

## THE YIELD CURVE AND BOND PORTFOLIO IMMUNIZATION

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### SUMMARY

Choosing the periodicity for reinvesting funds coming from a bond portfolio is a subject scarcely treated by the theory of portfolio management.

One year is the reference date usually chosen when calculating the yield rate of a portfolio. Published papers related to this subject show a concern about the fact of this rate being constant or not along the investment horizon. However, I believe that the problem arisen by the variability in the yield rate during the year has not been treated with enough accuracy.

When one considers both the reinvestments of resources and the rates variations during the year, it appears that the theory of portfolio immunization based on the duration must be reformulated in order to provide the best strategy for immunization.

In this paper, I have tried to study the effects produced by the increase in the frequency on the yield curve, and how do they affect the modifications which result in the immunization policy.

I do not think that we are dealing with a closed subject. On the contrary, I am convinced that, should this line of thought be accepted, we should get more deeply into it in order to attain more complete formulations of the bond portfolio immunization theory based on the duration.

## **La courbe des rendements et l'immunisation de portefeuilles d'obligations**

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### **Résumé**

Le choix de la périodicité pour le ré-investissement de fonds provenant d'un portefeuille d'obligations est un sujet rarement traité par la théorie de la gestion des portefeuilles.

Un an constitue la période de référence généralement retenue pour le calcul du taux de rendement d'un portefeuille. Les documents publiés relativement à ce sujet soulèvent la question de savoir si ce taux est constant ou non dans le cadre de l'investissement. Je crois cependant que le problème soulevé par la variabilité dans le taux de rendement au cours de l'année n'a pas été traité de façon suffisamment précise.

Lorsque l'on considère à la fois le ré-investissement des ressources et les variations des taux au cours de l'année, il apparaît qu'il faille reformuler la théorie de l'immunisation du portefeuille fondée sur la durée de façon à permettre la meilleure stratégie d'immunisation possible.

Dans le présent document, j'ai tenté d'étudier les effets produits par l'augmentation de la fréquence sur la courbe du rendement et la façon dont ils affectent les modifications en résultant dans la politique d'immunisation.

Je ne pense pas que nous ayons affaire à une question résolue une fois pour toutes. Je suis convaincu, au contraire que, si nous acceptons ce raisonnement, il faudrait l'étudier de façon plus approfondie pour pouvoir obtenir des formulations plus complètes de la théorie de l'immunisation des portefeuilles d'obligations fondée sur la durée.

## THE YIELD CURVE AND BOND PORTFOLIO IMMUNIZATION

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When the duration is applied to bond portfolio management and, specially, when it is used to determine the best immunization policy for the accumulated value and/or for the obtained yield rate, there are some restrictive hypothesis which, if not interpreted correctly, may even invalidate the conclusions that could be inferred.

Some of these hypothesis basically affect the yield to maturity, so that if we denote by  $F_1, F_2, \dots, F_n$ , the cash flow of the portfolio, the present value at the rate  $i$  is:

$$V_0(i) = \sum_{t=1}^n F_t(1+i)^{-t} \quad (1)$$

The year is the unit of time usually chosen, which requires  $\{F_t\}$  and  $i$  referring to such unit; it is assumed that all the reinvestments have been done at the same rate  $i$ . This is expressed by saying that the yield curve is flat and, finally, that the yield rate  $i$  is obtained if the investment in the portfolio is maintained until the end of the estimated horizon. Under such conditions, the very best immunization policy considers that the investment horizon (the remaining period of time for owning the portfolio) has to meet always the duration of the portfolio:  $D(i)$ , with a value that can be calculated using the Macaulay formula:

$$D(i) = \frac{\sum_{t=1}^n t \cdot F_t \cdot (1+i)^{-t}}{V_0(i)} \quad (2)$$

The required adjustments have to be made every year, so that:

*Duration of the portfolio = Remaining time to complete the horizon h*

should always apply.

Following this policy, we obtain that if the yield rate  $i$  varies as a consequence of a modification in the interest rates and/or of the market yield, the portfolio manager is sure that the accumulated value of the investment in such a portfolio is immunized (against variations of the above mentioned rates), so that if he goes on with this policy, he will obtain at least, after  $h$  years, the value:

$$V_h(i) = V_0(i) \cdot (1+i)^h \quad (3)$$

this value being the minimum of the function  $V_h(i')$ , with  $i'$  the new yield rate of the portfolio. The demonstration of this proposition may be found in many papers, being the most commonly accepted reference the paper by Fisher and Weil (1971), which first offered an accurate and complete demonstration of the immunization theorem of the accumulated value of a portfolio.

Fisher and Weil admit and base their analysis on the hypothesis that the annual yield rates are not constant along the investment horizon, so that if, following their nomenclature, we assign  $r_1, r_2, \dots, r_n$ , to the annual yield rates<sup>1</sup> which are in force during the first, second, ...,  $n$  years, we may calculate the duration of an asset with a cash flow that should be  $F_1, F_2, \dots, F_n$ , as follows:

We first calculate the actual value of this flow which will be:

$$V_0 = F_1(1+r_1)^{-1} + F_2 \left[ (1+r_1)(1+r_2) \right]^{-1} + \dots + F_n \left[ (1+r_1) \dots (1+r_n) \right]^{-1} \quad (4)$$

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<sup>1</sup> Such yield rates may be calculated applying the annual yield curves. See, among others, Hauguen (1986) Chapter 14.

Secondly, we apply the concept of duration to this cash flow, obtaining:

$$D_{FW} = \frac{1}{V_0} \cdot \left[ 1 \cdot F_1(1+r_1)^{-1} + 2 \cdot F_2 \left[ (1+r_1)(1+r_2) \right]^{-1} + \dots + n \cdot F_n \left[ (1+r_1) \dots (1+r_n) \right]^{-1} \right] \tag{5}$$

Although it is a more general formula than (2), the application of (5) to bond portfolios, may arise the same (or similar) problems that Fisher and Weil tried to solve, so that, as we may see in this paper, almost the same mistakes are made than when using directly the Macaulay duration formula.

#### CASH FLOW PERIODICITY AND THE YIELD CURVE

The immunization policy based on the Macaulay duration has (among others) the following restrictions:

- 1) the (annual) yield curve is flat
- 2) the yield to maturity is the same for all the securities

These two restrictions imply that everything is reinvested at the same rate and that the various items of the cash flow are reinvested yearly. However, as it happens in a portfolio, if the income is not produced in a regular and periodic way, but with irregularity, which is the meaning of the annual yield rate  $i$ ?; and more specifically, which is the reference unit of time of the internal yield rate?

The duration <sup>2</sup> has to be always calculated at the yield to maturity, so that equation (1) must be fulfilled. This means that the unit of time of  $F_t$  and  $i$  is the same. If  $\{F_t\}$  stands for the cash flow of a portfolio, this has to be reinvested immediately, so that the yield along the investment horizon should be really  $i$ .

As we already discussed, such income is not produced in a periodic way. Thus, it seems logical to think that the portfolio manager will accrue for and invest

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<sup>2</sup> In what follows we will refer to the Macaulay duration as the duration; when we will need to use other duration formulae, we will particularly mention it.

regularly at the current yield rates of the market; this means that  $i$  will not have an annual periodicity; we cannot imagine that the manager had to wait days, weeks, or even months to be able to invest the amounts coming from the cash flow  $\{F_t\}$ .

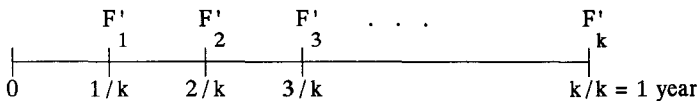
Let  $1/k$  be the proportional part of the year in which the manager makes the net reinvestments; and let  $i^{(k)}/k$  be the effective interest rate per  $k$ th of the year. When  $i^{(k)}/k$  is constant, it will be fulfilled that:

$$\left[ 1 + \frac{i^{(k)}}{k} \right]^k = 1 + i$$

If we denote by  $\{F'_h\}$ ,  $h = 1, 2, \dots, nk$ , the cash flow per  $k$ th of the year, the actual value of such investment at the rate  $i^{(k)}/k$  will be:

$$V_0 \left[ \frac{i^{(k)}}{k} \right] = \sum_{h=1}^{nk} F'_h \cdot \left[ 1 + \frac{i^{(k)}}{k} \right]^{-h} \tag{6}$$

In this case, to obtain the complete equivalence between (6) and (1) we will have to take into account that  $F_1$  of (1) should be the accumulated value of  $F'_1, F'_2, \dots, F'_k$  at the rate  $i^{(k)}/k$ , as it appears in the following diagram:



which gives rise to the equation:

$$F'_1 \cdot \left[ 1 + \frac{i^{(k)}}{k} \right]^{k-1} + F'_2 \cdot \left[ 1 + \frac{i^{(k)}}{k} \right]^{k-2} + \dots + F'_k = F_1 \tag{7a}$$

or:

$$F'_1 \cdot (1+i)^{1-1/k} + F'_2 \cdot (1+i)^{1-2/k} + \dots + F'_k = F_1 \tag{7a}$$

where we took into account that  $\left[1+i^{(k)}/k\right] = (1+i)^{1/k}$

In general, for any year t, it should be fulfilled that:

$$F'_{t-k+1} \cdot \left[1+\frac{i^{(k)}}{k}\right]^{k-1} + F'_{t-k+2} \cdot \left[1+\frac{i^{(k)}}{k}\right]^{k-2} + \dots + F'_t = F_t \tag{7b}$$

If any of the terms of the annual cash flow  $\{F_t\}$  were to be always equal to the accumulated value of the partial flows per kth of the year according to (7), the equivalence between (1) and (6) would be absolute. But, what happens if the interest rate changes from a kth to another in the same year?

Let  $i_1^{(k)}/k, i_2^{(k)}/k, \dots, i_k^{(k)}/k$ , be the various interest rates in force per kth of the year during the first year; then, it will obviously happen that:

$$\left[1+\frac{i_1^{(k)}}{k}\right] \cdot \left[1+\frac{i_2^{(k)}}{k}\right] \cdot \dots \cdot \left[1+\frac{i_k^{(k)}}{k}\right] = 1+i_1 \tag{8a}$$

and  $i_1$  will be the equivalent annual interest rate during the first year.

Let  $i_{k+1}^{(k)}/k, i_{k+2}^{(k)}/k, \dots, i_{2k}^{(k)}/k$ , be the various interest rates in force per kth of the year during the second year, and  $i_2$  its annual equivalent; we should have that:

$$\left[1+\frac{i_{k+1}^{(k)}}{k}\right] \cdot \left[1+\frac{i_{k+2}^{(k)}}{k}\right] \cdot \dots \cdot \left[1+\frac{i_{2k}^{(k)}}{k}\right] = 1+i_2 \tag{8b}$$

Finally, let  $i_{nk-k+1}^{(k)}/k, i_{nk-k+2}^{(k)}/k, \dots, i_{nk}^{(k)}/k$ , be the various rates in force per kth of the year during the nth of the year, and  $i_n$  its annual equivalent, so that we would have:

$$\left[1 + \frac{i_{nk-k+1}^{(k)}}{k}\right] \cdot \left[1 + \frac{i_{nk-k+2}^{(k)}}{k}\right] \cdot \dots \cdot \left[1 + \frac{i_{nk}^{(k)}}{k}\right] = 1 + i_n \tag{8c}$$

We can always find an annual interest rate  $i$  that is equivalent to all of them, for if we multiply (8a), (8b), ..., (8c), it appears that:

$$\left[1 + \frac{i_1^{(k)}}{k}\right] \cdot \dots \cdot \left[1 + \frac{i_k^{(k)}}{k}\right] \cdot \dots \cdot \left[1 + \frac{i_{nk-k+1}^{(k)}}{k}\right] \cdot \dots \cdot \left[1 + \frac{i_{nk}^{(k)}}{k}\right] = (1+i_1) \cdot \dots \cdot (1+i_n) \tag{9}$$

From it, we may calculate the annual interest rate equivalent to all of them:

$$(1+i_1) \cdot (1+i_2) \cdot \dots \cdot (1+i_n) = (1+i)^n$$

from which:

$$i = \left[ (1+i_1) \cdot (1+i_2) \cdot \dots \cdot (1+i_n) \right]^{\frac{1}{n}} - 1 \tag{10}$$

Interest rates may also be inferred from the rates per kth of the year:  $i_k^1/k, \dots, i_k^k/k, i_k^{k+1}/k, \dots, i_k^{2k}/k, \dots, i_k^{nk-k+1}/k, \dots, i_k^{nk}/k$ . The interest rate  $i^{(k)}/k$  equivalent to the annual  $i$  could be obtained from:

$$i^{(k)}/k = (1+i)^{\frac{1}{k}} - 1$$

Nevertheless, it is obvious that there is a difference between applying the constant interest rate  $i^{(k)}/k$  to an cash flow accrued per kth of the year, instead of



the real structure of the interest rates per kth:

$$i_1^{(k)}/k, \dots, i_k^{(k)}/k, \dots, i_{nk-k+1}^{(k)}/k, \dots, i_{nk}^{(k)}/k \tag{11}$$

and that the only possibility to get an equivalence among the present values of the same flow should be for the rates (11) to be constant.

When one speaks about the equivalence formula of the yield curve (and of interest) in financial literature, we do not usually refer to an equation like (9), which expresses the intra-annual and inter annual structure, which we shall call from now on generalized structure of the interest rates, but only to an inter annual structure. That is to say, if  $r_1, r_2, \dots, r_n$ , are the (annual) yield rates in force during the years 1, 2, ..., n, we have that:

$$(1+r_1) \cdot (1+r_2) \cdot \dots \cdot (1+r_n) = (1+r)^n \tag{12}$$

from which it is easy to infer the annual yield rate equivalent  $r$ .

**EXAMPLE**

To see more in detail the meaning and the relevance of what we just explained, let us consider the following example: a portfolio produces a quarterly flow of income during three years, which is shown in the second column of table 1; let us consider three scenarios of quarterly interest rates:

- a) random with a geometric mean approximately equal to 3 %.
- b) growing in arithmetic progression at the rate of 5 basic points with a geometric mean equal to 3.275 %.
- c) decreasing in arithmetic progression at the rate of 5 basic points with a geometric mean equal to 2.725 %.

Assuming that any item of the cash flow is invested always at the quarterly rate in force in the market, the present value of the portfolio is shown in the fourth column of the table, being the present values of the portfolio in each one of the

scenarios the following ones:

$$\text{a): } 629.90 ; \text{ b) } 621.68 ; \text{ c) } 637.26$$

However, if instead of calculating with the quarterly interest rates, we wish to apply the "equivalent" annual interest rate, the interest rates in each one of the scenarios may be obtained by applying equation (8c). Therefore, in scenario a), the annual interest rate is obtained taking into account that the rates in force are: 3%, 2.9%, 2.8%, and 3%, from which it appears that:

$$(1+0.03) \cdot (1+0.029) \cdot (1+0.028) \cdot (1+0.03) - 1 = 0.1222$$

which is the rate found in the fourth column called "annual interest rates", from which results a quarterly interest rate:

$$(1+0.1222)^{0.25} - 1 = 0.02925$$

which is the rate applied during the four quarters of the first year. The equivalent rates of the remaining months can be calculated in a similar way; in addition, we may check that the average quarterly interest rate found in any scenario is correct using a simple application of the associative property of the geometric mean:

$$\left[ \prod_{t=1}^n (1+r_t) \right]^{\frac{1}{n}} = \left[ \left[ \prod_{t=1}^4 (1+r_t) \right]^{\frac{1}{4}} \cdot \left[ \prod_{t=5}^8 (1+r_t) \right]^{\frac{1}{4}} \cdot \dots \cdot \left[ \prod_{t=n-3}^n (1+r_t) \right]^{\frac{1}{4}} \right]^{\frac{4}{n}}$$

which in our case would be:

$$\left[ \prod_{t=1}^{12} (1+r_t) \right]^{\frac{1}{12}} = \left[ \left[ \prod_{t=1}^4 (1+r_t) \right]^{\frac{1}{4}} \cdot \left[ \prod_{t=5}^8 (1+r_t) \right]^{\frac{1}{4}} \cdot \left[ \prod_{t=9}^{12} (1+r_t) \right]^{\frac{1}{4}} \right]^{\frac{1}{3}}$$

That is to say, the quarterly yield rates obtained in the fifth column of the table produce the same geometric mean as the quarterly rates of each one of the scenarios:

3.008% en el a); 3.275% en el b) y 2.725% en el c).

However, when these rates are applied to the cash flow, they produce different present values and, consequently, the yield to maturity is different, a serious mistake being made in the average annual yield rates in all the cases and in all the scenarios b) and c), as we may well see:

in a)	12.542% as opposed to 12.497%
in b)	12.955% as opposed to 13.359%
in c)	12.108% as opposed to 11.742%

Thus, it is demonstrated that to correctly consider the yield curve it is not enough to choose an annual rate equivalent to the rates per  $k$ th independently of the frequency at which  $\{F_t\}$  is produced. The equivalence equation has to take into account the frequency of the cash flow, adequately accruing it for, so that the yield rate (the only one) obtained were to be a real reflection of the actualized value of the portfolio. *Frequency is a decision variable, and in its election basically have an influence the more significative cash incomes (numerically speaking) of the financial assets that conform the portfolio, for these will condition the reinvestments policy.*

TABLE 1

## SCENARIO A

QUAR- TERS (1)	CASH FLOW (2)	QUARTERLY RATES % (3)	INTEREST RATES % ANNUAL (4)	QUARTERLY (5)	CASH FLOW (6)
0	629.9	-	-	-	629.46
1	35	3.-	-	2.92	35
2	30	2.9	-	2.92	30
3	50	2.8	-	2.92	50
4	40	3.-	12.22	2.92	40
5	150	3.1	-	3.13	150
6	55	3.2	-	3.13	55
7	30	3.1	-	3.13	30
8	40	3.1	13.1	3.13	40
9	60	2.9	-	2.97	60
10	170	2.8	-	2.97	170
11	60	3.-	-	2.97	60
12	55	3.2	12.44	2.97	55

TABLE 1

## SCENARIO B

QUAR- TERS (1)	CASH FLOW (2)	QUARTERLY RATES % (3)	INTEREST RATES % ANNUAL (4)	QUARTERLY (5)	CASH FLOW (6)
0	621.68	-	-	-	625.51
1	35	3.-	-	3.07	35
2	30	3.05	-	3.07	30
3	50	3.1	-	3.07	50
4	40	3.15	12.88	3.07	40
5	150	3.2	-	3.27	150
6	55	3.25	-	3.27	55
7	30	3.3	-	3.27	30
8	40	3.35	13.76	3.27	40
9	60	3.4	-	3.47	60
10	170	3.45	-	3.47	170
11	60	3.5	-	3.47	60
12	55	3.55	14.64	3.47	55

TABLE 1

## SCENARIO C

QUAR- TERS (1)	CASH FLOW (2)	QUARTERLY RATES % (3)	INTEREST RATES % ANNUAL (4)	QUARTERLY (5)	CASH FLOW (6)
0	637.26	-	-	-	633.67
1	35	3.-	-	2.92	35
2	30	2.95	-	2.92	30
3	50	2.9	-	2.92	50
4	40	2.85	12.22	2.92	40
5	150	2.8	-	2.72	150
6	55	2.75	-	2.72	55
7	30	2.7	-	2.72	30
8	40	2.65	11.35	2.72	40
9	60	2.6	-	2.52	60
10	170	2.55	-	2.52	170
11	60	2.5	-	2.52	60
12	55	2.45	10.49	2.52	55

## CASH FLOW PERIODICITY: DURATION AND IMMUNIZATION

How much does what we just explained have an influence in the calculation of the duration? Will the duration still be useful to measure the sensitivity of the price facing the variations in the interest rate of the market?

The duration has two interpretations which may be inferred separately, leading to two equivalent formulae: the statistical and the economic interpretation. The statistical interpretation, being (2) its equation, which may be generalized for any unit of time <sup>3</sup> even when the interest rate is variable; let  $\{f_t\}$   $t = 1, 2, \dots, nk$ , be a cash flow, and let  $\{i^{(k)}/k\} = i_1^{(k)}/k, i_2^{(k)}/k, \dots, i_{nk}^{(k)}/k$  be up to the succession of interest rates by  $k$ th.

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<sup>3</sup> It has to be taken into account that the temporal unity of the interest rate has to be the same than the one used to measure time.

The *generalized duration* will be given by:

$$D_k \left[ \{i^{(k)}/k\} \right] = \frac{\sum_{t=1}^{nk} t \cdot f_t \cdot \prod_{j=1}^t \left[ 1 + \frac{i_j^{(k)}}{k} \right]^{-1}}{\sum_{t=1}^{nk} f_t \cdot \prod_{j=1}^t \left[ 1 + \frac{i_j^{(k)}}{k} \right]^{-1}} \quad (13)$$

Applying this expression to the cash flow of table 1, and taking into account the yield curve in each one of the scenarios, the following values for the duration may be obtained:

scenario a): 6.88609 quarters

scenario b): 6.85672 quarters

scenario c): 6.91954 quarters

The economic interpretation is tied to the concept of price elasticity (which is identified with the actual value of the net cash flow) in respect to the unitary capitalization factor  $(1+i)$ , obtaining as the equation of the duration:

$$D = - \frac{dP}{di} \cdot \frac{1+i}{P} \quad (14)$$

If the price of the asset or the value of the portfolio is calculated by applying (1), it will be easy to check the equivalence between the interpretation we just attributed to duration with formula (2). However, if we try to calculate the duration from (14) taking as the only rate  $i$  the annual equivalent to  $\{i^{(k)}/k\}$  given by (10), the obtained value is different from the one obtained by applying (13); such mistake is maintained even in the case where the succession  $\{i^{(k)}/k\}$  were to be of constant terms. The reason for such a divergence may be found in the difference between the weights in each one of the cases; taking the year as the reference, the weights in (13) would be  $\frac{1}{k}, \frac{2}{k}, \dots, \frac{nk}{k}$ , while in (2) or in (14) the weights are 1, 2, ...,  $n$ . In addition, we have to find the annual equivalent of each

one of the items of the cash flow  $\{F_t\}$  by applying the formula proposed in (7). Nevertheless, even under such conditions, there is a difference between the values of the duration calculated according to (13) or according to (14). In the Appendix, we show the origin of such a difference for two kinds of portfolio, as well as the general formula of the duration of a portfolio with constant income per  $k$ th of the year assuming that the yield curve would be flat.

In order to obviate such a discrepancy among results, we may calculate the value of the generalized duration by applying (13) and then use it in (14) with the aim of making a prediction, so that the difficulty set by the uniqueness of the required interest rate to find duration by applying (14) disappears.

This is what is done in practice, for if in (14) we know  $D$ ,  $P$  and  $i$ , we may be able to calculate which should be the relative variation in the price of a financial asset:  $\Delta P/P$  as the reaction in front of a change (relatively small) in the market rate:  $\Delta i/(1+i)$ , so that by using an approximate interpretation of (14), we could write:

$$\frac{\Delta P}{P} \approx -D \cdot \frac{\Delta i}{1+i} \quad (15)$$

In addition to the inherent limitations to the approximate nature of (15) which have been very much debated, we will have to speak more in detail about the practical application of such a formula.

Once we have determined the cash flow in a portfolio and we have defined the reinvestments policy in relation to periodicity and amounts: on which base the relative increase of  $i$  is calculated in order to determine the sensitivity of the portfolio's value? The references I know about, refer always to annual rates, since "once the equivalence between the annual rate and the rate per  $k$ th is set out (and solved), it makes no difference to apply one rate or another"; this would be true if the real yield rates per  $k$ th were to be constant along the year and were to vary only from one year to the other.

I think that here (as in many other cases) an extrapolation of the bond theory to the portfolio theory has been made, without much thinking on the consequences. A bond has a frequency in the cash flow that uses to be annual or, at the most, semiannual; therefore, the reinvestment takes place once or twice a year and, in addition, in small amounts specially in respect to the bond nominal, which minimizes the influence of the intra-annual structure of the yield rates. The nominal has to be reinvested only once, at the moment of amortization, and this will be done at the yield rate in force at that moment; as we may well see, the reinvestment is not a serious problem when the bond is considered separately.

However, in a portfolio the reinvestments take place with a higher frequency than the one of a bond, so that the changes in the interest rate will considerably affect the following reinvestments (reinvestment risk), as will happen in the portfolio evaluation (price risk). Therefore, it means little to consider variations in the annual interest rate (were the yield curve to be flat or not); what is really significant and determinant in the immunization policy is to correctly determine the yield curve so that the periodicity will meet the reinvestments policy.

Questions raised by some authors in relation to whether in (15) the variations in the interest rate,  $\Delta i$ , depend on any other variable or follow a specific stochastic process in their evolution, lack of meaning if we consider the yield curve within the year. In addition, when these authors speak of a variation in the interest rate, they always refer to a modification in the annual rate; that is to say, if  $i$  is the annual rate, then  $\Delta i$  is the variation of the interest or yield rate within the year; in this sense,  $i$  is a discrete variable <sup>4</sup>; therefore, the smaller the reference unit of time, the smaller the mistake made in (15).

It is then suitable to choose the smallest possible unit of time, since it will express the relative variation of the interest rate, for it is always possible to go

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<sup>4</sup> We will only have to remind here that when in the original definition of the duration, the derivative is calculated from  $(1+i)$ , and not from  $i$ , what happens is that  $d(1+i) = di$ ; we have to remind also that  $1+i$  is the capitalization factor per unit of time, that is to say,  $(1+i) = (1+i)^t$ , with  $t=1$ .



from a split structure to a more generalized one, by only applying (9) and (10), while it is not always possible to do the contrary. For instance, if a portfolio manager revises every three months, he will be interested in knowing which is the foreseen variation in the quarterly interest rate since, from a variation in the annual rate, he would like to know which is the time distribution in each one of the four quarters.

Consequently, it is more useful to take as the reference a unit of time for the interest rate which meets the chosen frequency in the policy of portfolio reinvestments. This is precisely what, with a bigger or lesser degree of conscience, does a portfolio manager, despite the fact of speaking about durations or rate (of isolated bonds, as well as of portfolios) expressed in years; he will make adjustments in the composition of the portfolio so that, following the designed immunization policy, he may, in the best possible way, obtain a benefit of the market situation, in relation to the yield rates as well as in relation to the prices.

To sum it up, I have tried in these pages to show that the uniqueness of the yield rate is a problem that must be faced by the bond portfolio manager if he wants to really make good use of the portfolio immunization theory; and, even though in some specific cases the yield to maturity should be only one, the periodicity has to be defined for the income to be reinvested in. This will be useful to specify the time frame of the yield curve; if the right decision in relation to the frequency in which the reinvestments will be done is not taken, then the immunization policy to be followed might even be invalidated.

## APPENDIX: DURATION OF A PORTFOLIO

## A.1 TWO-BONDS PORTFOLIO

Let us consider a portfolio made up of two bonds belonging to the same kind of risk (and, therefore, with the same market yield); a bond of annual coupon:  $c_a$  and another bond of semiannual coupon:  $c_s$ , both with the same nominal value:  $F$ , and with the same amortization date: at the end of  $n$  years. The semiannual diagram of the cash flow of the portfolio is:

$$c_s, c_a + c_s, c_s, \dots, c_s, c_a + c_s + 2F$$

The present value of each one of the bonds is:

$$P_a = \sum_{t=1}^n c_a \cdot (1+i_a)^{-t} + F(1+i_a)^{-n} \quad y \quad P_s = \sum_{t=1}^{2n} c_s \cdot (1+i_s)^{-t} + F(1+i_s)^{-2n} \quad (16)$$

being  $i_a$  the annual yield rate of the first bond and  $i_s$  the semiannual yield rate of the second, with the following relationship between rates:  $(1+i_a) = (1+i_s)^2$

When the portfolio is made of a bond of each type, the present value of the portfolio is  $P_a + P_s$ , and the annual yield rate of the portfolio will be given by the solution  $i$  of the equation:

$$P_a + P_s = \sum_{t=1}^n (c_a + c_s) \cdot (1+i)^{-t} + \sum_{t=1}^n c_s \cdot (1+i)^{-t+\frac{1}{2}} + 2F(1+i)^{-n} \quad (17)$$

Once the global  $i$  of the portfolio is known, we may calculate the duration in two ways: the first one, directly, and the second one through the annual

equivalent.

*Directly*

$$D = \frac{\sum_{t=1}^n (c_a + c_s) \cdot t \cdot (1+i)^{-t}}{P_a + P_s} + \frac{2 \cdot n \cdot F \cdot (1+i)^{-n}}{P_a + P_s} + \frac{\sum_{t=1}^n \left[ t - \frac{1}{2} \right] \cdot c_s \cdot (1+i)^{-t + \frac{1}{2}}}{P_a + P_s} \quad (18)$$

*Through the annual equivalent:*

The annual coupon equivalent to the semiannual coupon will be given by:

$$c'_a = c_s (1+i)^{\frac{1}{2}} + c_s \quad (19)$$

We have then a portfolio with an annual coupon  $c = c'_a + c_a$ , and its duration

is:

$$D' = \frac{\sum t \cdot c \cdot (1+i)^{-t}}{P_a + P_s} + \frac{2 \cdot n \cdot F \cdot (1+i)^{-n}}{P_a + P_s} \quad (20)$$

However, this duration must be corrected, since the two coupons of a year have been multiplied by 1 when the first coupon should have been multiplied by  $\frac{1}{2}$ ; the third coupon has been multiplied by 2, but should have been multiplied by  $2 - \frac{1}{2} = \frac{3}{2}$ , etc.; therefore, we have to subtract from the expression (20):

$$\frac{1}{2} \cdot c_s (1+i)^{-\frac{1}{2}} + \frac{1}{2} \cdot c_s (1+i)^{-\frac{3}{2}} + \dots + \frac{1}{2} \cdot c_s (1+i)^{-n + \frac{1}{2}}$$

from which, adding the terms of the progression and after some algebra, we get:

$$\frac{\frac{1}{2} \cdot c_s \cdot (1+i)^{\frac{1}{2}} \cdot a_{n|i}}{P_a + P_s} \quad (21)$$

Remaining as the definitive expression for the duration:

$$D = \frac{\sum t \cdot c \cdot (1+i)^{-t}}{P_a + P_s} + \frac{2 \cdot n \cdot F \cdot (1+i)^{-n}}{P_a + P_s} - \frac{\frac{1}{2} \cdot c_s \cdot (1+i)^{\frac{1}{2}} \cdot a_{n|i}}{P_a + P_s} \quad (22)$$

## A.2 THREE-BONDS PORTFOLIO

Let us assume that we add a bond with a quarterly coupon  $c_q$  to the portfolio with the same amortization date  $n$ ; the present value of this bond at the quarterly rate  $i_q$  is:

$$P_q = \sum_{t=1}^{4n} c_q \cdot (1+i_q)^{-t} + F(1+i_q)^{-4n} \quad (23)$$

If the portfolio is now made of a bond of annual coupon, another semiannual, and a third one with a quarterly coupon, its present value will be:

$$P_a + P_s + P_q$$

and to calculate the annual yield rate of the portfolio, we have to solve for  $i$  the

following equation:

$$\begin{aligned}
 P_a + P_s + P_q &= \\
 &= \sum_{t=1}^n (c_a + c_s + c_q) \cdot (1+i)^{-t} + \sum_{t=1}^n (c_s + c_q) \cdot (1+i)^{-t+\frac{1}{2}} + \\
 &+ \sum_{t=1}^n c_q \cdot (1+i)^{-t+\frac{1}{4}} + \sum_{t=1}^n c_q \cdot (1+i)^{-t+\frac{3}{4}} + 3F(1+i)^{-n} \quad (24)
 \end{aligned}$$

As we did before, the duration may be calculated directly or through the annual equivalent, subtracting the correction factor: the annual coupon  $c_a''$ , equivalent to the quarterly  $c_q$ , and to the semiannual  $c_s$  is given by:

$$c_a'' = c_s \cdot \left[ (1+i)^{\frac{1}{2}} + 1 \right] + c_q \cdot \left[ (1+i)^{\frac{3}{4}} + (1+i)^{\frac{1}{2}} + (1+i)^{\frac{1}{4}} + 1 \right] \quad (25)$$

and the annual global coupon will be equal to:

$$c = c_a'' + c_a$$

If we want to calculate the duration by using the annual yield rate and the annual equivalent cash flow, we should subtract from the expression:

$$D' = \frac{\sum t \cdot c \cdot (1+i)^{-t}}{P_a + P_s + P_q} + \frac{3 \cdot n \cdot F \cdot (1+i)^{-n}}{P_a + P_s + P_q} \quad (26)$$

the correction factors, the one already obtained in (21) and the corresponding to the

bond of the quarterly coupon:

$$\frac{\frac{1}{4} \cdot c_q \cdot (1+i)^{\frac{1}{4}} \cdot a_{n|i} + \frac{1}{2} \cdot c_q \cdot (1+i)^{\frac{1}{2}} \cdot a_{n|i} + \frac{3}{4} \cdot c_q \cdot (1+i)^{\frac{3}{4}} \cdot a_{n|i}}{P_a + P_s + P_q} \quad (27)$$

obtaining as the expression of the duration:

$$D = \frac{\sum tc(1+i)^{-t}}{P_a + P_s + P_q} + \frac{3nF(1+i)^{-n}}{P_a + P_s + P_q} - \frac{\frac{1}{2}c_s(1+i)^{\frac{1}{2}}a_{n|i}}{P_a + P_s + P_q} - \frac{\left[ \frac{1}{4}(1+i)^{\frac{1}{4}} + \frac{1}{2}(1+i)^{\frac{1}{2}} + \frac{3}{4}(1+i)^{\frac{3}{4}} \right] \cdot c_q \cdot a_{n|i}}{P_a + P_s + P_q} \quad (28)$$

### A.3 GENERALIZATION OF THE CORRECTION FACTOR FOR ANY PERIOD OF TIME

Let us assume that a financial asset which produces an income per  $k$ th of the year equal to  $c_k$ , and that the yield to maturity (YTM) of such asset is equal to  $i$  (annual). If we are dealing with a bond, it will amortize at the end of  $n$  years by its nominal and, consequently, the price of this bond  $P_k$ , will be given by:

$$P_k = c_k \cdot a_{nk|i} + F \cdot (1+i^{(k)}/k)^{-nk} \quad (30)$$

with:

$$i' = i^{(k)}/k = (1+i)^{\frac{1}{k}} - 1 \quad (31)$$

being the real amount per kth equivalent to YTM  $i$ .

The duration  $D_k$ , (expressed in kths of the year) of this asset will be given by:

$$D_k = \frac{\sum_{t=1}^{nk} t \cdot c_k \cdot \left[1 + \frac{i^{(k)}}{k}\right]^{-t}}{P_k} + \frac{k \cdot n \cdot F \cdot \left[1 + \frac{i^{(k)}}{k}\right]^{-nk}}{P_k} \tag{32}$$

which will be equivalent to  $D = \frac{D_k}{k}$  years.

If we intend to use directly the annual rate  $i$ , we will first calculate the equivalent annual coupon:

$$c'_a = c_k(1+i)^{\frac{k-1}{k}} + c_k(1+i)^{\frac{k-2}{k}} + \dots + c_k(1+i)^{\frac{1}{k}} + c_k \tag{33}$$

Since the generalized correction factor is:

$$\frac{\sum_{j=1}^{k-1} \frac{j}{k} \cdot (1+i)^{\frac{j}{k}} \cdot a_{n| i} \cdot c_k}{P_k} \tag{34}$$

we will have that the duration will be given by the formula:

$$D = \frac{\sum_{t=1}^k t \cdot c'_a \cdot (1+i)^{-t}}{P_k} + \frac{n \cdot F \cdot (1+i)^{-n}}{P_k} - \frac{\sum_{j=1}^{k-1} \frac{j}{k} \cdot (1+i)^{\frac{j}{k}} \cdot a_{n| i} \cdot c_k}{P_k} \tag{35}$$

or, performing the previous additions:

$$D = \frac{c'_a}{P_k} \left[ \left[ 1 + k + \frac{1}{i} \right] \cdot a_{k|i} - \frac{k}{i} \right] + \frac{n \cdot F \cdot (1+i)^{-n}}{P_k} -$$

$$- \frac{c_k \cdot a_{n|i}}{P_k} \cdot \left[ \frac{1}{i^{(k)}} \cdot (k-2)(1+i) + \frac{1 + \frac{i^{(k)}}{k}}{i^{(k)} \cdot i^{(k)}} + \frac{1+i}{k + i^{(k)}} \cdot \left[ 1 - \frac{1}{[i^{(k)}/k]^2} \right] \right]$$

#### A.4 EXAMPLE

Let us consider a portfolio made of the following financial assets:

- A bond which is amortized within 5 years at par with an annual 10% coupon
- A bond which is amortized within 5 years at par with a semiannual 6% coupon
- A bond which is amortized within 20 quarters at par with a quarterly 2% coupon
- A loan which is amortized with a unique payment within 5 years, with a monthly interest allowance at a monthly 1% rate.

It is considered that the yield rate of the portfolio is 10%. To simplify, we will assume that the nominal value of each of the described assets is 100.

Using equation (1), the theoretical value of each one of them is:

$$P_a = 100; P_b = 108.69; P_c = 93.53; P_d = 109.63$$

Therefore, the total value of the portfolio is:

$$P = P_a + P_b + P_c + P_d = 411.855$$

If we want to use the annual duration with the annual interest rate, we will



have to find first the amounts of the equivalent annual interests by using (25) for the semiannual and quarterly coupons. We get:

$$c''_a = 6 \cdot \left[ (1.1)^{\frac{1}{2}} + 1 \right] + 2 \cdot \left[ (1.1)^{\frac{3}{4}} + (1.1)^{\frac{1}{2}} + (1.1)^{\frac{1}{4}} + 1 \right]$$

$$c''_a = 12.292 + 8.294 = 20.586$$

Concerning the monthly interests of the loan, its annual equivalent is obtained from (33) assigning to  $k$  the value 12, giving:

$$c'_a = 1 \cdot (1.1)^{\frac{11}{12}} + 1 \cdot (1.1)^{\frac{10}{12}} + \dots + 1 \cdot (1.1)^{\frac{1}{12}} + 1 = 12.54$$

From that we get that the equivalent annual coupon of the portfolio is equal to the coupon of the annual bond: 10, plus the equivalents to the semiannual bond 12.292, plus the quarterly one 8.294, and, finally, the loan interest 12.54, giving:

$$c = 10 + 12.292 + 8.294 + 12.54 = 41.126$$

From now on, we are ready to obtain an approximate value of the duration by means of a simple generalization of (26):

$$D' = \frac{\sum_{t=1}^5 t \cdot c \cdot (1+i)^{-t}}{P_a + P_b + P_c + P_d} + \frac{4 \cdot n \cdot F \cdot (1+i)^{-5}}{P_a + P_b + P_c + P_d}$$

Substituting  $c$  by 41.126,  $i$  by 0.1,  $F$  by 100, and  $P_a + P_b + P_c + P_d = 411.855$ , and performing the indicated algebra, we obtain: 4.1307.

Secondly, we have to calculate the correction factors, by using (34) with  $k$  having the values 2, 4 and 12 and with  $i_2 = 0.0488$ ;  $i_4 = 0.02411$ ;  $i_{12} = 0.007974$ , we get that:

correction factor corresponding to b) : 0.0290  
 correction factor corresponding to c) : 0.0292  
 correction factor corresponding to d) : 0.0538

If we now deduct the previous amounts from the uncorrected annual duration, we obtain the value of the real duration:

$$D = 4.13072476 - 0.02896024 - 0.02919576 - 0.05381348 = 4.01875528$$

If we calculate the duration directly, we get that the smallest time periodicity is the one corresponding to the loan cash flow. It follows that the monthly cash flow of the portfolio is:

$$\underbrace{1, 1, 3, 1, 1, 9, 1, 1, 3, 1, 1, 19, 1, \dots, 19, 1, \dots, 19, 1, \dots, 19, 1, \dots, 419}_{1\text{st year}} \quad \underbrace{\quad}_{2\text{nd}} \quad \underbrace{\quad}_{3\text{rd}} \quad \underbrace{\quad}_{4\text{th}} \quad \underbrace{\quad}_{5\text{th}}$$

Applying the equation corresponding to the duration to such cash flow at the monthly rate 0.0797414 equivalent to the annual 0.1, we obtain 48.2250364 months, which are 4.01875528 years. This is the same result that we obtained before.

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