Some Aspects of the Martingale Approach to the Term Structure of Interest Rates

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Summary

The martingale approach to modelling the term structure of interest rates is an important theoretical tool. We illustrate this approach through the derivation of bond and bond option prices using direct techniques from stochastic calculus for a class of Gaussian models. By carrying out the analysis of the model in this manner, one may see how the spot rate process, the information structure, and the market price of risk come together to build the model. Also, a direct approach requires no a priori assumptions about differentiability and the form of the bond price function which is necessary to carry through an analysis based on differential equations. We provide a formal statement and proof of how one may induce a multi-factor term structure model from a choice of independent factors, thereby generating a closed form multi-factor model based on the closed formula for the one-factor case for each of the factors. The theory behind this technique cannot be easily seen in the context of partial differential equations but is easily understood in the martingale context.

Certains aspects de la méthode des martingales utilisée dans la modélisation à termes des taux d'intérêt

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Résumé

La méthode des martingales utilisée dans la modélisation à termes des taux d'intérêt constitue un outil théorique précieux. Nous illustrons cette méthode par la dérivation des prix des obligations et des options d'obligations à l'aide de techniques directes tirées du calcul stochastique utilisé pour une classe de modèles gaussiens. En procédant ainsi à l'analyse du modèle, on peut voir comment le processus du cours au comptant, la structure des informations et le prix courant du risque s'allient pour construire le modèle. De plus, une méthode directe ne requiert aucunes assomptions a priori sur les possibilités différentielles et sur la forme de la fonction du prix de l'obligation nécessaire à une analyse fondée sur des équations différentielles. Nous apportons une déclaration formelle et la preuve de la façon dont on peut induire un modèle à termes à facteurs multiples à partir d'un choix de facteurs indépendants, créant ainsi un modèle à facteurs multiples sous forme fermée et fondé sur la formule fermée pour le cas à facteur unique et pour chacun des facteurs. La théorie sur laquelle repose cette technique ne peut être aisément perçue dans le contexte d'équations différentielles partielles mais est aisément comprise dans le contexte des martingales.

1. Introduction

The earliest papers [Vasicek (1977), Dothan (1978), Richard (1978), and Cox, Ingersoll, and Ross (1985)] in continuous time term structure modelling solved partial differential equations to get closed formulas for term structure models and bond option prices. As the mathematical economics underlying these models gradually became formalised, the martingale approach to the construction of arbitrage-free term structure models emerged. Artzner and Delbaen (1989) provides a thorough discussion of these issues. The martingale approach to term structure modelling offers some advantages over the partial differential equation method and one of the most important of these is that we have some general formulas that can be manipulated without ever having to solve partial differential equations.

A useful technique that was developed from the martingale point of view is the forward risk-adjusted measure of Jamshidian (1990, 1991). This technique provides a way to simplify option pricing in stochastic interest rate environments and is based directly on the expectation formula used in the martingale approach to term structure modelling. A discussion of Jamshidian's forward risk adjusted measure may be found in Pedersen and Shiu (1993). A related bond option pricing formula is given in Brenner and Jarrow (1993). Many authors [Jamshidian (1993), Dybvig (1989), Hull and White (1990)] have remarked on how one obtains a multi-factor model from several independent one-factor models. The martingale framework is ideal for providing a formal statement and its proof of how one may induce a multi-factor term structure model from a choice of independent factors.

Section 2 of the paper gives a brief review of the martingale approach to term structure modelling. Section 3 of the paper derives the multi-factor Gaussian model of Chaplin and Sharp (1993) and indicates how to obtain the bond option prices in this model. Gaussian models are popular because of their mathematical tractability. The explicit computations in section 3 also serve to provide an example of how the pieces of the martingale approach fit together to build a term structure model. Section 4 provides a formal statement and proof of how one may induce a multi-factor term structure model from a choice of independent factors.

2. A Summary of the Martingale Approach

In this section we give a brief description of what is involved in the martingale approach to term structure modelling. The essential point is that an arbitrage-free model of the term structure may be prescribed by the equation (1) presented below. The work of

Artzner and Delbaen (1989) is a thorough, though demanding, reference on this topic. The book of Duffie (1992, chapter 7) may also be consulted for this theory.

Uncertainty in the model is captured by a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}$. The filtration is an increasing family of sub σ-algebras of $\mathcal{F}$; by increasing is meant $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$; and the filtration is interpreted as the way in which the information in the model evolves or is revealed through time. The filtration is often referred to as the information structure. Also given is a spot rate process r. For $0 \le t \le T$, let P(t, T) denote the price at time t of a zero-coupon bond that pays 1 at time T. Note that P(T, T) = 1. If Q is another measure on this probability space then we say that Q is *equivalent* to P provided that Q has the same null sets as P. A probability measure Q is said to be an *equivalent martingale measure* if Q is equivalent to P and if all bond prices satisfy the equation

$$P(t, T) = E^Q[\exp(-\int_t^T r_u \, du) \mid \mathcal{F}_t], \qquad (1)$$

where E^Q denotes expectation with respect to the probability measure Q. The term "equivalent martingale measure" comes from the fact that the process $\{\exp(-\int_0^t r_u \, du)P(t, T)\}_{0 \le t \le T}$ is a martingale under Q.

A term structure model is arbitrage-free if there exists an equivalent martingale measure Q. The reader may consult Artzner and Delbaen (1989) for the theory behind this statement. Consequently, one method of prescribing an arbitrage-free model of the term structure of interest rates is to choose a spot rate process r and an equivalent probability measure Q and proceed to define bond prices by equation (1). This procedure is known as the martingale approach to constructing arbitrage-free models of the term structure. Another approach to constructing an arbitrage-free model of the term structure is to argue that in the absence of arbitrage there must exist a "market price of risk process" from which a partial differential equation is obtained which is then solved for the bond prices. This is the approach taken in Vasicek (1977) and Chaplin and Sharp (1993, appendix I). Implicit in this approach is an argument that an equivalent martingale measure must exist. The resolution of the question of whether there is any loss in generality in approaching term structure modelling from the martingale approach requires a significant amount of machinery. This issue is discussed in Duffie (1992, chapters 6, 7). We will leave this issue aside except to say that for practical purposes there is no loss in generality inherent in the martingale approach.

Although the martingale approach to term structure modelling is a theoretically useful tool, it must be admitted that the idea of choosing an equivalent probability measure Q does not readily appeal to one's intuition. In appendix 1 we offer an informal discussion of the intuition behind Q by relating it to the market price of risk process. We see from this discussion, particularly equations (A-1) and (A-2), that the equivalent measure Q governs the relationship between risk and return for the bond prices. This provides an intuition for what the role of the equivalent probability measure is as an input in the martingale approach to term structure modelling. In the remainder of the paper we will say nothing more about this intuition.

We have offered a brief indication as to why we may approach arbitrage-free term structure modelling by selecting a spot rate process and an equivalent probability measure Q and then proceed to define bond prices by the formula (1). We point out that this formula serves to define a particular term structure model for a particular choice of a spot rate process r and an equivalent probability measure Q. In the following section we will see how to evaluate this expectation for a particular choice of r and Q which leads to a Gaussian model.

Consider now a European call option with expiry date t and strike price K that is written on a discount bond with maturity T, t<T. When this call option expires at time t, it pays the maximum of zero and P(t, T) – K. It then follows that the price of this option at time 0 is given by

$$E^Q[\exp(-\int_0^t r_u\,du)\,\mathrm{Max}[0, P(t,T)-K]\,]\,. \qquad (2)$$

The pricing of the option in the martingale approach to term structure modelling that is described here is a two-step procedure. The first step is a prescription of the bond prices which is made by choosing a spot rate process and an equivalent probability measure with respect to which the bond prices are defined through equation (1). The second step is to employ these bond price processes to price the option using equation (2).

3. A Multi-factor Gaussian Model

The model that we present here comes from the work of Chaplin and Sharp (1993). In order to enable the reader to compare what we do here with the approach of Chaplin and Sharp, we follow their notation as far as possible.

We now specialise the probability space and information structure discussed in section 2. We take as primitive a probability space on which is defined an n-dimensional correlated Brownian motion. The information structure is taken to be the Brownian filtration generated by the correlated Brownian motion. This n-dimensional correlated Brownian motion is denoted by Z and we denote its components by $Z^{(1)}, Z^{(2)}, \ldots, Z^{(n)}$. The spot rate process will be taken as the sum of n factors,

$$r = r^{(1)} + r^{(2)} + \cdots + r^{(n)}.$$

Such a decomposition of the spot rate may arise from a variety of economic factors such as inflation rates and price indices. Chaplin and Sharp (1993) provide a discussion of the intuition behind this decomposition. Each of the factors is assumed to follow a stochastic differential equation

$$dr_t^{(i)} = \alpha_i(\gamma_i - r_t^{(i)})dt + \eta_i dZ_t^{(i)}, \quad i = 1, 2, \ldots, n \qquad (3)$$

where α_i, γ_i, and η_i are positive constants. The parameter γ_i is interpreted as the long run mean level for the factor $r^{(i)}$, while α_i is the speed with which this reversion to the mean occurs and η_i reflects the instantaneous variability in $r^{(i)}$. Note that there is one stochastic differential equation for each factor and that the i^{th} component of the correlated Brownian motion appears in the dynamics of the i^{th} factor.

By the method of integrating factors (Arnold, 1974, page 130) the solution to equation (3) may be expressed as

$$r_s^{(i)} = e^{-\alpha_i(s-t)} r_t^{(i)} + \int_t^s e^{-\alpha_i(s-u)} \alpha_i \gamma_i \, du + \int_t^s e^{-\alpha_i(s-u)} \eta_i \, dZ_u^{(i)}, \ s>t. \qquad (4)$$

In order to complete the specification of this model, we must select an equivalent probability measure Q, and having done so we may then turn to the computation of the expectation (1). However, it transpires that it is convenient to modify this expectation once we understand the form of the equivalent probability measure Q rather than to compute this expectation directly. This is the idea of risk-neutralising the factor processes that is mentioned in Chaplin and Sharp (1993, appendix I). Therefore, we specify the form of the equivalent probability measure Q in appendix 2 where we also discuss the modification of the expectation that we will now apply.

As shown in appendix 2, the problem of computing the bond prices for this term structure model becomes that of computing the expectation (1) under the original measure P with each γ_i adjusted appropriately. Specifically, the bond prices for this model are given by

$$P(t, T) = E[\exp(-\int_t^T r_u \, du) \mid \mathcal{F}_t] \tag{5}$$

where the factors satisfy the adjusted stochastic differential equations

$$dr_t^{(i)} = [\alpha_i(\gamma_i - r_t^{(i)}) - \eta_i \lambda_i] \, dt + \eta_i dZ_t^{(i)} . \tag{6}$$

(In the interests of simplicity, we have not used a new symbol for the correlated Brownian motion in equation (6). The nature of this abuse of notation can be seen from equations (A-5) and (A-6) in appendix 2.)

In order to get the bond prices for the Chaplin-Sharp model, it remains to evaluate the expectation in equation (5) for $r = r^{(1)} + r^{(2)} + \cdots + r^{(n)}$ where $r^{(i)}$ has the dynamics specified by equation (6). In the following it will be convenient to define the functions $F_i(s) := [1 - \exp(-\alpha_i s)]/\alpha_i$. One obtains the identity

$$\begin{aligned}\int_t^T r_s^{(i)} \, ds = {} & r_t^{(i)} F_i(T-t) + (\gamma_i - \eta_i \lambda_i/\alpha_i)(T-t) - (\gamma_i - \eta_i \lambda_i/\alpha_i) F_i(T-t) \\ & + \int_t^T \eta_i F_i(T-s) \, dZ_s^{(i)}\end{aligned} \tag{7}$$

by applying the stochastic calculus version of integration by parts (Karatzas and Shreve, 1988, page 155). Summing from i = 1, ... , n yields

$$\begin{aligned}\int_t^T r_s \, ds = {} & \sum_{i=1}^n \left\{ r_t^{(i)} F_i(T-t) + (\gamma_i - \eta_i \lambda_i/\alpha_i)(T-t) - (\gamma_i - \eta_i \lambda_i/\alpha_i) F_i(T-t) \right\} \\ & + \sum_{i=1}^n \int_t^T \eta_i F_i(T-s) \, dZ_s^{(i)} .\end{aligned} \tag{8}$$

We then see that the expectation in equation (5) is equal to

$$\exp\Big(-\sum_{i=1}^{n}\Big\{r_t^{(i)}F_i(T-t) + (\gamma_i - \eta_i\lambda_i/\alpha_i)(T-t) - (\gamma_i - \eta_i\lambda_i/\alpha_i)F_i(T-t)\Big\}\Big)$$
$$\times E\Big[\exp\Big(-\sum_{i=1}^{n}\int_t^T \eta_i F_i(T-s)\, dZ_s^{(i)}\Big)\Big]. \qquad (5')$$

Define $X := \sum_{i=1}^{n}\int_t^T \eta_i F_i(T-s)\, dZ_s^{(i)}$. The variance of X is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\int_t^T \eta_i\eta_j\rho_{ij}F_i(t-s)F_j(t-s)\, ds$$

(Karatzas and Shreve, 1988, page 140, (2.23)). Define $R_{ij} := \eta_i\eta_j\rho_{ij}/(\alpha_i + \alpha_j)$. Then a straightforward but laborious calculation involving completing the square shows that this variance is equal to

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\eta_i\eta_i\rho_{ij}}{\alpha_i\alpha_j}(T-t) - 2\sum_{i=1}^{n}F_i(T-t)\frac{1}{\alpha_i}\sum_{j=1}^{n}R_{ij}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{n}R_{ij}F_i(T-t)F_j(T-t). \qquad (9)$$

Since X has a normal distribution (Arnold, 1974, page 77, (4.5.6)) the expectation in expression (5') is E[exp(−X)] = exp(−E[X] + (1/2)Var(X)) by the normal moment generating function formula. The expectation of X is zero and the variance of X is given by (9) so that the expression (5') becomes

$$\exp\Big(-\sum_{i=1}^{n}\Big\{r_t^{(i)}F_i(T-t) + (\gamma_i - \eta_i\lambda_i/\alpha_i)(T-t) - (\gamma_i - \eta_i\lambda_i/\alpha_i)F_i(T-t)\Big\}$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\eta_i\eta_j\rho_{ij}}{\alpha_i\alpha_j}(T-t) - \sum_{i=1}^{n}F_i(T-t)\frac{1}{\alpha_i}\sum_{j=1}^{n}R_{ij} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}R_{ij}F_i(T-t)F_j(T-t)\Big).$$

Define $D_i := \gamma_i - \lambda_i\eta_i/\alpha_i - \eta_i^2/(2\alpha_i^2)$, and

$$H(s) := s[\sum_{i=1}^{n} D_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\eta_i\eta_j\rho_{ij}}{\alpha_i\alpha_j}] - \sum_{i=1}^{n} F_i(s)[D_i - \frac{1}{\alpha_i}\sum_{\substack{j=1\\ j\neq i}}^{n} R_{ij}]$$

$$+\sum_{i=1}^{n} F_i(s)^2 \frac{\eta_i^2}{4\alpha_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} R_{ij}F_i(s)F_j(s)\,,$$

then formula (5) becomes

$$P(t,T) = \exp(-\sum_{i=1}^{n} r_t^{(i)}F_i(T-t) - H(T-t))\,. \tag{10}$$

Equation (10) gives the bond prices for this model. Note that the bond prices are not a function of r, but rather of the vector of factors $(r^{(1)}, \dots, r^{(n)})$.

In the case of an n-dimensional standard Brownian motion, so that $\rho_{ij} = 0$ for $i\neq j$, we see that the function H reduces to $H(s) = \sum_{i=1}^{n} (sD_i - F_i(s)D_i + F_i(s)^2 \frac{\eta_i^2}{4\alpha_i})$, and the bond prices may be written in the form

$$P(t,T) = \prod_{i=1}^{n} \exp(-r_t^{(i)}F_i(T-t) - [(T-t)D_i - F_i(T-t)D_i + F_i(T-t)^2 \frac{\eta_i^2}{4\alpha_i}])\,. \tag{11}$$

The one-factor Vasicek (1977) model is a special case of the above model and is specified by the spot rate process

$$dr_t = \alpha(\gamma - r_t)dt + \eta dZ_t\,,$$

and the equivalent probability measure

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \exp(-\lambda Z_T - \frac{1}{2}\lambda^2 T)\,.$$

The resulting bond prices for this model are

$$P(t, T) = \exp(-r_t F(T-t) - [(T-t)D - F(T-t)D + F(T-t)^2 \frac{\eta^2}{4\alpha}]) . \qquad (12)$$

We now discuss the pricing of a European call option written on a bond for the general case of the Chaplin-Sharp model with bond prices as given by (10). Specifically, consider a European call option with expiry date t and strike price K that is written on a discount bond with maturity T, $t<T$. Then the price of this option is given by the expectation in (2) where we are of course using the spot rate process r, the equivalent probability measure Q, and the bond price function P(t, T) of this model. The computation of this expectation is complicated by the presence of the random variable $\exp(-\int_0^t r_s \, ds)$. One way to circumvent this difficulty is to use the forward risk-adjusted measure of Jamshidian (1990, 1991). This approach is discussed in the context of the martingale approach to term structure modelling in Pedersen and Shiu (1993). It is shown there that, provided that ln[P(t, T)], ln(dQ/dP), and $\int_0^t r_s \, ds$ are jointly normal under the original measure P, the price of the European option is

$$P(0, T)\Phi\left(\frac{1}{\sigma}\ln\left[\frac{1}{K}\frac{P(0, T)}{P(0, t)}\right] + \frac{\sigma}{2}\right) - P(0, t)K\Phi\left(\frac{1}{\sigma}\ln\left[\frac{1}{K}\frac{P(0, T)}{P(0, t)}\right] - \frac{\sigma}{2}\right)$$

where σ^2 is the variance of ln[P(t, T)] under the original probability measure P. It follows from Arnold (1974, page 77, (4.5.6)) that in the Chaplin-Sharp model ln[P(t, T)], ln(dQ/dP), and $\int_0^t r_s \, ds$ are jointly normal under the original measure P and so this option pricing formula applies. Furthermore, σ^2 can be given an analytic expression in terms of the model parameters which enables us to express the option price in terms of the model parameters. We now develop the expression for σ^2 in terms of these model parameters.

We see from equation (10) that the random part of ln[P(t, T)] is $-\sum_{i=1}^{n} r_t^{(i)} F_i(T-t)$.

We also see from equation (4) that the random part of $r_t^{(i)}$ is $\int_0^t e^{-\alpha_i(t-s)} \eta_i \, dZ_s^{(i)}$.

Consequently, the random part of ln[P(t, T)] is $-\sum_{i=1}^{n} F_i(T-t)\int_0^t e^{-\alpha_i(t-s)}\eta_i\, dZ_s^{(i)}$. It now follows that

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \eta_i\eta_j F_i(T-t)F_j(T-t)\int_0^t e^{-(\alpha_i+\alpha_j)(t-s)}\rho_{ij}\, ds$$

(Karatzas and Shreve, 1988, page 140, (2.23)) which simplifies to

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \eta_i\eta_j\rho_{ij}F_i(T-t)F_j(T-t)\frac{1-e^{-(\alpha_i+\alpha_j)t}}{\alpha_i+\alpha_j}.$$

One may then price the option by using this formula for σ^2 in the option formula above.

4. Multi-factor Models from One-factor Models

The martingale approach to term structure modelling provides a convenient framework in which to formalise the notion that one may induce a multi-factor model of the term structure from several independent one-factor models of the term structure. This principle is well known and one can find instances of it in the papers Jamshidian (1993), Dybvig (1989), and Hull and White (1990). The purpose of this section is to justify this principle by showing that the bond price function that it produces is the bond price function of an arbitrage-free term structure model. The practical application of this principle is that one may obtain closed-form solutions for certain multi-factor models from known closed form solutions for one-factor models by simply multiplying bond price functions together. This procedure will always lead to an additive factor model, $r = r^{(1)} + r^{(2)} + \cdots + r^{(n)}$.

Lemma: Let $r^{(1)}, r^{(2)}, \ldots, r^{(n)}$ be independent processes and let $\mathcal{F}^{(i)}$ denote the filtration generated by $r^{(i)}$. Let $\mathcal{F}$ denote the filtration generated by $r^{(1)}, r^{(2)}, \ldots, r^{(n)}$ so that $\mathcal{F}_t = \bigvee_{i=1}^{n} \mathcal{F}_t^{(i)}$. Let $Q^{(1)}, \ldots, Q^{(n)}$ be probability measures on $\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(n)}$, respectively. *Let $P^{(i)}(t, T)$ denote the bond price function from the one-factor term structure model* defined by $r^{(i)}$ and $Q^{(i)}$; hence

$$P^{(i)}(t, T) = E^{Q^{(i)}}[\exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t^{(i)}].$$

Then the function $P(t, T) := \prod_{i=1}^{n} P^{(i)}(t, T)$ is the bond price function for the multi-factor term structure model defined by the spot rate process $r = \sum_{i=1}^{n} r^{(i)}$ and the equivalent probability measure $\frac{dQ}{dP} = \prod_{i=1}^{n} \frac{dQ^{(i)}}{dP}$; hence

$$P(t, T) = E^{Q}[\exp(-\int_t^T r_s\, ds) \mid \mathcal{F}_t].$$

Proof: The proof depends on the role of independence in computing conditional expectations as stated in Williams (1991, page 88, (k)). From this result it follows that, if $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ are processes adapted to $\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(n)}$, respectively then for s<t

$$E[\prod_{i=1}^{n} X_t^{(i)} \mid \mathcal{F}_s] = \prod_{i=1}^{n} E[X_t^{(i)} \mid \mathcal{F}_s^{(i)}]. \qquad (*)$$

The assertion of the lemma amounts to the assertion that

$$E^{Q}[\exp(-\int_t^T r_s\, ds) \mid \mathcal{F}_t] = \prod_{i=1}^{n} E^{Q^{(i)}}[\exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t^{(i)}].$$

Using Bayes' rule (Karatzas and Shreve, 1988, page 193) we have

$$E^{Q}[\exp(-\int_t^T r_s\, ds) \mid \mathcal{F}_t] = \frac{1}{E[\frac{dQ}{dP} \mid \mathcal{F}_t]} E[\frac{dQ}{dP} \prod_{i=1}^{n} \exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t].$$

It follows from equation (*) that $E[\frac{dQ}{dP} \mid \mathcal{F}_t] = \prod_{i=1}^{n} E[\frac{dQ^{(i)}}{dP} \mid \mathcal{F}_t^{(i)}]$. Consequently,

$$\frac{1}{E[\frac{dQ}{dP} \mid \mathcal{F}_t]} E[\frac{dQ}{dP}\prod_{i=1}^{n} \exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t]$$

$$= E[\prod_{i=1}^{n} \frac{dQ^{(i)}/dP}{E[\, dQ^{(i)}/dP \mid \mathcal{F}_t^{(i)}\,]} \exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t]$$

$$= \prod_{i=1}^{n} E[\frac{dQ^{(i)}/dP}{E[\, dQ^{(i)}/dP \mid \mathcal{F}_t^{(i)}\,]} \exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t^{(i)}\,]$$

$$= \prod_{i=1}^{n} E^{Q^{(i)}}[\, \exp(-\int_t^T r_s^{(i)}\, ds) \mid \mathcal{F}_t^{(i)}\,],$$

where the second equality follows from equation ($*$) and the third equality follows from using Bayes' rule in reverse. QED.

As an illustration of this lemma we note that the bond prices that we have calculated in equation (11) are a product of the one-factor bond prices appearing in equation (12).

One example of the application of this lemma is provided by considering a multi-factor Cox-Ingersoll-Ross model. Jamshidian (1993) refers to this model as the separable Cox-Ingersoll-Ross model. Let us first recall the one-factor Cox-Ingersoll-Ross model. Let W be a Brownian motion and $\mathcal{F}$ its Brownian filtration. Let κ, θ, σ be positive constants and λ a constant. The one-factor Cox-Ingersoll-Ross model is defined by the spot rate process

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\, dW_t\,,$$

and the equivalent probability measure

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \exp(-\int_0^T (\lambda/\sigma)\sqrt{r_s}\, dW_s - \frac{1}{2}\int_0^T (\lambda/\sigma)^2 r_s ds)\,.$$

Since λ and σ are both constants, the reader may wonder why we do not denote λ/σ by the constant λ instead. This is because we want our bond pricing formula to agree exactly with Cox, Ingersoll, and Ross (1985, equation (23)). The market price of risk in their model is $(\lambda/\sigma)\sqrt{r_t}$ as can be seen from the sentence following their equation (22). If we were to denote λ/σ by the constant λ instead, then we would get the formula that is presented in Hull and White (1990). Of course, either notation is correct.

Under the equivalent probability measure the spot rate process satisfies the stochastic differential equation

$$dr_t = (\kappa+\lambda)[\kappa\theta/(\kappa+\lambda) - r_t]dt + \sigma\sqrt{r_t}\,d\widetilde{W}_t,$$

where $\widetilde{W}$ is a Brownian motion under Q. The expectation (1) may be evaluated for this model by using this form for the spot rate and an expectation formula noted in Delbaen (1993, page 126). This yields the bond prices (Cox, Ingersoll, and Ross (1985, equation (23))

$$P(t, T) = A(t, T)\exp(-r_tB(t, T)),$$

where

$$A(t, T) := \left(\frac{2\gamma\exp[(\kappa+\lambda+\gamma)(T-t)/2]}{(\kappa+\lambda+\gamma)(\exp[\gamma(T-t)] - 1) + 2\gamma}\right)^{2\kappa\theta/\sigma^2},$$

$$B(t, T) := \frac{2(\exp[\gamma(T-t)] - 1)}{(\kappa+\lambda+\gamma)(\exp[\gamma(T-t)] - 1) + 2\gamma},$$

$$\gamma := \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}.$$

Let us now consider a multi-factor Cox-Ingersoll-Ross model. To this end, define the following functions for $i = 1, \dots, n$

$$A_i(t, T) := \left(\frac{2\gamma_i\exp[(\kappa_i+\lambda_i+\gamma_i)(T-t)/2]}{(\kappa_i+\lambda_i+\gamma_i)(\exp[\gamma_i(T-t)] - 1) + 2\gamma_i}\right)^{2\kappa_i\theta_i/\sigma_i^2},$$

$$B_i(t, T) := \frac{2(\exp[\gamma_i(T-t)] - 1)}{(\kappa_i+\lambda_i+\gamma_i)(\exp[\gamma_i(T-t)] - 1) + 2\gamma_i},$$

$$\gamma_i := \sqrt{(\kappa_i+\lambda_i)^2 + 2\sigma_i^2},$$

where κ_i, θ_i, σ_i are positive constants and λ_i is a constant. Define the bond price function

$$P(t, T) := \left(\prod_{i=1}^{n} A_i(t, T)\right) \exp\left(-\sum_{i=1}^{n} r_t^{(i)} B_i(t, T)\right), \tag{13}$$

This is the bond price function for a multi-factor Cox-Ingersoll-Ross model to which Jamshidian (1993) refers as the separable CIR model. Indeed, if we now let W denote an n-dimensional standard Brownian motion with components $W^{(1)}, \dots, W^{(n)}$, then the lemma tells us that the bond price function in equation (13) is the bond price function for the multi-factor model with the spot rate process $r = r^{(1)} + r^{(2)} + \cdots + r^{(n)}$,

$$dr_t^{(i)} = \kappa_i(\theta_i - r_t^{(i)})dt + \sigma_i \sqrt{r_t^{(i)}}\, dW_t^{(i)},$$

and the equivalent probability measure

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \prod_{i=1}^{n} \exp\left(-\int_0^T (\lambda_i/\sigma_i)\sqrt{r_s^{(i)}}\, dW_s^{(i)} - \frac{1}{2}\int_0^T (\lambda_i/\sigma_i)^2 r_s^{(i)} ds\right).$$

Of course, one is not necessarily restricted to use only factors from the same class of model. For example, one could mix independent one-factor Vasicek and one-factor Cox-Ingersoll-Ross models to get a two-factor model.

5. Concluding Remarks

We have given a brief discussion of the martingale approach to the construction of arbitrage-free term structure models and have derived the Gaussian model of Chaplin and Sharp (1993) within the martingale framework. We have shown how the martingale approach can be used to formalise the principle that one may induce a multi-factor model from several independent one-factor models. We have intended that this paper give some concrete examples of the martingale approach without insisting on rigorous proofs. Unfortunately, the intersection of the term structure models that are mathematically tractable and those that fit the empirical term structure data well is a thin set. In practice one may use techniques such as those in Tilley (1992) to help build acceptable models.

Although the martingale approach provides a theoretical perspective on term structure modelling it has nothing to say about what our choice of r and Q should be.

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Appendix 1

In this appendix we offer an informal argument to show how the choice of an equivalent probability measure Q governs the relationship between risk and return for the bond price processes. We hope that this will help make the role of the equivalent probability measure Q in the construction of arbitrage-free term structure models seem less sterile. A related argument may be found in Artzner and Delbaen (1989, section 2.2).

In this appendix we assume that the information structure is a Brownian filtration generated by an n-dimensional standard Brownian motion W. The components of W are denoted by $W^{(1)}, \dots, W^{(n)}$. The spot rate process r is adapted to this filtration and bond prices are defined by equation (1):

$$P(t, T) = E^Q[\exp(-\int_t^T r_u \, du) \mid \mathcal{F}_t].$$

Define $P^*(t, T) := \exp(-\int_0^t r_s \, ds) P(t, T)$. It then follows from the preceding equation that $P^*(t, T)$ is a martingale under Q. If Q is equivalent to P on $\mathcal{F}_T$ then there is some process λ such that Q is of the form

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \exp(-\sum_{i=1}^{n} \int_0^T \lambda_s^{(i)} \, dW_s^{(i)} - \frac{1}{2}\sum_{i=1}^{n} \int_0^T (\lambda_s^{(i)})^2 \, ds).$$

Define $\widetilde{W}_t^{(i)} := W_t^{(i)} + \int_0^t \lambda_s^{(i)}\, ds$. $\widetilde{W}$ is a Brownian motion under Q by Girsanov's theorem. Now consider a particular bond with maturity T. Since $P^*(t, T)$ is a positive martingale under Q, there exists an n-dimensional process $(\phi^{(1)}(T), \ldots, \phi^{(n)}(T))$ such that

$$P^*(t, T) = P^*(0, T)\exp\Big(\sum_{i=1}^{n} \int_0^t \phi_s^{(i)}(T)\, d\widetilde{W}_s^{(i)} - \frac{1}{2}\sum_{i=1}^{n} \int_0^t (\phi_s^{(i)}(T))^2\, ds \Big).$$

By Itô's lemma

$$\frac{dP^*(t, T)}{P^*(t, T)} = \sum_{i=1}^{n} \phi_t^{(i)}(T)\, d\widetilde{W}_t^{(i)} = \sum_{i=1}^{n} \phi_t^{(i)}(T)\, dW_t^{(i)} + \sum_{i=1}^{n} \phi_t^{(i)}(T)\lambda_t^{(i)}\, dt,$$

so that

$$\frac{dP(t, T)}{P(t, T)} = \sum_{i=1}^{n} (\phi_t^{(i)}(T)\lambda_t^{(i)} + r_t)\, dt + \sum_{i=1}^{n} \phi_t^{(i)}(T)\, dW_t^{(i)}.$$

If we denote the drift term in this equation by $\mu_t(T)$ then we can sum up by saying that $P(\cdot, T)$ satisfies the stochastic differential equation

$$\frac{dP(t, T)}{P(t, T)} = \mu_t(T)\, dt + \sum_{i=1}^{n} \phi_t^{(i)}(T)\, dW_t^{(i)}, \tag{A-1}$$

with

$$\mu_t(T) - r_t = \sum_{i=1}^{n} \lambda_t^{(i)}\phi_t^{(i)}(T). \tag{A-2}$$

We note that this equation holds for bonds of all maturity dates T, with the same process λ applying for all bonds. We remind the reader that the λ appearing in this equation is the same λ that defines Q. Equation (A-2) corresponds to the prices of risk in Chaplin and Sharp (1993, page 12, (AI-8)).

The relationship (A-2) may be given the following interpretation. The left hand side of this equation is the instantaneous excess expected return on the bond while the right hand side is a measure of the local risk in the bond. This equation thus restricts the relationship between risk and return through the process λ. For this reason the process λ is referred to as the market price of risk. Note that if the local risk in the bond is zero then the instantaneous expected return on the bond is equal to the spot rate r_t. We see then that the role of the equivalent probability measure Q in constructing arbitrage-free models of the term structure is to impose a particular relationship between risk and return.

Appendix 2

In this appendix we show how the formulation of the term structure model in section 3 can be reduced to the formulation specified in equations (5) and (6). As we will need some simple facts about correlated Brownian motion, let us review these here. In the following we treat all vectors as column vectors and we denote the transpose of M by M^t.

An *n-dimensional correlated Brownian motion* is an n-dimensional continuous stochastic process with independent increments such that the increment $Z_t - Z_s$ is normally distributed with mean zero and covariance matrix (t–s)C, where C is a symmetric positive definite matrix. It is clear that $C = E[Z_1 Z_1^t]$, the covariance matrix of Z_1. Elements of the matrix C will be denoted by ρ_{ij}. In the notation of Karatzas and Shreve (1988), we have $\langle Z^{(i)}, Z^{(j)} \rangle_t = \rho_{ij} t$. An *n-dimensional standard Brownian motion* is thus an n-dimensional correlated Brownian motion with C being the identity matrix. Since C is a symmetric positive definite matrix, there exists a lower triangular matrix Σ such that $C = \Sigma\Sigma^t$ (Choleski factorisation of C (Burden and Faires, 1989)). It may be checked that $B := \Sigma^{-1}Z$ is a standard Brownian motion which generates the same filtration as Z.

By an application of Itô's lemma, the relationship $B = \Sigma^{-1}Z$, and the martingale representation theorem for Brownian motion (Karatzas and Shreve, 1988, pages 182-184) one can show that any equivalent probability measure Q is of the form

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = \exp\left(\sum_{i=1}^{n} \int_0^T v_s^{(i)}\, dZ_s^{(i)} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^T v_s^{(i)} v_s^{(j)} \rho_{ij}\, ds \right), \qquad \text{(A-3)}$$

from which it follows that the specification of an equivalent probability measure Q is tantamount to a choice of the vector process $v = (v^{(1)}, \ldots, v^{(n)})$. We now state the form of Q for the model of section 2.

Lemma: Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in R^n$ be chosen. Define the n-vector $\zeta := C^{-1}\lambda$. Set

$$\ln\left(\frac{dQ}{dP}\bigg|_{\mathcal{F}_T}\right) = -\sum_{i=1}^{n} \zeta_i Z_T^{(i)} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \zeta_i \zeta_i \rho_{ij} T = -\zeta^t Z_T - \frac{1}{2}\zeta^t C \zeta T.$$

Then this choice of Q taken with the factor processes specified by (3) leads to the model described by equations (5) and (6).

Proof: In the following, elements of the matrix Σ will be denoted by σ_{ij} and elements of the matrix Σ^{-1} will be denoted by τ_{ij}. Recall that $B = \Sigma^{-1}Z$. Matrix algebra shows that

$$-\sum_{i=1}^{n} \zeta_i Z_T^{(i)} = -\zeta^t Z_T = -(C^{-1}\lambda)^t(\Sigma B_T) = -\lambda^t[(\Sigma\Sigma^t)^{-1}]^t \Sigma B_T = -\lambda^t(\Sigma^{-1})^t B_T,$$

and from this identity we obtain

$$-\sum_{i=1}^{n} \zeta_i Z_T^{(i)} = -(\lambda_1\tau_{11} + \lambda_2\tau_{12} + \cdots + \lambda_n\tau_{1n})B_T^{(1)} - \cdots$$

$$-(\lambda_1\tau_{n1} + \lambda_2\tau_{n2} + \cdots + \lambda_n\tau_{nn})B_T^{(n)}.$$

Let $\tilde{B}_t^{(i)} := B_t^{(i)} + (\lambda_1\tau_{i1} + \lambda_2\tau_{i2} + \cdots + \lambda_n\tau_{in})t$. Then by Girsanov's theorem (Karatzas and Shreve, 1988, page 191) $\tilde{B}$ is a standard Brownian motion under Q. We now see that

$$\begin{aligned} dZ_t^{(i)} &= \sigma_{i1}dB_t^{(1)} + \sigma_{i2}dB_t^{(2)} + \cdots + \sigma_{in}dB_t^{(n)} \\ &= -\sigma_{i1}(\lambda_1\tau_{11} + \lambda_2\tau_{12} + \cdots + \lambda_n\tau_{1n})dt \\ &\quad - \cdots - \sigma_{in}(\lambda_1\tau_{n1} + \lambda_2\tau_{n2} + \cdots + \lambda_n\tau_{nn})dt \\ &\quad + \sigma_{i1}d\tilde{B}_t^{(1)} + \sigma_{i2}d\tilde{B}_t^{(2)} + \cdots + \sigma_{in}d\tilde{B}_t^{(n)} \\ &= -\sigma_{i1}(\lambda_1\tau_{11} + \lambda_2\tau_{12} + \cdots + \lambda_n\tau_{1n})dt \\ &\quad - \cdots - \sigma_{in}(\lambda_1\tau_{n1} + \lambda_2\tau_{n2} + \cdots + \lambda_n\tau_{nn})dt + d\tilde{Z}_t^{(i)}, \end{aligned} \tag{A-4}$$

where $\tilde{Z}$ is a correlated Brownian motion under Q. It is clear that

$$\sigma_{i1}(\lambda_1\tau_{11} + \lambda_2\tau_{12} + \cdots + \lambda_n\tau_{1n}) + \cdots + \sigma_{in}(\lambda_1\tau_{n1} + \lambda_2\tau_{n2} + \cdots + \lambda_n\tau_{nn})$$

$$= (\sigma_{i1}, \dots, \sigma_{in})^t\Sigma^{-1}\lambda = \lambda_i .$$

Thus we see from (A-4) that

$$dZ_t^{(i)} = -\lambda_i dt + d\tilde{Z}_t^{(i)} . \qquad \text{(A-5)}$$

It now follows from equations (3) and (A-5) that the factors satisfy the stochastic differential equations

$$dr_t^{(i)} = [\alpha_i(\gamma_i - r_t^{(i)}) - \eta_i\lambda_i]\, dt + \eta_i d\tilde{Z}_t^{(i)} \quad i = 1, 2, \dots, n \qquad \text{(A-6)}$$

where $\tilde{Z}$ is a correlated Brownian motion under Q with the same covariance matrix C as Z has under the original measure P. It is therefore evident that the expectation (1) for our choice of r and Q is the same as the expectation defined by (5) and (6). QED.

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