NON-PARALLEL YIELD CURVE SHIFTS AND IMMUNIZATION

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Summary

The traditional approaches to immunization involve conditions on the durations and convexities of assets and liabilities. However, because the very definitions of duration and convexity reflect an underlying assumption of parallel yield curve shifts, the associated immunization strategies will generally fail when this assumption is not realized. This article explores the relationship of immunization to the underlying yield curve shift assumption, both in theory and with a detailed example of immunization of a surplus position. Results are also presented regarding immunization of the net worth asset ratio. From these analyses, it becomes clear that immunization against one type of shift may allow a great deal of exposure and vulnerability to other types.

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Déplacements divergents de la courbe du rendement et immunisation

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Résumé

Les politiques traditionnelles relatives à l'immunisation mettent en jeu des conditions imposées aux durées et convexités des actifs et passifs. Cependant, étant donné que la définition même de la durée et de la convexité reflète une assomption sous-jacente de déplacements parallèles de la courbe du rendement, les stratégies d'immunisation qui lui sont associées échouent généralement quand cette assomption ne se réalise pas. Le présent article explore le rapport existant entre l'immunisation et l'assomption sous-jacente des déplacements de la courbe du rendement, à la fois en théorie et sous forme d'exemple détaillé d'immunisation d'une position excédentaire. Des résultats sont également présentés relatifs à l'immunisation du rapport des actifs en capitaux propres. Ces analyses démontrent clairement que l'immunisation contre un type de déplacement peut permettre d'importants risques et une certaine vulnérabilité à d'autres types.

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Non-Parallel Yield Curve Shifts and Immunization

Why Immunization Need Not, and Often Does Not, Work

A common goal for asset/liability managers is to maintain the modified duration of assets equal to a multiple of the modified duration of liabilities, where this multiple equals the ratio of liability to asset market values. Duration calculations are often made with respect to yield curves which reflect the average qualities of the respective portfolios. For more precision, each quality sector is valued on the appropriate yield curve, and the portfolio duration values determined by taking weighted averages of the individual components. As is well known, these weights reflect the relative market values of the individual components.

The principle underlying this duration management approach is that the asset and liability portfolio values will move in tandem as the underlying yield curves move in parallel. That is, each portfolio will

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change by approximately the same absolute amount for yield curve shifts for which each yield point moves by the same absolute amount. Consequently, the surplus or net worth position will remain relatively stable. Put another way, this duration management approach assures that the duration of surplus will be zero.

Subject to additional conditions on the respective portfolio inertias or convexities [Bierwag (1987), Grove (1974), Reitano(1990b, 1991c)], this surplus value will be 'immunized'. That is, parallel yield curve shifts will only stabilize or improve its value.

Another common management approach is to maintain the duration of assets equal to the duration of liabilities. Parallel yield curve shifts will then cause assets and liabilities to change by approximately the same relative amount. Consequently, the surplus or net worth position will also change by this common relative amount, and the net worth asset ratio, or ratio of surplus to assets, will remain approximately constant. Again subject to conditions on asset and liability inertias or convexities (see above references and Kaufman (1984)), the net worth asset ratio will in fact be immunized against parallel yield curve shifts.

Analyzed from the perspective of surplus management, the above strategies disguise a number of risks. First of all, there are practical difficulties in maintaining perfect durational targets, and even small duration mismatches have the potential to create great surplus sensitivity (Messmore (1990)). In addition, managing duration values while ignoring convexities has the potential to 'reverse immunize' the account, in that parallel yield curve shifts will only stabilize or decrease the value of surplus or the net worth asset ratio under the respective strategies.

As it turns out, the underlying yield curve shift assumption poses the greatest potential for risk. In a series of articles (Reitano (1989, 1990a, 1991a,b,1992)), the limitations of the parallel shift assumption have been analyzed, and models developed which generalize the notions of duration and convexity to arbitrary yield curve shifts. In the process, it has become clear that the traditional measures can greatly disguise duration risk, as well as obscure the effects of convexity.

It is no surprise, therefore, that classical immunization theories, which rely on the parallel shift assumption underlying duration and convexity, can disguise risk and the potential for immunization to fail.

In this article, we explore this potential through the detailed analysis of an example of the immunization of a surplus position. For more generality and mathematical rigor, see Reitano

(1990b, 1991c). Although we focus on surplus immunization, the shortcomings of the traditional strategy to immunize the net worth asset ratio are comparable and readily illustrated with a second example, introduced in Reitano (1990a). For an example of the immunization of future values of surplus, see Reitano (1991c).

An Example - Surplus Immunization

Assets are composed of a $43.02 million, 12%, 10 year bond, and $25.65 million, 6 month commercial paper. The single liability is a $100 million guaranteed investment contract (GIC) payment in year 5. The current yield curve, on a bond yield basis, equals 7.5%, 9.0%, and 10.0% at maturities of .5, 5 and 10 years, respectively. Yields at other maturities are assumed to be interpolated, and spot rates derived in the usual way. That is, they are derived as to price the various bonds suggested by the bond yield curve to par.

Given these assumptions, we then obtain:

	Market Value	Duration	Convexity
Assets:	73.25	4.243	34.94
Liabilities:	63.97	4.858	25.89
Surplus:	9.28	0	96.85

It is easy to check that the asset duration equals the *liability duration times the ratio of liability to asset market values.*

Also, the astute reader may notice that this example is very similar to that introduced in Reitano (1990a). The difference here is a change in the mix of bonds and commercial paper to achieve the required asset duration. In the original example, the mix was chosen to reproduce the duration of liabilities.

How Immunization Works

Let's denote by $S(\Delta i)$, *the value of surplus if the yield curve moves in parallel by* Δi. *That is, using vector notation, the yield curve shifts as follows:*

$$(.075, .090, .100) \longrightarrow (.075 + \Delta i, .090 + \Delta i, .100 + \Delta i).$$

Of course, $S(0) = 9.28$ *as noted in the above table. A standard calculation produces the following approximation for* $S(\Delta i)$:

$$S(\Delta i) \simeq S(0)(1 - D^S \Delta i + 1/2\, C^S (\Delta i)^2), \qquad (1)$$

where D^S *is the duration of surplus,* $D^S = -S'(0)/S(0)$ *,and* C^S *its convexity,* $C^S = S''(0)/S(0)$ *(see Reitano (1989, 1991b) for details).*

It is clear from (1) that in order to have $S(\Delta i)$ no smaller than $S(0)$, we must have $D^S = 0$. This is because if D^S is positive, say, negative shifts would be favorable, but positive shifts unfavorable and $S(\Delta i)$ could fall below $S(0)$. Although the C^S term could help, the $(\Delta i)^2$ factor significantly dampens its effect.

In addition to $D^S = 0$, we require C^S to be positive to assure immunization. The approximation in (1) then becomes:

$$S(\Delta i) \simeq S(0)(1 + 1/2\, C^S(\Delta i)^2), \tag{2}$$

and the right-hand side of (2) can clearly be no smaller than $S(0)$. Consequently, we can be confident that the surplus value is immunized at least for moderate values of Δi. We say 'moderate' because for very large values of Δi, the $(\Delta i)^3$ and higher powered terms ignored in (1) and (2) can become significant.

To implement this surplus immunization, we require relationships between D^S and C^S and the corresponding values for assets and liabilities. A calculation shows that D^S is a weighted average of D^A and D^L, while C^S is a weighted average of C^A and C^L:

$$D^S = w_1 D^A + w_2 D^L, \tag{3}$$

$$C^S = w_1 C^A + w_2 C^L. \tag{4}$$

Here, $w_1 = A/S$, the reciprocal of the net worth asset ratio, while $w_2 = -L/S$, or minus one times the financial leverage ratio.

From (3) and (4), we see that in order to have D^S equal to 0, and C^S positive, we require that the duration of assets equals that of liabilities times L/A, and that the convexity of assets exceeds that multiple of liabilities:

$$D^A = \frac{L}{A} D^L, \tag{5}$$

$$C^A > \frac{L}{A} C^L. \tag{6}$$

From the values for the above example, we see that both (5) and (6) are satisfied. For this example, the approximation in (1) becomes:

$$S(\Delta i) \simeq 9.28\,[1 + 48.43\,(\Delta i)^2]. \tag{7}$$

Calculating actual surplus values and those estimated by (7), denoted $S^e(\Delta i)$, we obtain the results in Table 1. Note that immunization against parallel shifts is successful, and that the estimates obtained with (7) provide good approximations to the actual resulting $S(\Delta i)$ values.

How Immunization Fails In Theory

The above example illustrates an important point: traditional immunization can not fail in theory if the underlying assumptions are satisfied. To the extent it fails, it must fail because at least one of the assumptions underlying the model fails to hold. In practice, this failing assumption is typically the assumption of parallel shifts.

Does this mean that immunization is impossible if yield curve shifts are non-parallel? The answer is: No, but you have to change the model, which will in turn change the conditions necessary for immunization.

Two approaches are in fact possible. First of all, one can change the yield curve shift assumption from parallel to another explicit shift type, and develop conditions under which immunization is then achieved. Secondly, the more general question of immunization against arbitrary yield curve shifts can be explored.

In this article, we examine the first approach because it represents a mathematically more straightforward generalization of the classical theory, yet provides deep insight to immunization theory and practice (see Reitano (1990b, 1991c) for more generality).

To this end, assume that $\bar{N} = (n_1, n_2, n_3)$ *specifies the* *yield curve shift 'direction' of interest. For the above classical model, $\bar{N} = (1,1,1)$ is the assumed direction vector. In general, a shift of Δi 'in the direction of $\bar{N}$' will mean that the 6 month rate of .075 shifts by $n_1 \Delta i$, the 5 year rate of .09 by $n_2 \Delta i$, and the 10 year rate of .10 by $n_3 \Delta i$. Given this direction vector, one can define the notions of 'directional duration' and 'directional convexity' in the direction of $\bar{N}$. When $\bar{N} = (1,1,1)$, these notions reduce to the classical definitions of duration and convexity (see Reitano (1989, 1991b,d) for details).*

As it turns out (see Reitano (1990b, 1991c)), denoting by $S_N(\Delta i)$ the surplus value given this shift of Δi in the direction of $\bar{N}$, equation (1) still holds. The only difference is that the directional duration and convexity values, D_N^S and C_N^S, must be used. Analogous to the classical definitions,
$D_N^S = -S_N'(0)/S_N(0)$, *and* $C_N^S = S_N''(0)/S_N(0)$.
Further, (3) and (4) still hold, as do (5) and (6) as the appropriate conditions for immunization. That is, if:

$$D_N^A = \frac{L}{A} D_N^L, \tag{8}$$

$$C_N^A > \frac{L}{A} C_N^L, \tag{9}$$

the directional duration of surplus, D_N^S, will be zero, and the *directional convexity,* C_N^S*, will be positive. As in (2), therefore, surplus will be immunized against shifts in the direction of* $\bar{N}$*.*

Consequently, the classical theory generalizes naturally to immunization against shifts of any specified direction. Unfortunately, structuring the portfolio so that (5) and (6) are satisfied does not generally imply that (8) and (9) will be satisfied for other direction vectors $\bar{N}$*. More generally, structuring the portfolio to satisfy (8) and (9) for a given* $\bar{N}$ *does not imply that these constraints are satisfied for other direction vectors. The reason for this is that both* D_N *and* C_N *can vary greatly as* $\bar{N}$ *changes, and can vary differently for assets and liabilities.*

In theory, one can identify conditions under which 'complete immunization' is achieved. That is, conditions under which immunization is achieved for every direction vector $\bar{N}$ *simultaneously (see Reitano (1990b, 1991c)). Unfortunately, the condition on the durational structures of assets and liabilities is very restrictive and potentially difficult to implement, as is that for the convexity structures. Consequently, in practice some immunization exposure may be inevitable.*

Returning to the above example, which satisfied immunizing conditions for $\bar{N} = (1,1,1)$*, we investigate the potential range of* values for D_N^S and C_N^S, as the direction vector $\bar{N}$ changes. Because these ranges depend on the length of the vector $\bar{N}$, that is, the square root of the sum of the squares of its components, it is necessary to restrict this value. Since we wish to compare the resulting ranges of values to the values produced in the classical model where $\bar{N} = (1,1,1)$, we restrict the length of $\bar{N}$, denoted $|\bar{N}|$, to equal $|(1,1,1,)| = \sqrt{3}$.

Given $\bar{N} = (n_1,n_2,n_3)$, the directional duration of the exemplified surplus function, $S_N(\Delta i)$, is given by:

$$D_N^S = 4.55n_1 - 35.43n_2 + 30.88n_3. \qquad (10)$$

The coefficients in (10) are the 'partial durations' of surplus, viewed as a function of the 6 month, 5 year and 10 year yield rates. A calculation shows that for $\bar{N} = (1,1,1)$, the classical parallel shift assumption, we obtain $D_N^S = D^S = 0$ as expected.

For non-parallel yield curve shifts, however, the directional duration of surplus can be much different from 0. Specifically, restricting our attention to direction vectors of the same length as the parallel shift (1,1,1), we have:

$$-81.78 \leq D_N^S \leq 81.78, \qquad |\bar{N}| = \sqrt{3}. \qquad (11)$$

That is, the durational sensitivity of surplus can be as large as 81.78, and as small as -81.78, when yield curve shifts are allowed to be non-parallel. The v-shaped non-parallel shift, $\bar{N} = (.167, -1.300, 1.133)$, has length $\sqrt{3}$ and produces the extreme positive duration, $D_N^S = 82.19$. Similarly, $-\bar{N}$ is an extreme negative shift.

As has been noted in the above referenced articles, all extreme shifts are proportional to the 'total duration vector', $\bar{D}^S = (4.55, -35.43, 30.88)$, made up from the partial durations used in (10). For example, the extreme 'positive' shift $\bar{N}$ above is about 3.7% of $\bar{D}^S$.

Mathematically, the inequalities in (11) are produced using the Cauchy-Schwarz inequality for the size of an inner product or dot product. Because the expression for D_N^S in (10) equals an inner product of $\bar{D}^S$ with $\bar{N}$, the Cauchy-Schwarz inequality states that this value is less than or equal to the product of the lengths of these vectors, and greater than or equal to -1 times this value. In addition, the extremes of this inequality are achieved when the given vectors are parallel (see Reitano (1989), (1991b) for details).

Analogous to (10), the general formula for C_N^S is:

$$C_N^S = 7.14n_1^2 - 126.21n_2^2 - 127.64n_3^2$$

$$+ 2(-25.80n_1n_2 + 9.63n_1n_3 + 60.31n_2n_3). \quad (12)$$

The coefficients in (12) are the 'partial convexities' of surplus. A calculation shows that when $\bar{N} = (1,1,1)$, $C_N^S = 96.85$, *which equals the* C^S *value noted above.*

For non-parallel yield curve shifts, the directional convexity value produced by (12) can be significantly different from this parallel shift value, and even negative. In particular, restricting our attention to direction vectors $\bar{N}$ *of length* $\sqrt{3}$, *the length of* $(1,1,1)$, *we have:*

$$-434.15 \leq C_N^S \leq 424.04, \qquad |\bar{N}| = \sqrt{3}. \qquad (13)$$

In addition, the yield curve shifts of extreme convexity are $\bar{N}_1 = (-.306, -1.662, .379)$ *and* $\bar{N}_2 = (.049, .376, 1.690)$.

A simple calculation shows that except for rounding, both shift vectors have length equal to $\sqrt{3}$, *and using (12),* $\bar{N}_1$ *produces the negative lower bound in (13), while* $\bar{N}_2$ *produces the positive upper bound.*

Mathematically, the inequalities in (13) are developed from (12) by noting that this expression for C_N^S *is in fact a quadratic form in the vector* $\bar{N}$. *That is, this expression equals* $\bar{N}^T \bar{C}^S \bar{N}$, *where* $\bar{C}^S$ *is the matrix of partial convexities, or the 'total convexity matrix'. Standard analysis techniques then reveal that this* function is less than or equal to $|\bar{N}|^2$ times one constant, and greater than or equal to $|\bar{N}|^2$ times another constant. These constants are the largest and smallest 'eigenvalues' of $\bar{C}^S$, respectively, and the function in (12) assumes these outer bounds when $\bar{N}$ is proportional to the associated 'eigenvectors' (see Reitano 1991b for details).

How Immunization Fails in Practice

In theory, it is clear from (11) that D_N^S need not be close to zero, even though it equals zero when $\bar{N} = (1,1,1)$. Similarly, from (13) we see that C_N^S need not be positive, even though it equals 96.85 when $\bar{N} = (1,1,1)$. Consequently, since we have as in (1):

$$S_N(\Delta i) \simeq S(0)(1 - D_N^S \Delta i + 1/2\, C_N^S (\Delta i)^2), \quad (14)$$

it is clear that the surplus value need not be immunized in theory for general shift directions $\bar{N}$ other than (1,1,1). That is, it need not be the case that $S_N(\Delta i)$ will equal or exceed $S(0) = 9.28$, in theory.

What about in practice, with actual observable yield curve shifts? Certainly, if yield curve shifts never occurred which made

D_N^S large, or C_N^S negative, the above theory would provide little insight to immunization practice.

As a historic data base, we investigated monthly movements in the Treasury yield curve from the end of August, 1984 to June 1990, at maturities of 6 months, 5 and 10 years. Both one-month and overlapping 6 month yield curve change vectors, $\bar{N}$, were analyzed. Based on 65 overlapping half-year change vectors, normalized to have $|\bar{N}| = \sqrt{3}$, we observed that:

$$
\begin{aligned}
-12.37 &\leq D_N^S \leq 30.38, \\
-203.12 &\leq C_N^S \leq 338.41.
\end{aligned}
\tag{15}
$$

Comparing the D_N^S values produced during this period to the theoretical range in (11), we conclude that while significant duration values were observed, the real world was relatively tame to this example compared to what theory suggests, covering only 26% of the potential range of values. Similarly, the observed C_N^S values, while clearly not all positive, were again somewhat tamely distributed compared to (13), though covering a larger percentage of possible values (63%) than did the associated D_N^S values.

Similar conclusions can be drawn from the 70 monthly change vectors, which produced the following somewhat larger ranges:

$$- 20.53 \leq D_N^S \leq 35.69,$$
$$- 228.80 \leq C_N^S \leq 368.73. \qquad (16)$$

Turning next to the corresponding estimates of the surplus values using (14), the following range of values was produced using half-year change vectors:

$$8.25 \leq S_N^e(\Delta i) \leq 10.52. \qquad (17)$$

The range for monthly change vectors was very similar, extending from 8.67 to 10.21. Both ranges compare unfavorably to the initial surplus value, S(0) = 9.28, implying that immunization was often not successful.

As for the distribution of results, Table 2 provides percentile data for the half-year change vectors. Almost half (30) of the 65 change vectors produced negative duration values, placing $D_N^S = 0$ *when* $\bar{N} = (1,1,1)$ *at about the 46th percentile of results. In addition, only 4 change vectors (6%) produced duration sensitivities less than 2.0 in absolute value, implying the extent to which the traditional value,* $D^S = 0$*, disguised surplus risk.*

For directional convexities, only about 23% of the sample yield curve changes produced negative values, which may appear at odds *with the symmetry of the theoretical interval in (13). However, the theoretical interval provides no information about the expected distribution of results; it only defines its possible range. On the other hand, the yield curve shifts experienced during this relatively short period should not be interpreted as constraining those possible in other periods. The traditional value,* $C_N^S = 96.85$ *when* $\overline{N} = (1,1,1)$, *is seen to be at about the 60th percentile of this distribution.*

From the distribution of estimated surplus values, $S_N^e(\Delta i)$, *we observe that the initial value,* $S(0) = 9.28$, *is at about the 54th percentile. That is, immunization was unsuccessful in a little more than half of the 6 month periods studied. Also, the relative changes in surplus caused by these yield curve shifts are seen to be substantial, extending from -11.1% to +13.3%.*

In general, the above comments on the Table 2 distributions apply equally well to the distribution of monthly change vectors on Table 3. One exception relates to D_N^S, *in that about 16% of the yield curve vectors produced duration sensitivities less than 2.0 in absolute value, compared with 6% in the distribution of half-year results. Also, almost 40% of the sample* C_N^S *values were negative, though skewness to positive values is still evident in this distribution. Finally, while still unfavorable about 50% of the time, the distribution of surplus changes is more tightly distributed,* reflective of the shorter time frame used for yield curve changes.

It is natural to inquire into the accuracy of the surplus approximation in (14), which was used to evaluate the efficacy of immunization on Tables 2 and 3. Table 4 contains actual and estimated values of $S_N(\Delta i)$ for 11 non-overlapping 6 month periods from January, 1985 to June, 1990.

As can be seen, the approximation in (14) produced very good accuracy in all cases. In addition, we see that the range of resulting S_N values span the range produced on Table 2. Finally, based on Table 4, immunization was unsuccessful during 6 of the 11 periods.

An Example - Immunization of the Surplus Ratio

As noted in the introduction, the net worth asset ratio, $r^S = S/A$, can be immunized against parallel yield curve shifts by matching the asset to the liability duration, and maintaining more asset convexity:

$$D^A = D^L, \qquad C^A > C^L. \tag{18}$$

As in the surplus immunization case above, immunization against shifts in the direction of $\bar{N}$ can be insured by (18), if directional durations and convexities are used in these constraints (Reitano (1990b, 1991c)).

Unfortunately, the problem here is the same as that illustrated above. That is, structuring the portfolio to satisfy (18) for one assumption about $\bar{N}$ (for example, $\bar{N} = (1,1,1)$), does not insure that such conditions are satisfied for other assumptions due to the potential for D_N and C_N to vary as in (11) and (13).

Consider the example above, only changing the mix of assets to \$50 million of the bond, and \$17.48 million of the commerical paper, as in Reitano (1990a). The duration of assets (4.857) then equals that of liabilities, while the convexity (40.41) exceeds that of the liabilities. The initial net worth asset ratio, r^S, then equals .12669.

While the same detailed analysis as above is possible, we present only the counterpart to Table 4. That is, on Table 5 is shown the values of the net worth asset ratios after actual 6 month yield curve changes, $R_N(\Delta i)$, as well as those estimated by a formula comparable to (14).

As for Table 4, immunization was unsuccessful during 6 of the

11 periods. In addition, the actual net worth asset ratios were well approximated by the approximating formulas over the full range of results.

Summary and Conclusions

Classical immunization strategies, which explicitly reflect the assumption of parallel yield curve shifts, cannot in theory be expected to provide immunization when the yield curve shifts do not cooperate with this defining assumption. However, these conditions readily generalize to conditions which insure immunization against any given yield curve shift assumption. Unfortunately, these conditions are not compatible in general. That is, immunization against a given type of shift will often create exposure to other types of shifts, causing immunization to fail as other shifts are realized.

An ancillary benefit of the theoretical analysis, however, is that one can develop estimates of the degree of immunization risk. Inequalities such as in (11) and (13) provide the theoretical unit exposures to duration and convexity risk. In addition, these values were seen to capture much of the potential for immunization to fail, as the approximations for $S_N(\Delta i)$ *in (14) and those for* $R_N(\Delta i)$ *accurately estimated actual values over a wide range of yield curve movements.*

Of course, once immunization risk is quantified, the first step toward its reduction has been taken.

Table 1

Δi	$S(\Delta i)$	$S^e(\Delta i)$
-.02	9.481	9.460
-.01	9.327	9.325
-.005	9.291	9.291
0	9.280	9.280
.005	9.290	9.291
.01	9.322	9.325
.02	9.440	9.460

Table 2

Distribution of D_N^S, C_N^S and $S_N^e(\Delta i)$

65 Overlapping 6 month Periods

August, 1984 - June, 1990

Percentile	D_N^S	C_N^S	$S_N^e(\Delta i)$	$\frac{\Delta S}{S}$ X 100%
1.5%	-12.37	-203.12	8.25	-11.1%
10%	- 8.56	- 52.32	8.61	- 7.2%
20%	- 6.06	- 15.37	8.76	- 5.6%
30%	- 3.82	17.10	8.95	- 3.5%
40	- 2.49	38.88	9.04	- 2.6%
50%	2.05	61.79	9.15	- 1.4%
60%	3.82	95.79	9.51	+ 2.4%
70%	4.24	137.62	9.69	+ 4.4%
80%	6.47	176.85	9.97	+ 7.5%
90%	8.94	212.49	10.06	+ 8.4%
100%	30.38	338.41	10.52	+13.3%

Note: D_N^S and C_N^S are normalized so that $|\vec{N}| = |(1,1,1)| = \sqrt{3}$.

Table 3

Distribution of D_N^S, C_N^S and $S_N^e(\Delta i)$

70 One Month Periods

August, 1984 - June, 1990

Percentile	D_N^S	C_N^S	$S_N^e(\Delta i)$	$\frac{\Delta S}{S}$ X 100%
1.5%	-20.53	-228.80	8.67	- 6.6%
10%	-16.10	- 70.60	8.86	- 4.5%
20%	- 9.27	- 54.66	9.05	- 2.4%
30%	- 5.91	- 30.19	9.16	- 1.3%
40%	- 2.44	2.48	9.22	- 0.6%
50%	- 0.35	52.32	9.30	+ 0.2%
60%	2.13	105.86	9.36	+ 0.8%
70%	4.26	131.68	9.41	+ 1.4%
80%	10.23	162.78	9.50	+ 2.4%
90%	12.52	206.92	9.54	+ 2.8%
100%	35.69	368.73	10.21	+10.1%

Note: D_N^S and C_N^S are normalized so that $|\bar{N}| = |(1,1,1)| = \sqrt{3}$.

Table 4

Actual v. Estimated Values

Surplus Values After Yield Curve Changes

from Non-overlapping 6 month Periods

6 months beginning	$S_N(\Delta i)$	$S_N^e(\Delta i)$	$S_N^e(\Delta i)$ Percentile
1/1/85*	8.868	8.861	25th
7/1/85*	9.132	9.127	48th
1/1/86	10.529	10.517	100th
7/1/86*	8.382	8.383	5th
1/1/87	9.878	9.883	77th
7/1/87*	9.040	9.040	40th
1/1/88*	9.219	9.219	52nd
7/1/88	10.001	10.000	83rd
1/1/89*	8.899	8.896	27th
7/1/89	9.328	9.328	55th
1/1/90	9.508	9.509	60th

* Immunization unsuccessful: $S(0) = 9.280$

Table 5

Actual v. Estimated Values

Net Worth Asset Ratios at Ends

of Non-overlapping 6 month Periods

6 months beginning	$R_N(\Delta i)$	$R_N^e(\Delta i)$	$R_N^e(\Delta i)$ Percentile
1/1/85*	12.140%	12.142%	23rd
7/1/85*	12.556%	12.557%	51st
1/1/86	14.267%	14.272%	100th
7/1/86*	11.434%	11.436%	3rd
1/1/87	13.480%	13.478%	78th
7/1/87*	12.325%	12.324%	35th
1/1/88*	12.626%	12.624%	52nd
7/1/88	13.760%	13.752%	89th
1/1/89*	12.206%	12.208%	26th
7/1/89	12.728%	12.729%	56th
1/1/90	12.964%	12.963%	60th

* Immunization unsuccessful: r^S = 12.669%

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