Modeling and Measuring Volatility in the Black-Scholes Economy: A Bayesian Approach*

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Summary

The Black-Scholes economy model with constant drift and volatility is used as a first attempt to statistically analyze stock volatility. A posterior distribution of market volatility given past stock prices is derived using a Bayesian approach. A market analyst would incorporate his/her personal knowledge through the specification of a prior distribution. The natural conjugate prior is a Normal-Generalized Inverse Gaussian distribution with five hyperparameters. The posterior mean and mode take a credibility form that is a weighted average of three different volatility measurements. Specifying these weights along with other easily calculated market measurements allows the analyst to indirectly obtain the prior distribution. European options are priced by calculating the expected value of the Black-Scholes formula using the posterior distribution of stock volatility. A set of historical S&P 500 data is numerically analyzed to illustrate results.

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Modèle et mesure de la volatilité dans l'économie Black-Scholes : Une méthode bayesienne*

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Résumé

Le modèle économique Black-Scholes de volatilité et de dérive constante est utilisé pour tenter pour la première fois d'analyser statistiquement la volatilité des actions. Une distribution postérieure de la volatilité du marché tenant compte du prix antérieur des actions est dérivée par méthode bayesienne. Un analyste du marché incorporerait ses connaissances personnelles en spécifiant la distribution antérieure. Le conjugat antérieur est une distribution gaussienne inverse normale-généralisée comportant cinq hyperparamètres. Le mode et la moyenne postérieure se présentent sous forme prévisible qui est la moyenne pondérée de trois mesures de volatilité différentes. La spécification de ces pondérances et d'autres mesures du marché aisément calculées permet à l'analyste d'obtenir indirectement la distribution antérieure. Le prix des options européennes est établi en calculant la valeur anticipée de la formule Black-Scholes à l'aide de la distribution postérieure de la volatilité des actions. Un ensemble de données S&P 500 historiques est analysé numériquement pour illustrer les résultats.

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1. Introduction

Stock options have become important instruments in portfolio management and other fields of finance. The pricing of these and many other derivative securities requires the use of formulae that rely heavily on measurements of stock volatility. For example, the well-known Black-Scholes pricing formula for European options does not require the use of a stock's mean return but does depend on stock volatility measurements in its calculations. Dynamic portfolio insurance strategies require variances and covariances of many assets to construct hedged portfolios. The magnitude of underlying stock volatility directly contributes to the cost of constructing and maintaining these hedged portfolios.

Empirical evidence by Black (1976) shows that stock market volatility is negatively correlated with market performance. In periods of market down turns, increases in market volatility are observed. When stock prices are rising, the volatility is significantly lower. The model proposed in this paper could reflect this observed behaviour but for simplifying reasons, and the lack of information available concerning this phenomenon (for various theories, see Malkiel(1979), Modigliani and Cohn (1979), Fama (1981), Pindyck (1984) and Chou(1988)), it will not be incorporated into the analysis.

Hence, volatility is not a simple parameter to estimate. Many innovative techniques have been introduced and empirically tested including Autoregressive Conditional Heteroskedasticity models (ARCH) and continuous stochastic volatility models. Bollerslev, Chou and Kroner (1992) present a survey of ARCH model in finance. This review of the literature includes both theoretical and empirical results.

Hull and White (1987) analyze the pricing of options under stochastic volatility. Wiggins (1987) also solves the call option valuation problem assuming a continuous stochastic process for return volatility and uses this model to derive statistical estimators. Mean reversion models have been considered and used by Cox, Ingersol and Ross (1985), Hull and White (1988) and Randolph (1991).

In this paper, the problem of measuring volatility in the Black-Scholes economy using a Bayesian statistical approach is considered. This first attempt assumes that the drift and volatility components of the underlying stock price are constant over time. This assumption will be relaxed in future papers. Observed fluctuations in stock prices will be used to derive a posterior distribution for the underlying volatility. The model used to obtain the likelihood for the observed data is the same model used to derive the Black-Scholes option formula.

The posterior distribution of the underlying volatility performs two useful purposes. First, it can be integrated with the Black-Scholes formula to obtain option prices of the underlying stock without requiring point estimates of the volatility. Unlike classical statistical approaches, this Bayesian analysis relies on a prior distribution so that the market analyst can incorporate his/her own beliefs in pricing option instruments. Second, it is extremely difficult for traditional methods to provide accurate measurements of volatility uncertainty. Variances of point estimators could be calculated but they do not completely reveal the uncertainty of information. Furthermore, it is extremely difficult to incorporate variances of volatility estimates into option pricing formulae. The posterior distribution graphically illustrates increases (or decreases) in uncertainty of the underlying stock volatility through increases (or decreases) in the dispersion of the distribution. This is illustrated through numerical analysis of historical S&P 500 observations.

The prior distribution can be thought of as the information or knowledge the professional market analyst believes about the underlying stock volatility. Although the functional form of the prior distribution proposed may seem complicated and too abstract for a practical user to specify (the prior distribution requires the specification of five parameters), it will be shown that the prior's parameters exhibit intuitive and natural interpretations that will be of assistance in specify their values.

Assume that the information structure in this economy is determined by a standard Brownian motion W(s) and the filtration process

$$F_t = \sigma\{W(s);\ 0 \le s \le t\}. \tag{1}$$

Associated with this filtration process are two securities. The first is a riskless asset whose value at time t is represented by

$$B(t) = B(0)e^{rt} \tag{2}$$

where r is a fixed constant representing the "force of interest". The second security is the stock and it value at time t is represented by

$$S(t) = S(0)e^{\left(\mu - 0.5\sigma^2\right)t + \sigma W(t)} \tag{3}$$

where μ is the drift constant and σ^2 is the underlying volatility.

This model was used by Black and Scholes (1973), Merton (1973), Cox and Huang (1986) and many others to develop the theory of option pricing. A "no arbitrage" argument either by the use of martingales or other discrete means in conjunction with the above model can be used to derive the Black-Scholes formula for pricing call options as

$$C\left(t|S(t), K, r, T, \sigma^2\right) = S(t)N\left(z + \frac{1}{2}\sigma\sqrt{T-t}\right) - Ke^{-r(T-t)}N\left(z - \frac{1}{2}\sigma\sqrt{T-t}\right) \tag{4}$$

where

$$z = \frac{\ln\left(S(t)/K\right) + r(T-t)}{\sigma\sqrt{T-t}}, \tag{5}$$

K is strike price, T is maturity date and N(x) is the cumulative standard Normal Distribution. All variables in this formula are known quantities (or may be estimated fairly accurately) except the volatility σ^2 of the underlying stock.

2. Likelihood Equation and Observed Data

The security is observed over a specified period of time. Sample values of the stock price are observed and recorded along with the exact time the observations are taken. Let $D = \{(t_i, S(t_i));\ i = 1, \ldots, n\}$ represent the set of data that is collected. Without loss of generality, assume $t_0 \equiv 0$ and $S(t_0) \equiv 1$. For this general model, observations need not be made at equal time intervals. From equation (3) and the well-known fact that the standard Brownian Motion has stationary and independent increments, the difference in logarithmic stock prices $\ln S(t_i) - \ln S(t_{i-1})$, given drift μ and volatility σ^2, are independent normally distributed with mean $(\mu - 0.5\sigma^2)(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$. Therefore, the likelihood of the observed data is

$$L(D|\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2(t_i - t_{i-1})}} \exp\left\{-\frac{1}{2}\frac{\left(\ln[S(t_i)/S(t_{i-1})] - (\mu - 0.5\sigma^2)(t_i - t_{i-1})\right)^2}{\sigma^2(t_i - t_{i-1})}\right\} \quad (6)$$

For notation convenience, let

$$r_i = \frac{S(t_i)}{S(t_{i-1})},\ R_1 = \frac{1}{t_n}\sum_{i=1}^{n} \ln r_i = \frac{1}{t_n}\ln\frac{S(t_n)}{S(t_0)} \text{ and } R_2 = \frac{1}{t_n}\sum_{i=1}^{n}\frac{(\ln r_i)^2}{t_i - t_{i-1}}. \quad (7)$$

Then, the likelihood can be reduced to

$$L(D|\mu,\sigma^2) \propto (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{t_n\mu^2}{2\sigma^2} + \frac{t_n\mu}{2} + \frac{t_n R_1 \mu}{\sigma^2} - \frac{t_n R_2}{2\sigma^2} - \frac{t_n\sigma^2}{8}\right\}. \quad (8)$$

From this equation, it is easily seen that all the data can be reduced to four sufficient statistics, namely n, t_n, R_1 and R_2.

3. Prior and Posterior Distributions

Given σ^2, a natural conjugate prior for μ is normal with mean $\alpha+\gamma\sigma^2$ and variance $\beta^2\sigma^2$. The observed negative correlation between stock volatility and stock returns can be accomplished through specifying a negative value for γ. This model can also be used to test for a negative correlation by specifying an uninformative symmetric prior around zero for γ and observing its posterior distribution. Unfortunately, this test will require data over a significant period of time where the assumption of constant stock volatility may not hold. For simplicity, the remainder of the paper will ignore this correlation by assuming $\gamma=0$ so that the mean of $\left(\mu\middle|\sigma^2\right)$ is not a function of σ^2 and is equal to α.

Since the formula for a call option does not depend on the value of μ, we can integrate out this undesired parameter to obtain a marginal likelihood distribution as a function of volatility only. After much computation, one arrives at the result

$$\begin{aligned} L\left(D\middle|\sigma^2\right) &\propto E_{\mu|\sigma^2}\left(L\left(D\middle|\mu,\sigma^2\right)\right) \\ &\propto \left(\sigma^2\right)^{-\frac{n}{2}} \exp\left\{-\sigma^2\frac{t_n}{8\left(\beta^2 t_n+1\right)}-\frac{t_n}{2\sigma^2}\left(\frac{\beta^2 t_n R_2-\beta^2 t_n R_1{}^2+R_2-2\alpha R_1+\alpha^2}{\beta^2 t_n+1}\right)\right\} \end{aligned} \tag{9}$$

Even though this marginal likelihood is no longer a function of μ, the hyperparameters α and β of the conditional random variable $\left(\mu\middle|\sigma^2\right)$ still need to be specified.

If $\left(\mu\middle|\sigma^2\right)$ is constant almost surely so that $\beta=0$, the above marginal likelihood reduces to

$$L\left(D\middle|\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{n}{2}} \exp\left\{-\sigma^2\frac{t_n}{8}-\frac{t_n}{2\sigma^2}\left(R_2-2\alpha R_1+\alpha^2\right)\right\}. \tag{10}$$

Another desirable property is the likelihood not rely on the hyperparameter $\alpha=E(\mu)$. One approach of accomplishing this requirement is to assume that the prior of $\left(\mu\middle|\sigma^2\right)$ be an improper uniform prior. This is equivalent to assuming beta large. As $\beta \to \infty$, the limiting posterior distribution is

$$L\left(D\middle|\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{n}{2}} \exp\left\{-\frac{t_n}{2\sigma^2}\left(R_2 - R_1^2\right)\right\}. \tag{11}$$

Returning to the general form of the likelihood in equation (9), the natural conjugate prior distribution for σ^2 takes the form

$$p\left(\sigma^2\right) \propto \left(\sigma^2\right)^{-\frac{A_0}{2}} \exp\left\{-\sigma^2 B_0 - \frac{1}{2\sigma^2} C_0\right\} \tag{13}$$

where A_0, B_0 and C_0 are chosen so that $p\left(\sigma^2\right)$ is a proper distribution. Using Bayes Theorem, the posterior distribution for σ^2 given the observed data D is

$$\begin{aligned} p\left(\sigma^2 \middle| D\right) &\propto E_{\mu|\sigma^2}\left(L\left(D\middle|\mu,\sigma^2\right)\right)p\left(\sigma^2\right) \\ &\propto \left(\sigma^2\right)^{-\frac{A_n}{2}} \exp\left\{-\sigma^2 B_n - \frac{1}{2\sigma^2} C_n\right\} \end{aligned} \tag{14}$$

where

$$A_n = A_0 + n,$$

$$B_n = B_0 + \frac{t_n}{8\left(\beta^2 t_n + 1\right)}$$

and

$$C_n = C_0 + t_n\left(\frac{\beta^2 t_n}{\beta^2 t_n + 1}\right)\left(R_2 - R_1^2\right) + t_n\left(\frac{1}{\beta^2 t_n + 1}\right)\left(R_2 - 2\alpha R_1 + \alpha^2 t_n\right). \tag{15}$$

The probability density function

$$f\left(x\middle|A,B,C\right) \propto x^{-\frac{A}{2}} \exp\left\{-Bx - \frac{C}{2x}\right\} \tag{16}$$

is the General Inverse Gaussian Distribution. With the integrating constant included, the probability density function is

$$f(x|A,B,C)=\frac{\left(\frac{C}{2B}\right)^{\frac{A}{4}-\frac{1}{2}}}{K_{\frac{A}{2}-1}\left(\sqrt{2BC}\right)}x^{-\frac{A}{2}}\exp\left\{-Bx-\frac{C}{2x}\right\}. \tag{17}$$

where $K_{\pm v}(x)$ is the Modified Bessel Function of the Second Kind. For a more thorough discussion of Bessel functions, refer to Abramowitz and Stegun (1965), Press et al. (1988) or Relton (1965). $K_{\pm v}(x)$ has a singularity at x=0. Thus, (17) is only valid for $B,C>0$. Jorgensen (1982) gives a detailed statistical analysis of this distribution.

If $C=0$ and $A'=-\frac{A}{2}+1>0$ then the density function takes the limiting Gamma Distribution form

$$f(x|A',B)=\frac{B^{A'}}{\Gamma(A')}x^{A'-1}e^{-Bx}. \tag{18}$$

If $B=0$ and $A'=\frac{A}{2}-1>0$ then the limiting distribution is the Inverse Gamma distribution.

With the change of variables $A=2(v+1)$, $B=\frac{\lambda}{2\mu^2}$ and $C=\lambda$, the distribution is transformed into

$$f(x|v,\lambda,\mu)=\frac{\mu^v}{K_v\left(\frac{\lambda}{\mu}\right)}\exp\left(-\frac{\lambda}{\mu}\right)x^{-(v+1)}\exp\left\{-\frac{\lambda(x-\mu)}{2\mu^2x}\right\}. \tag{19}$$

Note that this transformation is not a complete mapping between the (A,B,C) space and the (v,λ,μ) space. There are no corresponding points in the (v,λ,μ) space that corresponds to the values where $BC=0$ but B or $C>0$. This second parameterization has two advantages. If v=0.5 then one obtains the standard Inverse Gaussian Distribution

$$f(x|\lambda,\mu)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2x}\right\}. \tag{20}$$

Also, if $\mu=1$ then the above distribution reduces to the standard Wald Distribution (for more information regarding these distributions, see Johnson and Kotz (1970).) Second, the n^{th} moments of the random variable X takes the convenient form

$$E\left(X^n \middle| v,\lambda,\mu\right)=(\mu)^n \frac{K_{v-n}\left(\frac{\lambda}{\mu}\right)}{K_v\left(\frac{\lambda}{\mu}\right)}. \tag{21}$$

Using the original parameterization, these moments are given by

$$E\left(X^n \middle| A,B,C\right)=\left(\frac{C}{2B}\right)^{\frac{n}{2}} \frac{K_{\frac{A}{2}-n-1}\left(\sqrt{2BC}\right)}{K_{\frac{A}{2}-1}\left(\sqrt{2BC}\right)}. \tag{22}$$

From these equations, the mean and variance of X are found to be

$$E\left(X \middle| A,B,C\right)=\left(\frac{C}{2B}\right)^{\frac{1}{2}} \frac{K_{\frac{A}{2}-2}\left(\sqrt{2BC}\right)}{K_{\frac{A}{2}-1}\left(\sqrt{2BC}\right)} \tag{23}$$

and

$$\operatorname{Var}\left(X \middle| A,B,C\right)=\frac{C}{2B}\left\{\frac{K_{\frac{A}{2}-3}\left(\sqrt{2BC}\right)}{K_{\frac{A}{2}-1}\left(\sqrt{2BC}\right)}-\left(\frac{K_{\frac{A}{2}-2}\left(\sqrt{2BC}\right)}{K_{\frac{A}{2}-1}\left(\sqrt{2BC}\right)}\right)^2\right\}. \tag{24}$$

The mode of this distribution can easily be shown to be

$$\frac{C}{A}\left(\frac{A^2}{4BC}\right)\left[\left(1+\frac{8BC}{A^2}\right)^{\frac{1}{2}}-1\right]. \tag{25}$$

Finally, the moment generating function of X is

$$\Psi(t)=\left(1-\frac{t}{B}\right)^{\frac{A}{4}-\frac{1}{2}} \frac{K_{\frac{A}{2}-1}\left(\sqrt{2(B-t)C}\right)}{K_{\frac{A}{2}-1}\left(\sqrt{2BC}\right)}. \tag{26}$$

Combining the posterior distribution (14) with call option formula (4), numerical integration is used to evaluate

$$\begin{aligned} C\left(t \middle| S(t),K,r,T\right) &= E_{\sigma^2|D}\left\{C\left(t \middle| S(t),K,r,T,\sigma^2\right)\right\} \\ &= \int_0^\infty C\left(t \middle| S(t),K,r,T,\sigma^2\right)p\left(\sigma^2 \middle| D\right)d\sigma^2 \end{aligned} \tag{27}$$

By differentiating the Black-Scholes formula with respect to σ^2, it can easily be demonstrated that $C\left(t|S(t), K, r, T, \sigma^2\right)$ is a convex function for small values of σ^2 and concave for large values. That is, for

$$\sigma^2 > 2\left(1+\left\{\log\left(\frac{S}{Ke^{-r(T-t)}}\right)\right\}^2\right)^{\frac{1}{2}} - 2 = l \tag{28}$$

$C\left(\sigma^2\right)$ is a concave function. Hence, if $P\left(\sigma^2 > l|D\right)$ is close to one then by Jensen's Inequality,

$$C\left(E\left\{\sigma^2|D\right\}\right) \geq E_{\sigma^2|D}\left\{C\left(\sigma^2\right)\right\}. \tag{29}$$

When uncertainty in the market is small so that $P\left(\sigma^2 > l|D\right)$ is small,

$$C\left(\sigma^2_{\text{mode}}\right) \leq C\left(E\left\{\sigma^2|D\right\}\right) \leq E_{\sigma^2|D}\left\{C\left(\sigma^2\right)\right\}. \tag{30}$$

The conditional probability $P\left(\sigma^2 > l|D\right)$ is close to zero under one of the following two conditions: (i) stock volatility is small, or (ii) the value of l is large. If conditions (i) is satisfied then during periods of low stock volatility, the Black-Scholes formula tends to undervalue European call options. Similarly, conditions (ii) says that the Black-Scholes formula tends to undervalue European call options for deep-out-of-the-money or deep-in-the-money options. On the other hand, the Black-Scholes formula tends to overvalue European call options when market volatility is high or when the call option is at-the-money. These results extend findings obtained by Hull and White (1987) derived with stochastic volatility and agree with results obtained by Merton (1976) in conjunction with mixed jump-diffusion processes.

4. Parameter Interpretation of Prior Distributions

This section of the paper will discuss an approach that may be used to specify a prior distribution on σ^2 based on subjective beliefs of the stock market analyst. Recall from Section 3, the prior is parameterized by five variables that require specification. In particular, they are

A_0, B_0, C_0, α and β. It was also shown in Section 2 that the stock data can be summarized into the four summary statistics n, t_n, R_1 and R_2. From equation (15), the updating of the prior by the observed stock prices is given by

$$A_n = A_0 + n$$

$$B_n = B_0 + \frac{t_n}{8\left(\beta^2 t_n + 1\right)}$$

and
$$C_n = C_0 + t_n\left(\frac{\beta^2 t_n}{\beta^2 t_n + 1}\right)\left(R_2 - R_1^{\,2}\right) + t_n\left(\frac{1}{\beta^2 t_n + 1}\right)\left(R_2 - 2\alpha R_1 + \alpha^2 t_n\right). \tag{31}$$

We will begin the analysis with two special cases for the prior whose interpretation will help us in the general case. First, assume that observations are taken at equal time intervals of length Δ so that $t_n = \Delta n$. It can further be assumed that $\Delta = 1$ with the understanding that all market statistics referred to in the remainder of the paper are measured in the same unit of time as the observed stock prices. With this assumption, it is trivial to show that

$$\lim_{n\to\infty}\frac{A_n}{n} = 1,\ \lim_{n\to\infty}\frac{B_n}{n} = 0 \text{ and } \lim_{n\to\infty}\frac{C_n}{n} = \lim_{n\to\infty}\left(R_2 - R_1^2\right) = \sigma^2. \tag{32}$$

Also, the mode of the posterior distribution is

$$\frac{C_n}{A_n}\left(\frac{A_n^2}{4B_n}\right)\left\{\left(1 + \frac{8B_nC_n}{A_n^2}\right)^{\frac{1}{2}} - 1\right\}. \tag{33}$$

Since $(1+x)^{\frac{1}{2}} - 1 \sim x/2$ for small x and $\lim_{n\to\infty}\frac{B_nC_n}{A_n^2} = 0$, the above mode is approximately equal to $\frac{C_n}{A_n}$ for large values of n.

The first special case we will look at is the improper uniform prior for $\left(\mu\middle|\sigma^2\right)$. Recall that this is equivalent to letting $\beta \to \infty$. In this case, the mode of the posterior distribution can be written in the credibility form as

$$\frac{C_n}{A_n} = \left(\frac{A_0}{A_0 + n}\right)\left(\frac{C_0}{A_0}\right) + \left(\frac{n}{A_0 + n}\right)\left(R_2 - R_1^2\right). \tag{34}$$

If $\frac{B_0 C_0}{A_0^2}$ is small then $\frac{C_0}{A_0}$ is a good approximation of the mode of the prior distribution while $R_2 - R_1^2$ is the estimate of stochastic volatility using sample data only. The posterior mode is a weighted average of these two values, the size of the weights depending on the value of the prior specified parameter A_0 and the number of observations n.

Next, we will look at the other extreme of this prior distribution. Namely, when the prior value of $\left(\mu \middle| \sigma^2\right)$ is specified to be equal to a known constant α with no volatility. That is, $\beta = 0$. In this case, the mode of the posterior distribution takes the credibility form

$$\frac{C_n}{A_n} = \left(\frac{A_0}{A_0 + n}\right)\left(\frac{C_0}{A_0}\right) + \left(\frac{n}{A_0 + n}\right)\left(R_2 - 2\alpha R_1 + \alpha^2\right). \tag{35}$$

Again, $\frac{C_0}{A_0}$ is the mode of the prior distribution but this time, since the drift parameter μ is known to be equal to α with certainty, the estimate of stock volatility using observed returns is given by the expression

$$\frac{1}{n}\sum_{i=1}^{n}\left(\ln r_i - \alpha\right)^2 = R_2 - 2\alpha R_1 + \alpha^2. \tag{36}$$

Hence, the posterior mode is again the weighted average of the prior mode and the sample estimate of stock volatility using the observed data only with the same weights as before.

The two previous special cases can now be of assistance in analyzing the general case more easily. For any specified fixed value of β, the posterior mode can be written in the credibility form

$$\begin{aligned}\frac{C_n}{A_n} &= \left(\frac{A_0}{A_0 + n}\right)\left(\frac{C_0}{A_0}\right) \\ &+ \left(\frac{n}{A_0 + n}\right)\left\{\left(\frac{\beta^{-2}}{\beta^{-2} + n}\right)\left(R_2 - 2\alpha R_1 + \alpha^2\right) + \left(\frac{n}{\beta^{-2} + n}\right)\left(R_2 - R_1^2\right)\right\}\end{aligned} \tag{37}$$

In light of the above special cases, this equation tells us that the posterior mode is the weighted sum of three volatility measures. The first measure is the mode of the prior distribution. This measure is given the largest weight when little data is available. The next measure is the volatility measure with the hyperparameter α playing the role of the mean drift μ. The final measure of volatility in this formula is the sample estimate. If a large set of data is used, this last volatility measurement is given the greatest weighting in the posterior mode.

For v fixed and $x << v$, the Modified Bessel function asymptotically approaches

$$K_v(x) \sim \frac{\Gamma(v)}{2}\left(\frac{x}{2}\right)^{-v} \tag{38}$$

so that for small values of $\frac{B_n C_n}{A_n^2}$ (same condition as for the limiting behaviour of the mode), the mean is asymptotically equal to

$$E\left(\sigma^2 | A_n, B_n, C_n\right) = \left(\frac{C_n}{2B_n}\right)^{\frac{1}{2}} \frac{K_{\frac{A_n}{2}-2}\left(\sqrt{2B_n C_n}\right)}{K_{\frac{A_n}{2}-1}\left(\sqrt{2B_n C_n}\right)} \sim \frac{C_n}{A_n - 4}. \tag{39}$$

The mean of the distribution takes the an equivalent asymptotic form as the mode. Hence, all the credibility results stated earlier apply to the mean of the posterior distribution with the exception that A_0 is replaced with $A_0 - 4$.

The parameters of the prior distribution can be specified as follows. The market analyzer must first calculate his/her estimate of the expect market return $\alpha = E(\mu)$. Next, the individual must determine the positive weights p, q and r which sum to unity and which s/he would want to use in the point estimate of the posterior mode (or mean) of stock volatility. That is,

$$\begin{pmatrix}\text{Posterior Estimate}\\ \text{of Stock Volatility}\end{pmatrix} = p * \begin{pmatrix}\text{Prior Estimate}\\ \text{of Stock Volatility}\end{pmatrix} + q * \begin{pmatrix}\text{Estimate of Stock Volatility}\\ \text{assuming } \alpha \text{ known}\end{pmatrix} + r * \begin{pmatrix}\text{Population Estimate}\\ \text{of Stock Volatility}\end{pmatrix}. \tag{40}$$

If the analyst is unsure of his/her estimate of expected market return α, s/he can specify a small value for q, possibly zero. Setting q equal to zero is equivalent to specifying an improper uniform prior for $\left(\mu\middle|\sigma^2\right)$. If the analyst believes strongly in his/her estimate of stock volatility, s/he would specify a large value for p. If the analyst believes the market should primarily dictate the value of the stock volatility then r would be given a large value.

With p, q and r specified, equation (37) can be used to specify prior values for A_0 and β. Namely,

$$A_0 = \frac{pn}{1-p} \text{ and } \beta = \left(\frac{r}{qn}\right)^{\frac{1}{2}}. \tag{41}$$

C_0 can be found through the simple calculation

$$C_0 = A_0 * (\text{Prior estimate of stock volatility}). \tag{42}$$

Finally, the parameter B_0 does not seem to play any significant role in the analysis and thus can arbitrarily be set equal to any value as long as $\frac{B_0 C_0}{A_0^2}$ remains small. Otherwise, the analyst can specify a prior value for the mode and mean of the stock volatility and use the exact forms of equations (33) and (39) to solve for B_0 and C_0. This second approach is not recommended because the values of B_0 and C_0 are highly sensitive to little changes in the small difference between these two statistics.

5. Numerical Illustrations

For numerical illustrations, the daily closing values of the S & P 500 index were used from January, 1965 to December, 1992. For this analysis, it was assumed that volatility remained constant over a period of 10 working days so that n = 10. The same prior distribution was used for each calculation with hyperparameters determined as follows. The average yearly yield of the S & P 500, excluding dividends, from the beginning of 1965 to end of 1992 was 6.02%. The average daily return α over this time period was 2.33×10^{-4}. The weighting of the posterior mode was 20% for the prior mode, 30% for the volatility estimate assuming the mean return known and 50% for the population estimate of volatility. These weightings were arbitrarily chosen and allow the posterior distribution to reflex changes in market volatility while maintaining some stability over time through the prior distribution. The prior mode was calculated as the population estimate of daily volatility from January 1965 to December 1992. This value was 8.48×10^{-5}.

The hyperparameters for the prior distribution can now be found indirectly as was described in the previous section. They are $A_0 = 2.5$, $B_0 = 1$, $C_0 = 2.12 \times 10^{-4}$, $\alpha = 2.33 \times 10^{-4}$ and $\beta = 0.408$. The plot of the prior distribution of σ^2 is given in Figure 1. As can be seen in this diagram, the probability density function has a thick tail with very little mass near 0.

The posterior distribution for various years is shown in Figure 2. The updated values for A_0 and B_0 are the same for all graphs since they do not depend on the only two changing summary statistics R_1 and R_2. These values are $A_{10} = 12.5$ and $B_{10} = 1.469$. Since the graphs of posterior volatility are sensitive to the value of C_{10}, choosing particular dates would not be very useful for illustrative purposes. Thus, in order to obtain a "representative" graph for a particular year, the daily values of C_{10} were averaged over the entire year. For years 1967, 1972, 1977 and 1985, the posterior distribution is concentrated closer to zero than the prior

Figure 1: Prior Distribution

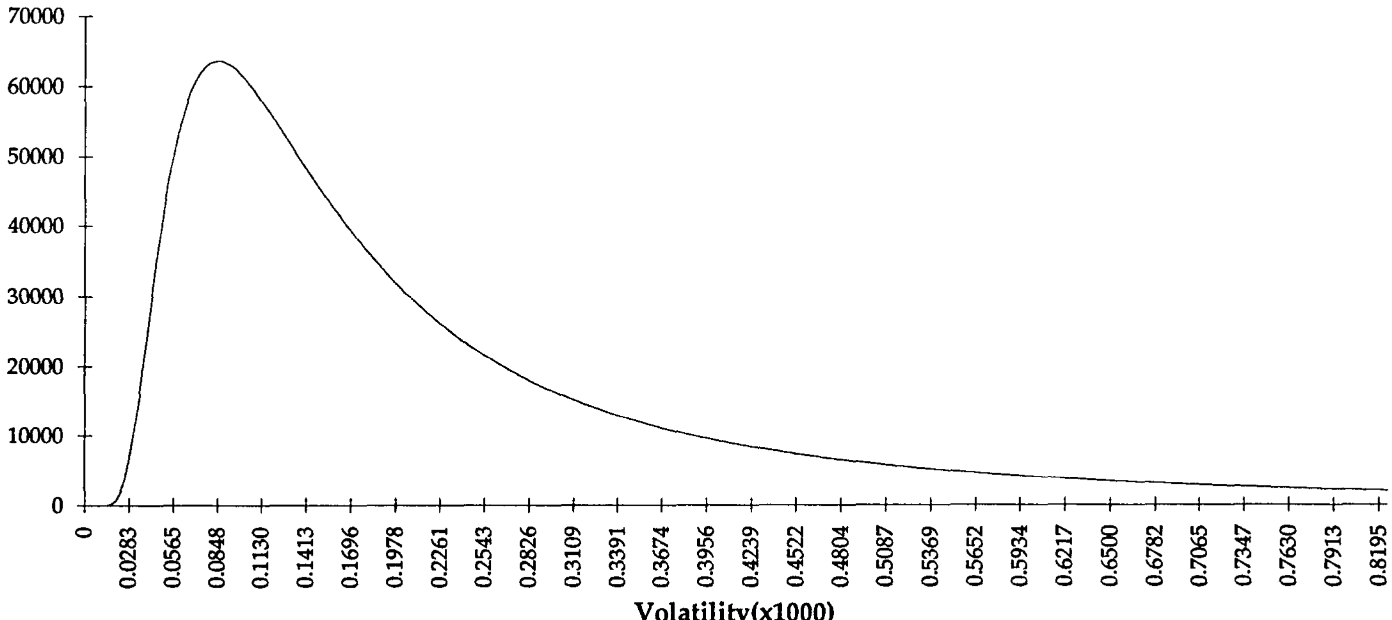
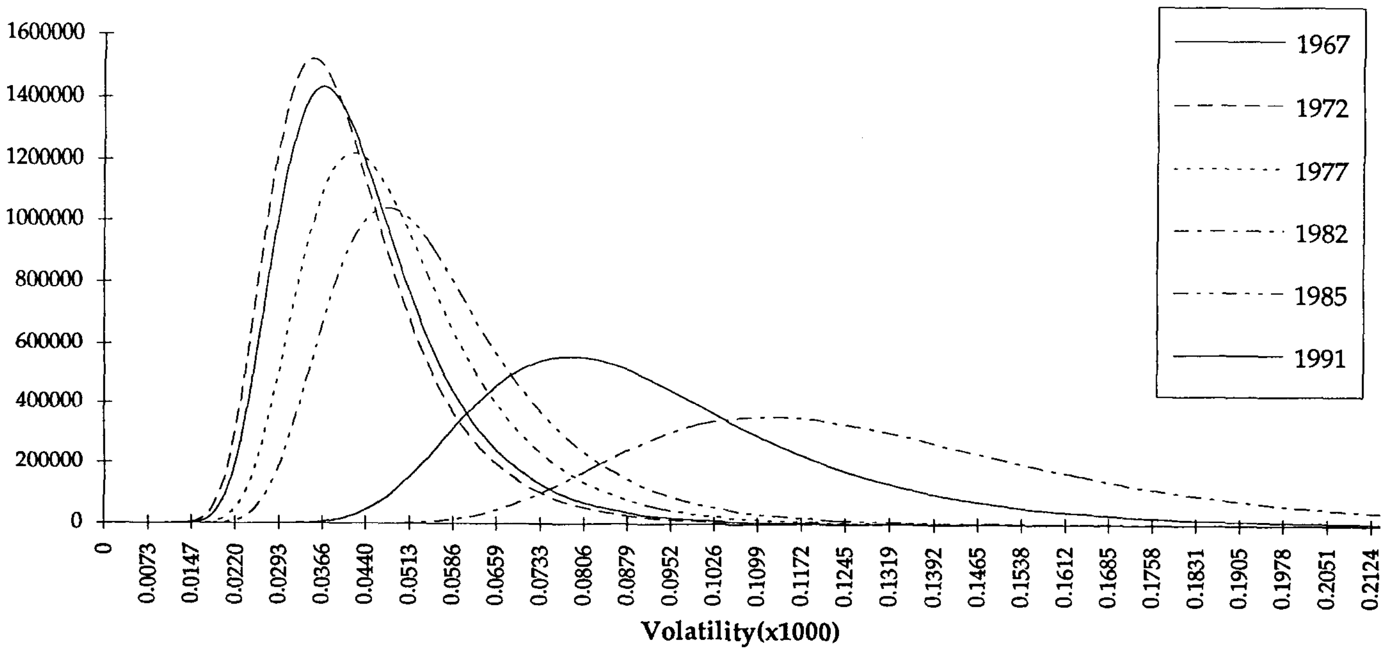


Figure 2: Volatility Over Selected Years

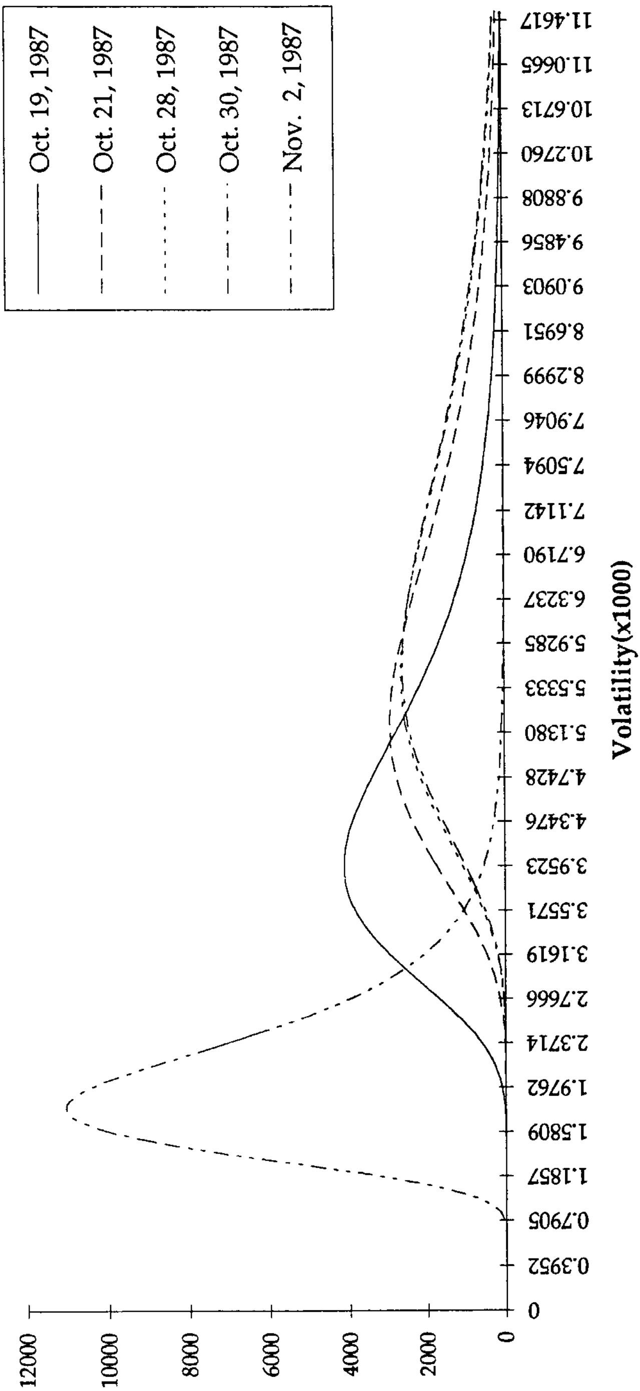


distribution. For year 1991, the posterior distribution is similar to the prior distribution. For the recession year of 1982, the posterior distribution is very thick tailed and has mass concentrated at large values of σ^2. Therefore, uncertainty in market volatility was smaller in years 1967, 1972, 1977 and 1985 as compared to 1991. In 1991, on average, stock prices did not contribute any information to reducing uncertainty in market volatility. In 1982, uncertainty in market volatility was increased after observing market prices. This analysis supports the observation that uncertainty in market volatility is smaller during periods of high market returns (bull markets) and larger during recessionary years when returns are not as great.

The third diagram illustrates posterior distributions of market volatility during and immediately following the stock market crash of 1987. Comparing Figure 2 to Figure 3, volatility during the stock market crash is 10 to 100 times greater than volatility observed during a typical year. Even during the year 1982, the average mode of market volatility is only 1.12×10^{-4} compared to 3.95×10^{-3} on October 19, 1987. There is an increase in market uncertainty in the week proceeding the crash but it is not significant enough to indicate an impending market collapse. Figure 3 also shows an improvement in market uncertainty two weeks after the market failure but it is still significantly larger than compared to the prior's uncertainty.

This example also illustrates one of the weaknesses of this model. Since volatility is assumed constant over an interval of 10 days, there is no distinction made in the calculations on the order in which market returns are observed. That is, on October 28 and October 30, these calculations would have produced the same graphs if the market crash occurred on October 19 or October 27!

Figure 3: Market Crash of 1987



6. Conclusion

The Black-Scholes option formula is today one of the most widely used formulae by market analysts. Unfortunately, stock volatility, which plays an integral part in this equation, is a difficult parameter to estimate.

The same model that sets the foundation for the Black-Scholes formula is used to derive a statistical model for the underlying stock volatility. For various reasons, the likelihood equation is analyzed using a Bayesian statistical approach. First, the market analyst will be able to incorporate his/her subjective beliefs of market volatility into the model through the specification of a prior distribution. The prior distribution, although requiring the specification of five hyperparameters, can be determined indirectly from certain natural characteristics of the model.

Second, graphing of the posterior distribution allows one to visually interpret uncertainty in market volatility. During periods of market uncertainty, the posterior distribution is thick tailed with a large mean (mode). Shifts in the posterior distribution represent changes in uncertainty of market volatility.

Third, since volatility is constantly changing, little data is available during any specific time period for an accurate estimate. This would make any large sample estimates that do not accurately model these changes in volatility inaccurate and therefore, not very useful.

Finally, uncertainty in market volatility is directly incorporated into the Black-Scholes model through the posterior distribution. Rather than substituting point estimates into the formula, the volatility is treated as a random variable. The expected value of the call option formula is found using the posterior distribution of market volatility. Compared to point estimates, this approach produces option prices that more closely reflect observed values.

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