SOLVENCY STANDARDS IN AN INSURANCE COMPANY

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ABSTRACT

The purpose of the paper is to apply Peccati's approach about "certainty equivalent and time" (3) to Van Slyke's framework about solvency standards in an insurance company. Van Slyke (5) proposes the Risk-Adjusted Present Value as a solvency standard. The problem is that random cash flows at different future times are to be dealt with correctly from the methodological point of view. Consequently, the "certainty equivalent" of the future random value of the company is proposed as a more correct solvency standard. Van Slyke's approach does not hold in the general case. However it may be accepted in some cases, in which the framework we propose comes down to his procedure.

1. INTRODUCTION

In his paper, Van Slyke presents a framework for evaluating the financial conditions of an insurance company. The Risk Adjusted Present Value, or RAV, is proposed as a solvency standard. It is determined on the future random cash flows of the company but, unlike the present value of expected cash flows, it takes the psychology of the evaluator (i.e., his/her attitude to risk) into consideration(1). If \( (X_t) \) are the random after-tax cash flows for future years \( t = 1, 2, \ldots, \infty \), the RAV is equal to(2):

\[
RAV = \sum_{t=0}^{\infty} v_t RAV_t = \sum_{t=0}^{\infty} v_t u^{-1} \left[ E(u(X_t)) \right],
\]

(1) "While solvency itself is not technically impaired unless cash outflows exhaust funds available for payments, solvency could be said to be at risk whenever the risk-adjusted present value is negative for reasonable levels of risk aversion" (Van Slyke (5), p. 411).

(2) The analysis might be easily extended to the multi-scenario hypothesis, as in Van Slyke (p. 420). We do not consider the existence of multiple scenarios in order to make the framework more understandable by the reader.
where \( v_t \) is the discount factor for time \( t \) and \( u \) is the \( uN - M \) utility function of the evaluator.

Note that the \( RAV_t \) is the "certainty equivalent" of the cash flows at time \( t \) and the \( RAV \) is the total sum of the present values of the certainty equivalents. In a word, Van Slyke proposes to apply the Net Present Value, or NPV, technique to the certainty equivalents of the cash flows at the different times \( t \). If the NPV is positive the solvency is not judged at risk.

This framework cannot be accepted because:
- it is not correct to determine the NPV of future random amounts;
- the evaluation depends on the position the evaluator will have with respect to the company.

The core of the problem is that the NPV criterion does not hold when the amounts are random. In our opinion it is not correct to determine either the expected values or the certainty equivalents of the random cash flows at different future times and then calculate their present values, as pretending that they are certain amounts. Neither is it correct to determine the present value of random future amounts and, then, calculate either the expected value or the certainty equivalent of the total sum of the present values. In fact, the NPV can be calculated only when the future cash flows are certain.

Both the certainty equivalent and the expected value should be determined taking account of the random value of the company at a given figure time (the time we suppose, for instance, the business will be over, i.e., when the company will be closed or, more likely, sold).

We define the value of the company as the total sum of internal funds available today in the firm plus the random cash flows at the different future times, which are supposed to be reinvested at a random rate of return until the end of the period.

Another fact is to be pointed out. The evaluation of the solvency depends on which position among those outlined by Peccati the evaluator will have with respect to the company. An internal evaluator's attitude to the problem would be the seller's in the mentioned framework,

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(3) "The discount factor \( v_t \) can be found by observing the current price of government-backed notes with various maturities, with adjustment for income taxes" (Van Slyke, p. 415).

(4) The certainty equivalent is defined in the sense of Keeney-Raiffa (1), chap. 4.
whereas an external evaluator would determine the certainty equivalent as the buyer would do.

The plan of the paper is as follows. In Section 2 we present a synthesis of Peccati’s framework whereas in Section 3 we introduce our framework and determine correct solvency standards for insurance companies.

2. PECCATI’S FRAMEWORK: A SYNTHESIS(5)

Given a random money amount $X$ (e.g., a lottery), the seller’s certainty equivalent, or SCE, is the solution $z_s$ of the equation in $z$:

$[\text{ii}] \quad E[u(X)] = u(z)$,

whereas the buyer’s certainty equivalent, or BCE, is the solution $Z_b$ of the equation in $Z$:

$[\text{iii}] \quad E[u(X - Z)] = u(0)$

where $u(x)$ is the individual's $vN - M$ utility function.

If $X$ is available at a future time $t$, the evaluation at time $\tau < t$ will be different. The SCE at time $\tau$ is the solution of the equation:

$[\text{iv}] \quad E[u(X)] = u(zf(\tau, t))$

whereas the BCE at time $\tau$ is the solution of the equation:

$[\text{v}] \quad E[u(X - Zf(\tau, t))] = u(0)$,

where $f$ is the financial factor used to transfer money amounts from one epoch to another(8).

(5) Peccati (3); (4), pp. 9–17. A mathematical systematization of the seller’s and of the buyer’s certainty equivalents may be found in La Valle (2).

(6) $z_s$ is the minimum selling price, i.e., the lowest price at which the sale of the random amount appears to the seller not to be unfavorable.

(7) $Z_b$ is the maximum buying price for the buyer of the lottery.

(8) For instance, it may be the market rate of interest.
If we consider a sequence $X = \{ (X_t, t) \}$ of random money amounts with maturities $t = 1, 2, \ldots, T$, the SCE at time $\tau < 1$ of $X$, denoted by $z_\tau$, is the certain amount which solves the equation in $z$:

\[[vi]\]
$$E\{u[z f(\tau, T)]\} = E \left\{ u \left[ \sum_{t=1}^{T} X_t f(t, T) \right] \right\}$$

whereas the equation in $Z$ giving the BCE $Z_\tau$ has a different form:

\[[vii]\]
$$u(0) = E \left\{ u \left[ \sum_{t=1}^{T} X_t f(t, T) - Z f(\tau, T) \right] \right\} ,$$

where $T$ is the horizon, i.e., the epoch at which the random amounts are transferred through a random financial factor $f$ with $f(x, z)f(z, y) = f(x, y)$ $\forall z \in [x, y]$; and $\tau < 1$ is the evaluation time.

Peccati shows that if $u$ is strictly monotone and continuous, equation [vi] possesses a unique solution and the BCE for $X$ exists and is unique ((3), pp. 5 and 15).

It is important to point out that the existence of both the SCE and the BCE does not mean that they are equal to each other. In fact, in general, the SCE and the BCE are different. Peccati shows that:

\[[viii]\]
$$z_\tau = Z_b$$

when the utility function is exponential ((3), p. 20).

3. The Mathematical Framework We Propose

3.1. The General Case

We consider the random after-tax cash flows $\{X_t\}$ for a given set of future years $t = 1, 2, \ldots, T$ (9), that may be reinvested by the company at random conditions with financial factor $f(x, y)$ (10) until the end of the period. A certain fraction $\alpha_t \in [0, 1]$ may be paid as a dividend to the shareholders.

\[\text{Note that we consider a finite set of future years.}\]
\[\text{With } f(x, z)f(z, y) = f(x, y) \forall z \in [x, y].\]
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We suppose that \( u(x) \), the evaluator's utility function, is continuous and monotonically increasing in \( R \); that every evaluator (either internal or external) agrees on the probability distributions of the future cash-flows and on the reinvestment opportunities of the firm; and that the utility function is the same for every evaluator.

The random value of the company at time \( T \) will be:

\[
V_T = \sum_{t=1}^{T} X_t(1 - \alpha_t)f(t, T) + P_r f(\tau, T)
\]

where \( P_r \) are the internal funds available in the firm at time \( \tau < 1 \), when the evaluation is made.

Both the expected value and the certainty equivalent determined on the base of expression [1] may be taken as a solvency standard. The certainty equivalent is to be preferred because it takes the psychology of the evaluator into consideration.

The expected value of the company at time \( \tau < 1 \) is the solution to:

\[
E[V_T f(\tau, T)] = E(V_T)
\]

and we get:

\[
V_\tau = \frac{\{E(V_T)\}}{E[f(\tau, T)]}.
\]

The certainty equivalent \( z_\tau \) at time \( \tau \) of \( V_T \) from the internal evaluator's view-point is the solution to the equation in \( z \):

\[
E[u(z f(\tau, T))] = E[u(V_T)],
\]

As for neutral predictions for cash flows in various times, Van Slyke states that "to the extent the company has cash inflows exactly equal to all cash outflows for various times [...], the net cash flow is zero and the company has removed the element of risk from both the asset portfolio and the portfolio of obligations" (p. 417). We remark that a neutral prediction is as risky as a prediction of positive or negative cash flows. Risk cannot be removed unless the business gives a certain cash flow, i.e., unless the business is risk free.

Of course, if there were not agreement on the probability distributions and on the reinvestment opportunities and if \( u \) were different among evaluators, the valuations would not coincide. But we want to point out that, *ceteris paribus*, the valuation depends on the position the evaluator will have with respect to the company.

They are the past cash flows (net of the dividend paid to the shareholders) reinvested at random conditions up to time \( \tau \).
which possesses a unique root, whereas the certainty equivalent $Z_\tau$ from an external evaluator's viewpoint is the solution to the equation in $Z$:

$$[5] \quad E[U(V_T - 2f(\tau, T))] = U(0),$$

for which again there is a unique root.

Note that $z_\tau$ is equal to $Z_\tau$ in one special case, i.e., when the evaluators have the same exponential utility function.

### 3.2. Certain Reinvestment Conditions

In the case of certain reinvestment conditions\(^{(14)}\) equation [4] can be solved as follows:

$$[6] \quad z_\tau = \phi(T, \tau)u^{-1}\left\{ E\left[u\left(\sum_{t=1}^T X_t(1 - \alpha_t)f(t, T) + P_\tau f(\tau, T)\right)\right]\right\},$$

whereas no interesting simplification is possible in the case of equation [5]\(^{(15)}\).

Equation [6] states that the certainty equivalent at time $\tau$ is equal to the present value of the certainty equivalent at time $T$, i.e., $z_\tau = \phi(T, \tau)z_T$.

### 3.3. Risk Neutrality

When the evaluator is risk-neutral, i.e., the utility function is linear, equation [4] becomes:

$$[7] \quad E[zf(\tau, T)] = E(V_T).$$

\(^{(14)}\)With $f(x, y) = f(x, y)$ certain and $1/f(x, y) = \phi(y, x)$, i.e., there is certainty about the financial factor of the company reinvestment opportunities whereas the company cash flows are still supposed to be random.

\(^{(15)}\)Note that $P_\tau f(\tau, T)$ is a certain amount. Van Slyke states that "amounts that are absolutely certain may be handled outside the RAV calculation" (p. 118). This sounds incorrect. If the consequence of a lottery is certain, its certainty equivalent must be equal to the consequence and, thus, there is no reason not to consider certain amounts inside the calculation.
The internal evaluator’s certainty equivalent at time $\tau$ is:

$$z_\tau = \frac{E(V_T)}{E[f(\tau,T)]} = \sum_{t=1}^{T} \frac{E[X_t(1 - \alpha_t)f(t,T)]}{E[f(\tau,T)]} + P_\tau,$$

which is the present value of $z_T$, the certainty equivalent at time $T$. The external evaluator’s certainty equivalent $Z_\tau$ is the solution to:

$$E[V_T - Zf(\tau,T)] = 0.$$

It can be easily seen that:

$$z_\tau = Z_\tau.$$

3.4. CERTAIN REINVESTMENT CONDITIONS AND RISK NEUTRALITY

If we consider both hypotheses of certain reinvestment conditions and of risk neutrality we reduce to Van Slyke’s framework.

In fact for the internal evaluator we get:

$$zf(\tau,T) = \sum_{t=1}^{T} E(X_t(1 - \alpha_t)f(t,T) + P_\tau f(\tau,T))$$

and the solution is:

$$z_\tau = \sum_{t=1}^{T} E(X_t)(1 - \alpha_t)f(t,\tau) + P_\tau.$$

For the external evaluator we get:

$$E \left\{ \sum_{t=1}^{T} X_t(1 - \alpha_t)f(t,T) + P_\tau f(\tau,T) - Zf(\tau,T) \right\} = 0$$

and the solution is:

$$Z_\tau = \sum_{t=1}^{T} E(X_t)(1 - \alpha_t)f(t,\tau) + P_\tau$$

which is obviously equal to $z_\tau$. As in Van Slyke’s paper we derive that the certainty equivalent at time $\tau$ of the value of the firm is the total sum of the present values of the cash flows at the different times $t$. 
3.5. A SPECIAL CASE

Note that under risk aversion (and proneness) only in one special case – i.e., when there is a unique random cash flow \( X \) at time \( t \) to evaluate – we get for the internal evaluator:

\[
I_{\tau} = \phi(t, \tau)u \{ E[u(X(1-\alpha)) + P\tau f(\tau, t)] \} = \phi(t, \tau)z_t .
\]

Expression [15] states that the time \( \tau \) certainty equivalent is equal to the present value of the time \( t \) certainty equivalent of the cash flow. The same result does not hold for the external evaluator.

3.6. CONCLUSIONS

In the general case – i.e., when we are in a multi-period scenario – it is not correct to determine the certainty equivalent for each time, to calculate the present value and then to consider the total sum of the present values as a solvency standard. Calculations should be made on the random value of the company determined at a given future time.

In conclusion, in this paper we show how to determine correct solvency standards taking into consideration the future cash flows of an insurance company. Note that the analysis could be extended to every company, whatever its business.

4. COMMENTS

In this paper we show a correct framework to evaluate, adapting the new way suggested by Van Slyke, the solvency of an insurance company. The certainty equivalent determined on the future random value of the company is taken as a solvency standard: if it is positive, the solvency is not judged at risk. The valuation will depend not only on the psychology of the evaluator but also on his/her position (either internal or external) with respect to the company. Van Slyke’s framework holds only under the hypotheses of certain reinvestment conditions and of risk neutrality of the evaluator: in this case the solvency standard is equal to the total sum of the present values of the certainty equivalents at the different future times. Under risk aversion (or proneness) we can think of a present value of the certainty equivalents of the cash flows only when there is a unique random cash flow at a future time.
BIBLIOGRAPHY


