variation in the sense of Hardy-Krause, $V(f)$ is the total variation of f and $D^*(P_N)$ measures discrepancy (or uniformity) of the point set P_N . This inequality effectively separates the integration error into two components: these correspond to the smoothness of the function and the discrepancy of the point set used in approximating the function. Hence for a fixed function f , point sets with lower discrepancy should lead to lower integration errors.

In practice, there are two major methods for constructing point sets that have low-discrepancy. They are the digital nets and integration lattices. We now briefly discuss the digital nets and delay the discussion on integration lattice to the following subsection. The digital nets construction produces (t, m, s) -nets and (t, s) -sequences with discrepancy of $O(N^{-1}(\log N)^{s-1})$ and $O(N^{-1}(\log N)^s)$, respectively. Explicit algorithms for constructing these sequences are given by van der Corput, Halton (1960), Sobol' (1967), Faure (1982), Niederreiter (1987, 1988), Tezuka (1995), Niederreiter and Xing (1996, 1998). For details, see Niederreiter (1992) and Tezuka (1995). It follows from the Koksma-Hlawka inequality that these nets and sequences achieve an error bound of $O(N^{-1}(\log N)^{s-1})$ and $O(N^{-1}(\log N)^s)$, respectively, and are more efficient than the Monte Carlo rate of $O(N^{-1/2})$ for large N . In recent years many researchers (see e.g. Joy, Boyle and Tan (1996), Paskov and Traub (1995), Ninomiya and Tezuka (1996), Tan and Boyle (2000)) have considered these low discrepancy point sets or sequences in high-dimensional finance problems and have documented their superiority over the standard Monte Carlo method.

2.3 Lattice Rules

In this subsection, we discuss another approach for approximating the integral (1) using lattice rules. Lattice rules are generalization of the number theoretic quadrature rules of Korobov (1959) for integration over an s -dimensional hypercube. They were first introduced by Sloan and Kachoyan (1987). To describe this method, we first introduce several definitions. For more details, see Sloan and Joe (1995) and Niederreiter (1992, chapter 5). An s -dimensional *integration lattices* L are discrete subsets of the real space \mathbb{R} , that contains integer vectors \mathbb{Z}^s , such that it is a set of all linear combinations with integer coefficients of s linearly independent vectors in \mathbb{R}^s . If the integration nodes P for (2) are a set of N points that lie in the intersection of the hypercube and an integration lattice; i.e. $P = L \cap [0, 1]^s$, then the resulting method is known as the lattice rule of order N , or simply an N -point lattice rule.

An important classification of lattice rules was established by Sloan and Lyness (1989) who showed that for any s -dimensional lattice rule L , there exists a unique integer r , $1 \leq r \leq s$ and positive integers n_1, \dots, n_r with $n_{i+1}|n_i$ for $i = 1, \dots, r-1$ and $n_r > 1$ such that the node set P can be derived from

$$\left\{ \frac{k_1}{n_1} \mathbf{z}_1 + \dots + \frac{k_r}{n_r} \mathbf{z}_r \right\} \quad \text{with } 1 \leq k_i \leq n_i \text{ for } 1 \leq i \leq r. \quad (5)$$

Here $\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{Z}^s$ are some suitable integer vectors and $\{\mathbf{z}\}$ is defined by

$$\{\mathbf{z}\} = (\{z_1\}, \dots, \{z_s\}) \in [0, 1]^s$$

where $\{z\}$ denotes the fractional part of z ; i.e. $\{z\} = z \pmod{1}$. The set of points in (5) are all distinct with $N = n_1 \cdots n_r$. The smallest integer r satisfying (5) is known as the rank of the lattice rule while n_1, n_2, \dots, n_r are the invariants. In this paper, we consider the lattice rules of rank 1. This particular class of lattice rules also corresponds to the method of good lattice points proposed by Korobov (1959) and Hlawka (1961).

We now discuss briefly the error bounds pertaining to the lattice rules. Let $f(\mathbf{u})$, $\mathbf{u} \in [0, 1]^s$ be a periodic function on \mathbb{R}^s with period 1 in each of its s variables. Let $\hat{f}(\mathbf{h})$, $\mathbf{h} \in \mathbb{Z}^s$, denote the Fourier coefficient of f so that

$$f(\mathbf{u}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \hat{f}(\mathbf{h}) \exp(2\pi i \mathbf{h} \cdot \mathbf{u}) \quad (6)$$

where

$$\hat{f}(\mathbf{h}) = \int_{[0,1]^s} f(\mathbf{u}) \exp(-2\pi i \mathbf{h} \cdot \mathbf{u}) d\mathbf{u}, \quad \mathbf{h} \in \mathbb{Z}^s, \quad (7)$$

and the inner product $\mathbf{h} \cdot \mathbf{u} = h_1 u_1 + h_2 u_2 + \dots + h_s u_s$. Then it can be shown that (see Theorem 2.8 of Sloan and Joe (1995)) the error arising from using the integration lattice L can be expressed in terms of Fourier coefficients as

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(\mathbf{x}_k) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| = \sum_{\mathbf{h} \neq \mathbf{0}, \mathbf{h} \in L^\perp} \hat{f}(\mathbf{h}), \quad (8)$$

where $\{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\} \in L \cap [0, 1]^s$ and L^\perp corresponds to the *dual lattice* of the s -dimensional integration lattice L ; i.e.

$$L^\perp = \{\mathbf{h} \in \mathbb{R}^s : \mathbf{h} \cdot \mathbf{x} \in \mathbb{Z} \text{ for all } \mathbf{x} \in L\}. \quad (9)$$

We now introduce additional regularity on the function f . More formally, we say f belongs to the class of continuous periodic function $\mathcal{E}_\alpha^s(C)$ for a fixed $\alpha > 1$ provided that for all nonzero $\mathbf{h} \in \mathbb{Z}^s$, we have

$$|\hat{f}(\mathbf{h})| \leq C \left[\prod_{i=1}^s \max(1, |h_i|) \right]^{-\alpha} = Cr(\mathbf{h})^{-\alpha} \quad (10)$$

where $r(\mathbf{h}) = \prod_{i=1}^s \max(1, |h_i|)$ and $C > 0$ is a constant does not depend on \mathbf{h} . The rate of decay of the Fourier coefficient of a function is related to the smoothness of the function. Hence $\mathcal{E}_\alpha^s(C)$ can be interpreted as characterizing a class of function of certain smoothness.

If $f \in \mathcal{E}_\alpha^s(C)$, it follows from (10) that the quadrature error in (8) is bounded from above by

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(\mathbf{x}_k) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq C \cdot \sum_{\mathbf{h} \neq \mathbf{0}, \mathbf{h} \in L^\perp} r(\mathbf{h})^{-\alpha}. \quad (11)$$

In the special case with lattice rules of rank 1, the dual lattice in (9) becomes

$$L^\perp = \{\mathbf{h} \in \mathbb{Z}^s : \mathbf{h} \cdot \mathbf{x} \equiv 0 \pmod{N}\} \quad (12)$$

so that for $f \in \mathcal{E}_\alpha^s(C)$, the error bound corresponding to (11) reduces to

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f\left(\left\{\frac{k}{N}\mathbf{z}\right\}\right) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq CP_\alpha(\mathbf{z}, N) \quad (13)$$

where

$$P_\alpha(\mathbf{z}, N) = \sum_{\substack{\mathbf{h} \neq \mathbf{0} \\ \mathbf{z} \cdot \mathbf{h} \equiv 0 \pmod{N}}} r(\mathbf{h})^{-\alpha}. \quad (14)$$

We now make several remarks regarding the applications of good lattice points.

- It has been established that if the point set P is a good lattice point corresponds to an optimal choice of \mathbf{z} , then error bound (13) becomes

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f\left(\left\{\frac{k}{N}\mathbf{z}\right\}\right) - \int_{[0,1]^s} f(\mathbf{u})d\mathbf{u} \right| \leq Cd(s, \alpha) \frac{(\log N)^{\beta(s, \alpha)}}{N^\alpha} \quad (15)$$

for function $f \in \mathcal{E}_\alpha^s(C)$ with $d(s, \alpha)$ and $\beta(s, \alpha)$ do not depend on N and $\alpha > 1$. When N is prime Bahvalov (1959) shows that $\beta(s, \alpha) = \alpha(s - 1)$. The work of Šargin (1963) indicates that the lower bound on $\beta(s, \alpha)$ is at least $s - 1$. It is only in the case with $s = 2$ where Šargin's lower bound is attainable. To-date, the best bounds on $P_\alpha(\mathbf{z}, N)$ have been those obtained by Niederreiter (1993). The bound in (15) also suggests that for function f with additional smoothness, the lattice rules could lead to a faster rate of convergence than those observed by using the digital nets.

- The last remark indicates that there exists g.l.p. for which the error bound (15) is attainable. It however makes no reference on how to construct such point set explicitly. It is only in the special case with $s = 2$ where there exists a simple construction. This 2-dimensional g.l.p. is based on the Fibonacci numbers and is the only case for which the Šargin lower bound is achieved. In higher dimensions, we only have existence theorems for g.l.p.. In 1959, Korobov demonstrated the existence of g.l.p. for N that is either prime or product of two primes. It also follows from his existence proof that for prime N , half or more of all the possible vector \mathbf{z} can be suitable choices for g.l.p.. Several authors have subsequently relaxed the restriction on N in the existence proof of Korobov. Keast (1973) shows the existence of g.l.p. of order N when N is a product of more than two primes while Niederreiter (1978) shows that g.l.p. exist for all N .

In practice, the g.l.p. for a fixed N and a fixed dimension s are constructed based on a computer search. This is achieved by searching the vector \mathbf{z} for which the error $\hat{P}_\alpha(\mathbf{z}, N)$ corresponding to the worst case function in $\mathcal{E}_\alpha^s(C)$ is minimum. To further reduce the possible combination of the vector \mathbf{z} , Korobov (196) considers a more restricted structure of the lattice points which is of the form

$$\mathbf{z}(l) = (1, l, l^2 \pmod{N}, \dots, l^{s-1} \pmod{N}), \quad 1 \leq l < N.$$

Each vector \mathbf{z} thus consists of only one parameter with a total of $N - 1$ possible choices, as opposed to N^s . Tables of g.l.p. obtained by minimizing $\hat{P}_2(s, N)$ can be found in Haber (1983). Other tables of g.l.p. can be found in Saltykov (1963) (which are reproduced in Stroud (1970), Hua and Wang (1981), Sloan and Joe (1995).

Recently, a more efficient way of obtaining the g.l.p. in high dimensions is proposed by L'Ecuyer (1999) who establishes a link between the points generated from the linear congruential generators and the g.l.p.. Good lattice points from this approach have been considered extensively by Lemieux and L'Ecuyer (1998, 1999).

- As noted in Sloan and Joe (1995) “*lattice rules are really designed to take advantage of the periodicity and smoothness properties of f* ”. Most functions, however, encountered in practice need not have the required periodicity and smoothness. A possible solution is to carry out a preliminary transformation so that a sufficiently regular non-periodic integrand can be converted into an integrand that has the needed periodicity. Good accounts on various approaches for periodizing the integrands can be found in Hua and Wang (1981), Zaremba (1972) and Beckers and Haegemans (1992). Here we just discuss one particular method we use in this paper. This approach is based on the standard change-of-variable technique such that for a given function f , the transformed periodized ϕ is given by

$$\phi(\mathbf{u}) = f(\psi(u_1), \dots, \psi(u_s))\psi'(u_1) \cdots \psi'(u_s),$$

where ψ is a smooth increasing function which maps $[0, 1]$ onto $[0, 1]$ and $\psi^{(j)}(0) = \psi^{(j)}(1) = 0$ for $1 \leq j \leq \alpha$. Possible choices of ψ are the polynomial-transformation $\psi(t) = 3t^2 - 2t^3$ or the sin-transformation (see Sidi (1993)) $\psi(t) = \frac{1}{2\pi}(2\pi t - \sin 2\pi t)$. Tan (1998) have considered in greater details the relative efficiency on various periodization approaches. In our illustration, we consider the sin-transformation.

- To provide a useful error estimate using the method of g.l.p., we use the “random shift” technique suggested by Cranley and Patterson (1976) and Joe (1990). For a given point set $P = \{x_0, \dots, x_{N-1}\}$ we generate m i.i.d. vectors of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ uniformly distributed numbers in $[0, 1]^s$. Let f_i be the estimate of the function f based on the set of N nodes $\{P + \mathbf{v}_i\}$; i.e. P is shifted by \mathbf{v}_i modulo 1. If $\mathbf{v}_i, i = 1, \dots, m$ are chosen independently and randomly from a multivariate uniform distribution on $[0, 1]^s$, then $f_i, i = 1, \dots, m$ are i.i.d. random variables with expected value equal to θ . This implies that the sample mean over m independent replications

$$\hat{\theta}^{\text{g.l.p.}} = \frac{1}{m} \sum_{i=1}^m f_i$$

is an unbiased estimator of θ and

$$\hat{\sigma}^2 = \frac{1}{m-1} \sum_{i=1}^m (f_i - \hat{\theta}^{\text{g.l.p.}})^2$$

is an estimator of the variance of $\hat{\theta}^{\text{g.l.p.}}$. The required confidence intervals can then be estimated based on $\hat{\theta}^{\text{g.l.p.}}$ and $\hat{\sigma}$.

It was also pointed out in Tuffin (1996) that the random shift technique can be applied to other low discrepancy sequences such as the Halton (1960), Sobol' (1967), Faure (1982), Niederreiter (1987, 1988) sequences in providing an estimate of the standard errors. In our comparison, we apply this technique to the Sobol' sequences.

3 Derivative Pricing

In this section, we provide a brief overview of derivative pricing and then discuss a special case that will be useful in our later examples. For a more detailed analysis see Duffie (1996). In modern finance, security prices are often modeled as stochastic processes to reflect future uncertainty. If we assume that there is no arbitrage in the financial market, then it is well established that this implies the market prices of securities suitably normalized are martingales. This martingale property means that we can write the current price of an asset as an expectation of its future cash flows under a probability measure known as the equivalent martingale measure. When the market is complete, the equivalent probability measure is unique and we shall make this assumption in the rest of the paper and call it the Q measure. We now illustrate how this approach is used to develop valuation formula for a European option based on several assets. A European option is a security which gives its owner the right to purchase a certain asset or receive a certain payoff at some fixed future date.

Consider an economy with s risky assets with prices $\mathbf{S}_t = (S_{1t}, \dots, S_{st})$ at time t . We assume the asset price processes satisfy

$$dS_{it} = (r - \delta_i)S_{it} dt + \sigma_i S_{it} dW_{it}, \quad 1 \leq i \leq s$$

under the equivalent martingale measure Q and where the parameters $\delta_i, \sigma_i, 1 \leq i \leq s$, are the annualized dividend yield and the volatility of asset i . The symbol r denotes annualized risk-free rate and we assume that the risk-free rate is constant. The process $\mathbf{W}_t = (W_{1t}, \dots, W_{st})$ are correlated Brownian motions, under Q , where each W_{it} has drift 0 and variance 1, and the i -th and j -th components have correlation ρ_{ij} . Note that each asset price follows a geometric Brownian motion process. Our set up is a generalization of the standard Black Scholes assumption to several underlying assets.

It is convenient to introduce some notation at this stage. Let $\theta_{it} = \log S_{it}, 1 \leq i \leq s$, then $\boldsymbol{\theta}_t = (\theta_{1t}, \dots, \theta_{st})$ is normally distributed with mean

$$\begin{aligned} \boldsymbol{\mu}_t &= (\mu_{1t}, \dots, \mu_{st}) \\ &= \left(\log S_{10} + (r - \delta_1 - \frac{1}{2}\sigma_1^2)t, \dots, \log S_{s0} + (r - \delta_s - \frac{1}{2}\sigma_s^2)t \right) \end{aligned} \quad (16)$$

and covariance matrix

$$\boldsymbol{\Sigma}_t = (\sigma_{ij}) = \rho_{ij}\sigma_i\sigma_j t. \quad (17)$$

The current price of any derivative security in this market is equal to the (discounted) expected value of its payoff under the Q measure. For example if the payoff on a European derivative security is $g(\mathbf{S}_T)$, where $g(\mathbf{S}_T) = g(S_{1T}, \dots, S_{sT})$ depends on the terminal asset prices (S_{1T}, \dots, S_{sT}) , the current price of this derivative security is

$$\begin{aligned} V_0 &= e^{-rT} \mathbb{E}_Q[g(\mathbf{S}_T)] \\ &= e^{-rT} \int_{\mathcal{A}(\mathbf{S}_T)} g(\mathbf{S}_T) f(\mathbf{S}_T) d\mathbf{S}_T \end{aligned} \quad (18)$$

where $\mathcal{A}(\mathbf{S}_T) \in \mathbb{R}^s$ is the integration domain and $f(\mathbf{S}_T)$ is a s -variate lognormal distribution with probability density function

$$f(\mathbf{S}_T) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}_T|}(2\pi)^s S_{1T} \cdots S_{sT}} \exp \left\{ -\frac{1}{2} (\log \mathbf{S}_T - \boldsymbol{\mu}_T)' \boldsymbol{\Sigma}_T^{-1} (\log \mathbf{S}_T - \boldsymbol{\mu}_T) \right\}$$

where $\log \mathbf{S}_T = (\log(S_{1T}), \dots, \log(S_{sT}))$, $\boldsymbol{\mu}_T$ and $\boldsymbol{\Sigma}_T$ are defined in (16) and (17).

The price of a European derivative security therefore reduces to evaluating integrals (18). The complexity of the problem depends on the structure of the payoff function $g(\mathbf{S}_T)$. It is only in rare cases that analytic solutions are available. In most situations, it is necessary to approximate (18) using numerical methods. We now turn to some specific examples and discuss different numerical methods of evaluating (18) in the context of these examples.

Our examples involve path dependent options based on a single underlying asset. It turns out that when the asset price follows geometric Brownian motion the option can be valued using the formula we developed in this section with a few minor adjustments. This is because when the asset price follows geometric Brownian motion the distribution of s monitoring points on the asset price path will also follow an s -dimensional multivariate lognormal distribution. Thus for the case of a discretely monitored lookback option with s observation points, we have an option whose payoff is a function of s correlated jointly lognormal variates.

4 Lookback Options

This subsection discusses the valuation of lookback options. More specifically, we consider a discrete fixed-strike lookback call option with the following payoff structure:

$$\max[\max(S_1, S_2, \dots, S_s) - K, 0]$$

where S_i denotes the price of the underlying asset at time t_i and $0 < t_1 < \dots < t_s = T$ are the s discrete monitoring time points. When the sampling frequency of the assets becomes continuous, the value of the corresponding option reduces down to simple Black-Scholes type expression as shown in Goldman, Sosin and Gatto (1979) and Conze and Viswanathan (1991). The discrete sampling case has been considered by Heynan and Kat (1995) and Dufresne, Keirstead and Ross (1996). The analytic expression for the value of the option in this case can be expressed in terms of multivariate normal probabilities as follows:

$$V_L(S_o, K, r, \delta, \sigma, T) = \sum_{i=1}^s H_i I_{n-i} S_o e^{(r-\delta)t_j - rT} - K e^{-rT} (1 - L), \quad (19)$$

where

$$\begin{aligned}
H_i &= N_i\left(\frac{\alpha_1}{\sigma}\sqrt{t_i - t_1}, \dots, \frac{\alpha_1}{\sigma}\sqrt{t_i - t_{i-1}}, d_1(X, t_i), R_i\right), \\
I_{s-i} &= N_{s-i}\left(-\frac{\alpha_2}{\sigma}\sqrt{t_{i+1} - t_i}, \dots, -\frac{\alpha_2}{\sigma}\sqrt{t_s - t_i}, \Psi_{s-i}\right), \\
L &= N_s(-d_2(K, t_1), \dots, -d_2(K, t_n), \Sigma) \\
d_1(K, t_i) &= \frac{\log(S_0/K) + (r + \sigma^2/2)t_i}{\sigma\sqrt{t_i}}, \\
d_2(K, t_i) &= d_1(X, t_i) - \sigma\sqrt{t_i}, \\
\alpha_1 &= r - \delta + \frac{\sigma^2}{2}, \\
\alpha_2 &= \alpha_1 - \frac{\sigma^2}{2}, \\
\Sigma &= (\gamma_{ik})_{n \times n}; \quad \gamma_{jk} = \sqrt{\frac{t_{j \wedge k}}{t_{j \vee k}}}, \quad 1 \leq j, k \leq n, \\
\Psi_{s-i} &= (\rho_{jk}^{(i)})_{(s-i) \times (s-i)}; \quad \rho_{jk}^{(i)} = \sqrt{\frac{t_{i+j \wedge k} - t_i}{t_{i+j \vee k} - t_i}}, \quad 1 \leq j, k \leq s-i, \\
R_i &= (r_{jk}^{(i)})_{i \times i}; \quad r_{jk}^{(i)} = \sqrt{\frac{t_i - t_{j \vee k}}{t_i - t_{j \wedge k}}}, \quad 1 \leq j, k \leq i-1, \\
&\quad r_{ij}^{(i)} = r_{ji}^{(i)} = \sqrt{\frac{t_i - t_j}{t_i}}, \quad 1 \leq j \leq i-1,
\end{aligned}$$

and N_m is the m -variate normal probability.

Hence pricing a lookback option using (19) boils down to evaluating different multivariate normal probabilities. There are very efficient algorithms for computing s -variate normal probability for small values of s . For high dimensional cases this can be a challenging numerical problem. See, for example, Genz (1992) for various approaches to this problem.

In this paper, we consider the relative efficiency of evaluating (19) using random points, low discrepancy Sobol' points and g.l.p.. Based on the technique suggested by Genz (1992), the Appendix discusses a sequence of transformations in order to convert the region of integration to the s -dimensional hypercube.

For the g.l.p., we evaluate the option prices with and without periodization. For the periodization, we consider the sin-transformation as discussed earlier. Comparing these two results allow us to assess the impact on periodizing the integrands. Table 1 reports the results. In this comparison, we consider 5-year lookback call options with annual monitoring and with the following parameter values $S_0 = 100$, $r = 10\%$, $\delta = 0$, $\sigma \in \{20\%, 30\%, 40\%\}$ and $K \in \{100, 110, 120\}$. For both random and g.l.p. points, we take $N = 562$ and 1142. The optimal \mathbf{z} correspond to these point sets are given in Hua and Wang (1981). For Sobol', we use $N = 512$ and 1024. We intentionally use a slightly different value of N for Sobol' since Sobol' sequences is a (t, s) -sequence in base 2 and has a better uniformity property when N is a power of 2. For each set of parameter values, we replicate the simulation independently

10 times using the random shift method discussed earlier in order to obtain an estimate of the standard error. Based on our numerical results we draw the following conclusions:

- In terms of numerical efficiency the g.l.p. method outperforms the Sobol'-based quasi-Monte Carlo method, which in turn is more efficient than the Monte Carlo method.
- The improvement in the efficiency of the g.l.p. approach when we use periodization is very striking. This improvement is amplified when we use more points. This is to be expected since the g.l.p. are designed to exploit the additional smoothness of the functions. In our examples with $N = 562$ points the standard errors in the g.l.p. approach with periodization are about 200 times smaller than those for the basic Monte Carlo approach, while without periodization they are about 16 times smaller. However when we use 1142 points, the standard errors for the g.l.p. approach with periodization are between 1000 and 3000 times smaller than those from standard Monte Carlo. Without the periodization, the same set of nodes only achieve an efficiency ratio of around 20.
- A feature common to both Sobol'-based and g.l.p.-based is that the efficiency of the method deteriorates as we increase the volatilities and/or the strikes.
- Note that we only consider at-the-money and out-of-the-money cases. This is because the in-the-money option can be formulated as an at-the-money case as follows: for $S_0 > K$,

$$\max[\max(S_0, S_1, \dots, S_s) - K, 0] = S_0 - K + \max[\max(S_1, \dots, S_s) - S_0, 0].$$

In the next phase of comparison, we consider the computation of option sensitivity parameters commonly known as the *Greeks*. For our illustration, we concentrate on the delta Δ , gamma Γ and vega \mathcal{V} . Formally, these Greeks are defined as:

$$\begin{aligned}\Delta &= \frac{\partial V}{\partial S}, \\ \Gamma &= \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}, \\ \mathcal{V} &= \frac{\partial V}{\partial \sigma},\end{aligned}$$

where V is the value of the derivative security. Consequently, the Greeks of a lookback option can be derived by taking the appropriate partial derivative in (19). Hence the option sensitivities are also expressed in terms of multiple integrals. The various sampling approaches considered earlier can be used to estimate the Greeks. Table 2 provides the comparison using the at-the-money discrete lookback options in Table 1 with $\sigma = 20\%$ and 30% . For random points and g.l.p. (with periodization), we use $N = 1142$ and 5003 while for Sobol' points, we consider 1024 and 4096 points. Similarly, the standard errors are estimated based on 10 independent replications. The conclusions are similar to the previous example. The method of g.l.p. is again the most efficient, with efficiency ratio in some cases over a thousand relative to the Monte Carlo method.

σ	K	Std.		Sobol'	Good Lattice Points	
		Random	Error		Periodized	Not Periodized
Random and g.l.p. use $N = 562$ while Sobol' $N = 512$						
0.2	100	47.027	0.421	47.373(5.3)	47.317(1000.6)	47.333(17.7)
	110	41.543	0.469	41.952(5.3)	41.884(290.3)	41.897(16.8)
	120	36.421	0.540	36.931(5.4)	36.843(106.8)	36.852(15.8)
0.3	100	56.861	0.496	57.336(5.7)	57.261(163.3)	57.276(17.3)
	110	51.864	0.535	52.397(5.7)	52.312(124.4)	52.323(16.6)
	120	47.230	0.582	47.838(5.6)	47.744(114.8)	47.748(15.8)
0.4	100	68.234	0.529	68.756(5.8)	68.680(193.3)	68.694(16.8)
	110	63.627	0.560	64.196(5.8)	64.114(205.2)	64.125(16.2)
	120	59.350	0.594	59.975(5.7)	59.887(220.6)	59.894(15.7)
Random and g.l.p. use $N = 1142$ while Sobol' $N = 1024$						
0.2	100	47.087	0.312	47.328(17.0)	47.318(2863.5)	47.314(17.9)
	110	41.644	0.348	41.900(16.8)	41.887(1952.9)	41.885(17.7)
	120	36.579	0.393	36.868(16.1)	36.850(1434.7)	36.849(18.0)
0.3	100	56.964	0.365	57.282(18.4)	57.265(1985.5)	57.259(19.9)
	110	51.996	0.394	52.334(17.8)	52.315(1568.2)	52.309(20.4)
	120	47.398	0.426	47.765(17.1)	47.744(1270.4)	47.738(21.1)
0.4	100	68.341	0.391	68.699(18.1)	68.681(1663.3)	68.673(21.3)
	110	63.755	0.414	64.131(17.7)	64.114(1302.3)	64.104(21.7)
	120	59.502	0.439	59.902(17.3)	59.884(1117.3)	59.874(22.2)

Table 1: Estimates of the Prices of Lookback Options based on Monte Carlo, Sobol' points and g.l.p.. The options evaluated are 5-Year lookback call with annual monitoring, $S_0 = 100$, $r = 10\%$, $\delta = 0$, $\sigma \in \{20\%, 30\%, 40\%\}$ and $K \in \{100, 110, 120\}$. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The Monte Carlo standard errors are reported in the fourth column. The values in parenthesis are the efficiency ratios as measured by the ratio of the standard error of the basic Monte Carlo method to the standard error of the method.

To complete our analysis, we also present two other results as reported in the last two columns of Table 2. These results are based on a widely used approach known as re-simulation.³ Let $\hat{V}(S)$ be the simulation estimate of the price of the option based on N simulation runs and for a given level of S . Suppose we are interested in estimating Δ at the initial asset price S_o . Using the forward finite difference method, delta can be approximated as

$$\Delta = \frac{\hat{V}(S_o + h) - \hat{V}(S_o)}{h}$$

for a small perturbation h . The above approximation represents a crude way of estimating the first derivative. An enhanced formula is to use the extended central difference formula as suggested Shamma (1995):

$$\Delta = \frac{\hat{V}(S_o - 2h) + 8\hat{V}(S_o - h) + 8\hat{V}(S_o + h) - \hat{V}(S_o + 2h)}{12h}.$$

To estimate vega, the above formula applies except that we perturb the volatility parameter, instead of the asset price. Similarly, the second derivative can be estimated as

$$\Gamma = \frac{-\hat{V}(S_o - 2h) + 16\hat{V}(S_o - h) - 30\hat{V}(S_o) + 16\hat{V}(S_o + h) - \hat{V}(S_o + 2h)}{12h^2}.$$

The results in the last two columns of Table 2 are based on these refined formulas. We consider both the random points and Sobol' points and in each case, we generate 10 independent batches with each replication consists of $N = 2^{20} = 1,048,576$ re-simulation. The values in parenthesis are the ratios of the standard errors from Monte Carlo-based direct evaluation of multiple integrals to those from the re-simulation approach (based on random or Sobol' points). Here we have intentionally used a much larger set of simulation runs in order for this method to be comparable to that based on direct evaluation of multiple integrals. In fact even with such enormous set of points, the efficiency gain in the case of Monte Carlo methods is only marginal, particularly with estimating Γ . It is well known that the re-simulation approach can be very inefficient for estimating the second derivative. The comparison favors the direct evaluation of multiple integrals approach even more if we take into account the computational time. Based on our implementation, the reported delta based on re-simulation is approximately 270 times more time consuming than that using the direct evaluation of multiple integrals with $N = 1142$.

The re-simulation technique combining with Sobol' points however, suggests that a substantial variance reduction can be achieved. (See last column of Table 2.) The Sobol'-based re-simulation nevertheless still performs badly in estimating Γ . The gains in efficiencies also become insignificant when taking the computational time into consideration and also when compared to the method of g.l.p. with a mere point set of 5003.

³See Broadie and Glasserman (1997) for comprehensive treatment of this subject along with other more efficient approaches.

σ	MC	Std. Error	Sobol'	g.l.p.	Re-simulation		
					MC	Sobol'	
— MC and g.l.p. use $N = 1143$, Sobol' $N = 1024$							
Δ	0.2	1.0317	0.00086	1.0333(2.81)	1.0334(70.82)	1.0335(6.75)	1.0334(102.86)
Δ	0.3	1.0843	0.00117	1.0859(8.34)	1.0859(106.09)	1.0861(5.51)	1.0859(70.95)
Γ	0.2	0.0023	0.00015	0.0031(5.96)	0.0031(99.33)	0.0030(4.03)	0.0031(4.81)
Γ	0.3	0.0035	0.00007	0.0036(10.78)	0.0036(64.15)	0.0035(1.80)	0.0036(1.76)
\mathcal{V}	0.2	87.34	0.88	88.41(8.36)	88.32(166.61)	88.34(14.69)	88.32(146.54)
\mathcal{V}	0.3	107.73	0.37	108.26(10.47)	108.24(120.24)	108.24(5.63)	108.24(34.29)
— MC and g.l.p. use $N = 5003$, Sobol' $N = 4096$							
Δ	0.2	1.0334	0.00073	1.0335(5.10)	1.0334(1211.4)	1.0335(5.70)	1.0334(86.62)
Δ	0.3	1.0857	0.00087	1.0860(8.83)	1.0859(312.27)	1.0861(4.08)	1.0859(52.59)
Γ	0.2	0.0030	0.00010	0.0031(8.73)	0.0031(1306.3)	0.0030(2.54)	0.0031(3.07)
Γ	0.3	0.0035	0.00005	0.0036(20.43)	0.0036(209.82)	0.0035(1.25)	0.0036(1.22)
\mathcal{V}	0.2	87.96	0.48	88.29(16.32)	88.37(825.3)	88.34(7.92)	88.32(79.14)
\mathcal{V}	0.3	108.20	0.26	108.24(39.09)	108.24(391.22)	108.24(3.90)	108.24(23.80)

Table 2: Estimates of Greeks: $\Delta, \Gamma, \mathcal{V}$. The parameter values for the options are 5-Year lookback call with annual monitoring, $S_0 = 100, r = 10\%, \delta = 0, K = 100, \sigma \in \{20\%, 30\%\}$. The estimates of Greeks in columns 3, 5 and 6 are based on direct evaluation of integrals using Monte Carlo, Sobol' and g.l.p. points. The last two columns are based on re-simulation using random and Sobol' points with $N = 1,048,576$. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The Monte Carlo standard errors are reported in the fourth column. The values in parenthesis are the efficiency ratios as measured by the ratio of the standard error of the basic Monte Carlo method to the standard error of the method.

5 Application to Equity-Indexed Annuities

Our next example concerns the valuation of the option embedded in the equity indexed annuities (EIA). These contracts have been very popular in the United States in recent years. (for details see Marrion, 2000a, 2000b). A typical EIA allows the contract holder to benefit from the growth of a stock index such as S&P 500 in addition to providing a basic guaranteed return. Different methods are used to tie the benefits under the contract to the performance of the index and these are known as indexing methods. One popular indexing method is known as the high water mark method. In this case the contract holder owns a fixed strike lookback option on the underlying index. Typically high water mark contracts have discrete monitoring: normally on a yearly basis. Since the maturity of an EIA generally is in the range from 3 to 10 years, this implies that the price of the discrete lookback option can be efficiently calculated using the g.l.p. method as we saw in the previous subsection.

We now consider in greater detail an EIA with high-water-mark indexing. We use the following notation:

P	amount invested initially, i.e. the pure premium,
T	the time to maturity of the contract from inception at time zero,
S_0, S_1, \dots, S_T	index levels at times $0, 1, 2 \dots T$,
g	the minimum guarantee rate compounded continuously,
β	proportion of the pure premium to be accumulated at the minimum interest rate,
α	the participation rate in the growth of the index.

Using these notations, the maturity value of this EIA can be represented as

$$\max \left[\beta P e^{gT}, P + \alpha P \max_{j=0,1,\dots,T} \left\{ \frac{S_j}{S_0} - 1 \right\} \right]. \quad (20)$$

The first component gives the required minimum guarantee at time T while the second component captures the contribution of high-water-mark feature. Rearranging the above expression yields

$$P \beta e^{gT} + \frac{P}{S_0} \alpha \max \left[\max_{j=0,1,\dots,s} S_j - \frac{S_0 \beta e^{gT} - (1 - \alpha) S_0}{\alpha}, 0 \right] \quad (21)$$

The first term arises from the minimum guarantee while the second term provides an explicit link to the lookback options considered in the previous subsection. From the above formulation, it is also obvious that issuing a high-water-mark EIA is equivalent to taking the following two positions simultaneously:

- invest in a T -year zero-coupon bond with initial amount $P \beta e^{(g-r)T}$ which yields a risk-free return of r . At maturity, this bond is worth $P \beta e^{gT}$ which coincides with the minimum guarantee.

- long $\frac{P}{S_0} \alpha$ units of T -year lookback call option with yearly discrete sampling and strike price $\frac{S_0 \beta e^{gT} - (1-\alpha)S_0}{\alpha}$.

From the no-arbitrage pricing theory, the total cost of investing an amount P in this EIA with initial index level S_0 must be equal to the sum of the above two positions.

We use the same parameter values as Lin (1999) who proposed an approximation algorithm for pricing EIA with high-water-mark features. For the first case, the parameter values are

$$S_0 = P = 100, r = 6\%, \delta = 0, \sigma = 20\%, \beta = 90\%, T = 5 \text{ years.}$$

The second set of examples are similar to the first case except that the interest rate, the volatility, and the maturity of the EIA are increased to 10%, 30% and 7 years, respectively.

The results are reported in Tables 3 and 4 for a variety of participation rates $\alpha \in \{65\%, 70\%, \dots, 95\%, 100\%\}$. These results show that the g.l.p. method again outperforms both standard Monte Carlo and Sobol'-based quasi-Monte Carlo methods. Relative to the basic Monte Carlo method, the gain in efficiencies for the Sobol' points are around 5.9 and 19.2 for the 5-year and 7-year EIA, respectively. In contrast, the efficiency ratios based on g.l.p. are in the range 115 to 133.

α	Random	Std. Error	Sobol'	g.l.p.
1.00	35.910	0.5451	36.493 (5.9)	36.413 (133.0)
0.95	33.975	0.5194	34.537 (5.9)	34.455 (133.2)
0.90	32.041	0.4937	32.576 (5.9)	32.498 (130.0)
0.85	30.107	0.4680	30.615 (5.9)	30.541 (130.0)
0.80	28.173	0.4424	28.655 (5.9)	28.584 (130.1)
0.75	26.240	0.4167	26.695 (5.9)	26.629 (126.3)
0.70	24.308	0.3911	24.736 (5.9)	24.674 (126.2)
0.65	22.377	0.3655	22.779 (5.9)	22.720 (121.8)

Table 3: 5-Year EIA with $S_0 = P = 100, r = 6\%, \delta = 0, \sigma = 20\%, \beta = 90\%$. Both random and g.l.p. use 562 points while Sobol' is based on 512 points. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The third column shows the standard errors correspond to the Monte Carlo methods. For the Sobol' and g.l.p. point sets, we report the estimated prices as well as the efficiency ratios (the value in parenthesis) as measured by the ratios of the standard errors of the basic Monte Carlo method to the standard errors of the method.

α	Random	Std. Error	Sobol'	g.l.p.
1.0	65.417	0.5209	65.756 (19.2)	65.775 (124.0)
0.95	61.918	0.4959	62.241 (19.2)	62.259 (124.0)
0.90	58.420	0.4709	58.727 (19.2)	58.744 (120.7)
0.85	54.922	0.4459	55.213 (19.2)	55.231 (120.5)
0.80	51.426	0.4209	51.702 (19.2)	51.718 (120.3)
0.75	47.931	0.3959	48.191 (19.2)	48.207 (116.4)
0.70	44.438	0.3709	44.682 (19.2)	44.697 (115.9)
0.65	40.948	0.3460	41.176 (19.2)	41.190 (115.3)

Table 4: 7-Year EIA with $S_o = P = 100$, $r = 10\%$, $\delta = 0$, $\sigma = 30\%$, $\beta = 90\%$. Both random and g.l.p. use 2129 points while Sobol' is based on 2048 points. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The third column shows the standard errors correspond to the Monte Carlo methods. For the Sobol' and g.l.p. point sets, we report the estimated prices as well as the efficiency ratios (the value in parenthesis) as measured by the ratios of the standard errors of the basic Monte Carlo method to the standard errors of the method.

6 Conclusion

We have shown that good lattice points can be used to exploit the additional smoothness exhibited in low-dimensional financial instruments. This method outperforms other numerical techniques for the examples we looked at. We also show that the g.l.p. method provides a very efficient way for computing the Greeks. We document that the use of periodization greatly enhances the efficiency of the g.l.p. method. It is natural to ask if these efficiency improvements can be also be realized for higher dimensional problems. Our preliminary work in this direction indicates that periodization algorithms do not seem as effective when the number of dimensions gets beyond ten or so. However our research suggest that the most dramatic gains are for low dimensional problems where periodization yields the greatest benefit.

Appendix: Multivariate Normal Probabilities

In this appendix, we consider the evaluation of the multivariate normal probabilities of the form:

$$N_s(a_1, a_2, \dots, a_s, \Sigma) = N_s(\mathbf{a}, \Sigma) = \frac{1}{\sqrt{|\Sigma|(2\pi)^s}} \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_s} e^{-\frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}} d\mathbf{x}.$$

This s -dimensional integral is the core to the pricing of the multivariate European derivative securities. To evaluate the above integral, Genz (1992) proposed a transformation of variables that results in $N_s(\mathbf{a}, \Sigma)$ being written as an integral over an $(s - 1)$ -dimensional unit cube with the new integration variable $\mathbf{w} = (w_1, w_2, \dots, w_{s-1})$.

We now summarize the 3 sequences of transformations:

1. In the first step, let $\Sigma = \mathbf{C}\mathbf{C}'$ be the Cholesky decomposition of the covariance matrix and applying the transformation $\mathbf{x} = \mathbf{C}\mathbf{y}$, we obtain

$$N_s(\mathbf{a}, \Sigma) = \frac{1}{\sqrt{(2\pi)^s}} \int_{-\infty}^{b'_1(\mathbf{y})} e^{-\frac{y_1^2}{2}} \cdots \int_{-\infty}^{b'_s(\mathbf{y})} e^{-\frac{y_s^2}{2}} d\mathbf{y} \quad (22)$$

with $b'_i(\mathbf{y}) = (b_i - \sum_{j=1}^{i-1} c_{i,j}y_j)/c_{i,i}$

2. Now use the substitution $y_i = \Phi^{-1}(z_i)$, for $i = 1, \dots, s$ where $\Phi(y_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_i} e^{-\frac{t^2}{2}} dt$. The integral (22) becomes

$$N_s(\mathbf{a}, \Sigma) = \int_0^{e_1} \int_0^{e_2(z_1)} \cdots \int_0^{e_s(z_1, z_2, \dots, z_{s-1})} dz \quad (23)$$

where

$$e_i(z_1, \dots, z_{i-1}) = \Phi \left(\frac{b_i - \sum_{j=1}^{i-1} c_{i,j} \Phi^{-1}(z_j)}{c_{i,i}} \right).$$

3. The last procedure is to transform the integration region to $[0, 1]^s$. This can be achieved by using the transformation $z_i = e_i w_i$, for $i = 1, \dots, s$ so that $dz_i = e_i dw_i$ and

$$N_s(\mathbf{a}, \Sigma) = e_1 \int_0^1 e_2(\mathbf{w}) \cdots \int_0^1 e_s(\mathbf{w}) \int_0^1 d\mathbf{w} \quad (24)$$

where

$$e_i(\mathbf{w}) = \Phi \left(\frac{b_i - \sum_{j=1}^{i-1} c_{i,j} \Phi^{-1}(e_j(\mathbf{w})w_j)}{c_{i,i}} \right).$$

The inner most integral over the variable w_n can be calculated analytically since e_s is independent of this variable. Consequently the problem is reduced to the computation of an $(s - 1)$ -dimensional integral instead of the computation of an s -dimensional integral.

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