Quick Solutions for Arithmetic Average Options on a Recombining Random Walk

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Summary

This paper develops a frequency distribution method for valuing European and American arithmetic average options on a discrete time, recombining multiplicative binomial stock price process. The method can value instruments with either time weighted or time unweighted averages, calculated using all or only some time points. The procedure first involves developing, for each possible ending price, a frequency distribution of the nodes through which all the paths ending at that price will pass. The set of paths associated with each ending price is called a bundle; at time T there are $T + 1$ bundles in a recombining binomial tree. Frequency data for all the nodes attained by the paths in any bundle is obtained analytically, as are each bundle's maximum and minimum path averages.

In any given bundle, the path averages may all exceed the option strike price, may all be less, or may lie both above and below it. Each bundle type is easily identified by its maximum and minimum path averages. Determining the contribution of the first two types of bundle to option value is immediate, but assessing the contribution of the third type, called a mixed bundle, requires further calculation to determine the conditional frequency distribution for the subset of paths whose averages exceed (fall below) the strike price of a call (put). Finally, using the (original or conditional) frequency distributions for each bundle, the means (or conditional means) of path averages can be computed and the option evaluated using them. This paper derives the analytic formulae and shows how the necessary subsidiary calculations can be done. Approximation methods are also discussed.

Solutions rapides aux options de moyenne arithmétique sur un modèle stochastique de recombinaison

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Résumé

Le présent exposé développe une méthode de distribution de fréquence pour l'évaluation d'options de moyenne arithmétique européennes et américaines en temps discret, par recombinaison du processus binôme multiplicateur du prix des actions. Cette méthode peut évaluer des effets présentant soit des moyennes pondérées en fonction du temps, soit des moyennes non pondérées en fonction du temps, calculées à l'aide de tous les points temporels ou seulement certains d'entre eux. Cette procédure implique d'abord le développement, pour chaque prix résultant possible, d'une distribution de fréquence des noeuds au travers desquels tous les parcours se terminant à ce prix vont passer. Les parcours associés à chaque prix résultant constituent ce qui est appelé un ensemble ; au point T, il y a T + 1 ensembles dans un arbre binôme de recombinaison. Les données de fréquence pour tous les noeuds atteints par les parcours d'un ensemble sont obtenues analytiquement, de la même façon que les moyennes maximum et minimum de parcours de chaque ensemble.

Dans un ensemble donné, les moyennes des parcours peuvent toutes dépasser le prix d'exercice des options, peuvent toutes lui être inférieures, ou peuvent lui être à la fois supérieures et inférieures. Chaque type d'ensemble est aisément identifié par ses moyennes de parcours maximum et minimum. La détermination de la contribution des deux premiers types d'ensemble à la valeur de l'option est immédiate, mais l'estimation de la contribution du troisième type d'ensemble, appelé ensemble mixte, requiert des calculs supplémentaires pour déterminer la distribution de fréquence conditionnelle pour le sous-ensemble de parcours dont les moyennes dépassent (sont inférieures) le prix d'exercice d'une option d'achat (vente). Enfin, à l'aide des distributions de fréquence (initiale ou conditionnelle) pour chaque ensemble, les moyennes (ou moyennes conditionnelles) des moyennes des parcours peuvent être calculées et permettre d'évaluer l'option. Le présent exposé dérive les formules analytiques et montre comment effectuer les calculs subsidiaires nécessaires. Des méthodes d'approximation sont également traitées.

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1. Introduction

Valuing arithmetic average options is a difficult problem largely because the conventional choice of price process in continuous time is a geometric diffusion, and the distribution of an arithmetic average option on it is a sum of lognormal variates that is not itself lognormal. Even so, Reiner [1991] generalizes Goldman, Sosin, and Shepp [1979] and Goldman, Sosin and Gatto [1979] to obtain analytic solutions for a wide variety of continuous time options whose averages are determined using continuous sampling.

In practice, however, most such options are based on discrete averaging. Moreover, implementing the continuous time solutions involves enough computation to warrant using discrete approximations. Several authors have proposed approximations: Turnbull and Wakeman [1991], Levy [1991], Curran [1992], Hull and White [1993]). Ho [1992] and Tilley [1993] suggest using simulation methods which rely on bundling to reduce calculations. Neave [1993] proposes a multiplicative binomial model for minimizing the necessary valuation calculations, but his exact methods become lengthy for $T > 20$.[2] Neave's approximations permit evaluating both European and American arithmetic average options, for values of T up to about 60. For the European options the accuracy of Neave's approximation method is

[1]We are indebted to Michael McIntyre for constructive suggestions and to Warren Tom for computational assistance.

[2]Neave's exact calculations become cumbersome on a 486 pc when the number of time points exceeds 24; the approximate calculations based on the geometric average permit using about 60 time points with resets every period. They do not seem to be appreciably worse for volatilities in excess of 0.40 per annum, and thus offer some improvements over other published work.

roughly comparable[3] to that obtained by Curran [1993]. Finally, while Hull and White [1993] describe their approximation methods for both European and American average options as quick and relatively accurate, they report few test results and offer no theoretical support for their methods.

Given the current state of the valuation routines, improved methods still seem to be needed, particularly because binomial models seem to converge more slowly as the volatility of the asset price process increases. It seems especially important to develop quick methods for processes with volatility of .40 per annum or greater, since as yet the literature reports few results for such cases.

Whatever the size of the volatility, this paper both offers new insights into the structure of the average option valuation problem and should prove substantially quicker than the exact methods of Neave [1993]. Moreover, the present method's approximations should be at least as accurate as those of HW, since this paper replace the HW recursively determined approximations with analytically determined bounds on the averages in each bundle. Finally, the present method can value options with either time weighted or unweighted averages, based either on all or only some time points.

The procedure is based on combinations of paths which, following Ho [1992] and Tilley [1993], are called bundles. We define a bundle as the set of paths all ending at the same price. Thus there are $T + 1$ bundles in a recombining binomial tree beginning at time 0 and ending at T. The size of a bundle depends on its ending price, and is described by the coefficients of the binomial expansion of $(1 + 1)^T$. By construction, all the paths in a given bundle have the same probability. We find both analytic expressions for the frequency distributions of the nodes through which all paths in any given bundle will pass,[4] and analytic expressions for each bundle's maximal and minimal path averages.

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[3]Curran does not value American options. Nevertheless the comparability of the two methods is not surprising, since both are based on the geometric mean. For American options, Neave's methods are relatively slow when $T > 20$.

[4]The sum of lognormal distributions on a continuous process is not lognormal, and our discrete processes face an analogous difficulty. We obtain the frequency distribution of paths through each node analytically. When calculating time unweighted averages, we sum these distributions; when time weighted averages are calculated, we use the frequency distributions for each point in time separately.

The procedure is to use the analytically determined frequency distributions, or conditional distributions derived from them, to value the option. The need for conditional distributions arises because bundle averages can all lie above a given strike price, below it, or both; we say the third circumstance involves mixed bundles. The paper first uses the maximal and minimal bundle averages to determine whether bundles are mixed or unmixed. Calculating the contribution of unmixed bundles to option value is trivial, but assessing the contribution of a mixed bundle requires determining a conditional frequency distribution for the subset of paths whose averages lie above (below) a call (put) option's strike price. Once a (conditional or unconditional) frequency distribution has been obtained for each bundle, means (or conditional means) of path averages and the valuation of the European option can be evaluated with very few additional calculations. The American option requires backward recursive application of the methods.

The paper's principal contributions are to demonstrate the use of the frequency distributions in valuation, to develop the analytic formulae described above, and to organize the additional calculations needed to complete the valuation procedure. On the basis of our computational experience with related problems, computation of exact results will be quick on a 486 PC for values of T up to about 50. For larger T, the calculations are likely to become too slow,[5] but in these cases the method leads naturally to fast approximations which are theoretically closer to true value than those developed by Hull and White (HW).

The approach can be applied to a variety of options, but to keep the exposition succinct the rest of the paper considers only the problem of valuing an arithmetic average spot price call. Section 2 specifies the model, Section 3 the conventions used to reduce calculations. Section 4 values a typical call, Section 5 concludes. Appendices describe further the calculations of Section 4.

2. The Model

This section defines the model and states necessary and sufficient conditions for existence of an unique martingale measure to be used in the valuation.

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[5]Values for somewhat larger T are easy to obtain on a work station. Usually, binomial models show relatively slow and non monotonic convergence (cf. Broadie and DeTemple [1993] but our computational experience with average options shows the values begin to converge monotonically for T approximately greater than 20, and that the values begin to converge, even for high volatility options, at about T = 30. Hull and White [1993] give no convergence results.

2.1 The Price Process and Average Spot Price Options

Let $\{ S_t \}$, the stock price process[6], be:

$$S_t = US_{t-1}; \tag{2.1}$$

$t \in \{1, 2, ..., T\}$, where U is a random variable which can either equal

$$u > 1 \text{ with probability } p$$

or

$$u^{-1} \text{ with probability } 1 - p \equiv q.$$

The realized stock price cannot become negative, and remains finite for finite values[7] of T and u.

To define the European options, let the exercise date be T, and let H be the time unweighted arithmetic average[8]

$$H = [\sum_{t=0}^{T} S_t]/(T+1) \tag{2.2}$$

Then the **arithmetic average spot price call** has a time T payoff of

$$C(H, K) = max \{ H - K, 0 \}, \tag{2.3}$$

and the **arithmetic average spot price put** has a time T payoff of

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[6]The continuous time limit of the price process is lognormal; see Cox and Rubinstein [1985] or Huang and Litzenberger [1988].

[7]For fixed T, the price can be kept below an upper bound M^* by choosing u such that $1 < u < (M^*)^{1/T}$.

[8]Reiner [1991] finds solutions for time weighted averages. After examining the frequency distributions of paths, it will be evident how the present method incorporates time weighted averages.

$$P(H\ ,K) = max\ \{\ K - H,\ \ 0\ \}. \tag{2.4}$$

2.2 The Martingale Measure

Assume

i) the absence of profitable arbitrage opportunities;

ii) dynamically complete markets; and

iii) zero transactions costs.

Define the normalized process

$$S_t{*} = S_t\ /\ R^t\ , \tag{2.4}$$

where $R^t \equiv (1 + r)^t$ indicates the t - period accumulation of \$1 at the single period risk free interest rate r. Then as is well known assumptions (i to (iii are necessary and sufficient for $\{\ S_t{*}\ \}$ to be a martingale under the measure $p*$, where:

$$p* = (\ R - u^{-1}\ )\ /\ (\ u - u^{-1}\ ). \tag{2.5}$$

3. Characteristics of Price Paths

The price paths of a recombining, multiplicative binomial process can be represented by their index values at each point in time. After dividing all possible price realizations by S_0, the paths of realized prices can be written, for $T = 1$:

$$\begin{matrix} u^0\ u^1 \\ u^0\ u^{-1}; \end{matrix} \tag{3.1}$$

and for $T = 2$:

$$\begin{matrix} u^0\ u^1\ u^2 \\ u^0\ u^1\ u^0 \\ u^0\ u^{-1}\ u^0 \\ u^0\ u^{-1}\ u^{-2}. \end{matrix} \tag{3.2}$$

The paths shown in (3.1) and (3.2) can be represented by a matrix of indices whose rows indicate individual paths. The matrix for $T = 2$ is

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{vmatrix},$$

and it is equally easy to write a similar matrix for higher values of T. Note the the path arithmetic and geometric averages can both be determined readily using the matrix of indices and a given value of u.

The number of paths ending at a given time - price node $(t, t-2k)$, is

$$b(t, k) \equiv t!/(t-k)!k!;\ k \in \{0, 1, \ldots, t\};\ t \in \{0, 1, \ldots, T\}.$$

It is not difficult to show inductively that, for any ending price t-$2k$, the $b(t, k)$ price path indices from $(0, 0)$ to $(t, t-2k)$ conform to $t + 1$ hypergeometric distributions. Since the formulae are somewhat cumbersome, we illustrate the distributions for the bundle of paths from $(0, 0)$ to $(8, 2)$ in Table 1. The row headings of the Table indicate price indices, the column headings time, and the entries the frequencies with which nodes are attained. Blank cells indicate infeasible time - price combinations for the given bundle.

Table 1. Frequency Distribution of Paths, (0,0) to (8,2)

Prices\t	0	1	2	3	4	5	6	7	8
6									
5						1			
4					4		6		
3				10		15		21	
2			20		24		30		56
1		35		30		30		35	
0	56		30		24		20		
-1		21		15		10			
-2			6		4				
-3				1					
-4									

To verify that the columns of Table 1 are hypergeometric distributions, write the value in the $t = 0$ column as $b(0,0)b(8,3)$. Similarly, the values in the $t = 1$ column are $b(1, 1)b(7, 3)$ and $b(1, 0)b(7, 2)$; in the $t = 2$ column $b(2, 2)b(6, 3)$, $b(2, 1)b(6, 2)$, and $b(2, 0)b(6, 1)$; in the $t = 3$ column $b(3, 3)b(5, 3)$, $b(3, 2)b(5, 2)$, $b(3, 1)b(5, 1)$ and $b(3, 0)b(5, 0)$. (After writing the first few columns of an array for any (T, j) in this fashion, it is easy to show inductively that, given any bundle, the frequency distributions for each time point are hypergeometric.)

The extremal paths of a given bundle define its maximal and minimal arithmetic average. In the present case the path with the maximal average is described by the price indices 012345432, that with the minimal average by 0-1-2-3-2-1012.[9] Next, using the row sums in the array and the fact that all paths ending at (8,2) have the common (martingale) probability $(p^*)^5(q^*)^3$, the mean of the (time unweighted) path averages in the bundle is

$$h_2 \equiv u^5 + 10u^4 + 46u^3 + \ldots + u^{-3} \,/\, (56)9. \tag{3.3}$$

Omitting the time subscript for brevity, denote the maximal and minimal arithmetic averages by h_2^+ and h_2^- respectively. If $K < h_2^-$, then the terms in (3.3) contribute

$$56(h_2 - K)(p^*)^5(q^*)^3/R^8$$

to the present value of a European arithmetic average spot price call, and zero if $K > h_2^+$. The cases

$$h_j^- < K < h_j^+ \,,$$

where j is any ending price, arise for about a third of the bundles for an at the money option. They require calculating a conditional frequency distribution for the subset of paths whose averages exceed the strike price. If T is relatively large, calculating the conditional frequency distributions exactly may prove excessively time consuming, since the calculations require enumerating some of the paths in each

[9]The same paths also have maximal and minimal sums of indices, and therefore maximal and minimal geometric averages, as explained and exploited below. Given the analytic expressions for the mean of each bundle's path averages and for the maximum and minimum path average in each bundle, the Hull and White [1993] forward induction estimates of the maxima and minima are not needed for valuation purposes.

mixed bundle.[10]

When exact calculation proves infeasible, the differences between path averages and exercise price can be approximated. Depending on the valuation context, the reduction in computating time may compensate for the lack of exactness. Experience with related binomial models suggests that fitting a convex cubic[11] between h_j^+ and h_j , and a concave cubic between h_j and h_j^- , while constraining the resulting piecewise function to have an inflection point at h_j , will provide a quick and relatively satisfactory approximation. This conjecture is based on examining data like those graphed in Appendix III, which shows the path averages for the bundle of 56 paths leading from (0, 0) to (8, 2) calculated for the binomial process examined in Section 4.

Given the distribution of averages in a bundle, a cubic approximation should give greater accuracy than the log linear methods described by HW. The present methods are also theoretically more accurate since HW use recursively estimated maximal and minimal averages while we employ analytically determined values.

4. Arithmetic average spot price options

Arithmetic averages can be computed using only the frequencies with which a bundle of paths attain the time - price nodes, thus eliminating the need to enumerate most of the individual paths. The chief difficulty is to finding the conditional frequency distributions for mixed bundles. Promising methods for carrying out these calculations are outlined below, with additional detail being given in Appendix II. As already mentioned, when T becomes so large that the additional exact calculations are cumbersome, approximate methods can still reduce the computational burden.

4.1 Example of exact calculations

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[10]If too many paths had to be evaluated, the method would offer little improvement over complete enumeration. But even in this worst case scenario, the approach still suggests a basis for closer approximations than proposed by others; cf. particularly HW. Moreover, our experiments suggest the actual proportion requiring enumeration is relatively small. (It depends on both the exercise price and the methods of calculation employed.) Finally, with additional computational experience we expect to refine our existing methods further.

[11]A power function may give a still closer fit.

The option valued in this section is a two year arithmetic average spot price call on a process with quarterly time intervals, an annual volatility of 0.23, an annual risk free rate of 0.10, an initial price of 1.00, and a strike price of 1.00. The example calculates the paths' arithmetic averages using the nine time points 0 through 8. The recombining binomial process has the parameters specified in Table 2.

Table 2. Parameters

Δt =	0.2500
σ =	0.2300 per annum
r =	0.1000 per annum
R =	1.0253 per quarter = exp(0.0250)
u =	1.1224 = exp (0.2300/(0.2500$^{.5}$))
p* =	0.5805 = $(R - u^{-1})/(u - u^{-1})$
q* =	0.4195 = $1 - p^*$
K =	1.0000

As a check on the subsequent calculations, valuation based on complete enumeration of the problem's 256 paths gives a time 0 European call value of 0.1178.

To implement the frequency distribution method, first record the possible time 8 price indices, along with the maximal and minimal index sums, $\{ v^+ \}$ and $\{ v^- \}$ respectively, as in Table 3. Index sums are useful because path geometric means can quickly be calculated from them, and path geometric means give good lower approximations to the arithmetic means (cf. Curran [1992], Neave [1993]).

Table 3. Prices and Index Sums

Prices	8	6	4	2	0	-2	-4	-6	-8
$\{ v^+ \}$	36	34	30	24	16	6	-6	-20	-36
$\{ v^- \}$	36	20	6	-6	-16	-24	-30	-34	-36

It is easy to see that the $\{ v^+ \}$, written in decreasing order, satisfy

$$b(9, 2) - 2b(k+1, 2);\ k \in \{0, 1, \ldots, T\},$$

while the $\{ v^{-} \}$ written in increasing order satisfy

$$-b(9, 2) + 2b(k+1, 2);\ k \in \{0, 1, \ldots, T\}.$$

(We employ the convention $b(j, k) = 0$ if $j < k$.)

For a call price of $1 = u^0$, it can quickly be verified that all the arithmetic averages of paths ending at prices of 8, 6, and 4 and none of the averages of paths ending at prices of -4, -6, and -8 will be included in the option value.[12] Thus the frequency distributions for the prices 8, 6, and 4 can be used without amendment, because they refer only to paths whose averages exceed the strike price. The frequency distributions for the bundles ending at -4, -6, and -8 can be ignored because they refer only to paths whose averages fall below the strike price. The frequency distributions for the mixed bundles defined by the ending prices of 2, 0, and -2 must be replaced by the conditional frequency distributions for the subsets of paths whose arithmetic averages exceed than the strike price.

A combination of procedures can minimize the calculations needed to obtain conditional frequency distributions. First, obtain an approximate conditional frequency distribution by using bounding arguments to eliminate (lower-valued) nodes through which no path could pass and still have an average greater than the strike price.[13] After determining which nodes can be eliminated, a combination of quick, simple forward and backward recursions generates an approximate conditional frequency distribution. (Appendix II details these procedures.) The resulting conditional distribution contains all paths whose averages exceed the strike price, and will generally also contain some paths with averages less than the strike price.

Next, refine the approximate conditional frequency distribution by determining which paths are candidates for having arithmetic averages lower than the strike price. The number of candidates can be minimized by observing that when the strike price is at least unity, any path with a geometric average at least equal to the strike price

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[12]In each bundle, the minimally extreme path has both a minimum arithmetic average and a minimum geometric average, which latter can be obtained from this paper's data using as $u^{V/T+1}$. This illustrates the close relations between arithmetic and geometric means and helps explain which Curran [1992] and Rogers and Shi [1993] find the geometric mean such a good approximation.

[13]Alternatively, use path sums to obtain a reduced frequency distribution which contains only a subset of the paths which exceed the exercise price.

will also have an arithmetic average which at least equals the strike price. Hence the candidate paths are those whose geometric averages are less than unity, and such paths can be identified using the sums of their path indices. These candidates are then enumerated using another quick recursion method (also illustrated in Appendix II), and their averages calculated to determine which candidates represent paths to be eliminated from the mixed bundle. The frequency distribution of nodes attained by the paths to be eliminated is then subtracted from the approximate conditional frequency distribution to obtain an exact conditional frequency distribution.

Table 4, which tabulates (both the unconditional and the conditional) frequency distributions contributing positively to the call value, displays the results of using the procedures just outlined. (Details of the calculations are presented in Appendix II.)

Table 4. Frequency Distributions after Modification

Ending Prices/ Nodes	8	6	4	2	0	-2
8	1					
7	1	1				
6	1	10	1			
5	1	10	10	1		
4	1	10	46	10	1	
3	1	10	46	46	10	1
2	1	10	46	126	46	8
1	1	10	46	126	110	18
0	1	10	46	120	146	20
-1		1	10	35	36	9
-2			1	4	2	7
# of Paths	1	8	28	52	39	7

Table 5 calculates the means (or conditional means) of path averages[14] from the frequency distributions of Table 4. The first part of the Table computes the contribution to the arithmetic averages, times the frequency of its occurrence. For example, $1(1.12241^8) = 2.5189$, the upper left hand entry in the body of Table 5. The remaining parts of the Table compute the number of paths times the difference

[14]If time weighted averages are required, it is necessary to write the distributions on a period by period basis rather than working with the row sums as done here.

between the path averages and the strike price, the martingale probabilities associated with paths ending at the nodes shown in Table 4, the Time 8 expected values of the contributions under the martingale, and finally the Time 8 sum and its Time 0 discounted equivalent, the call value.

Table 5. Calculations of Path Averages

2.5189	0.0000	0.0000	0.0000	0.0000	0.0000
2.2442	2.2442	0.0000	0.0000	0.0000	0.0000
1.9994	19.9944	1.9994	0.0000	0.0000	0.0000
1.7814	17.8138	17.8138	1.7814	0.0000	0.0000
1.5871	15.8711	73.0069	15.8711	1.5871	0.0000
1.4140	14.1402	65.0448	65.0448	14.1402	1.4140
1.2598	12.5980	57.9510	158.7353	57.9510	10.0784
1.1224	11.2241	51.6309	141.4237	123.4651	20.2034
1.0000	10.0000	46.0000	120.0000	146.0000	20.0000
0.0000	0.8909	8.9094	31.1829	32.0738	8.0185
0.0000	0.0000	0.7938	3.1751	1.5875	5.5564
Path Average Less Exercise Price (adjusted for number of paths)					
0.6586	3.6419	7.9056	7.6905	2.8672	0.2523
Martingale Probabilities					
0.0129	0.0093	0.0067	0.0049	0.0035	0.0025
Expected Values					
0.0085	0.0339	0.0532	0.0374	0.0101	0.0006

Call Values: Time 8, 0.1438; Time 0, 0.1178.

It is evident from the foregoing procedures how the calculations must be modified to value instruments whose averages are computed on a subset of the time points, and also how to compute time weighted rather than time unweighted averages.

5. Conclusions

This paper illustrates how to evaluate arithmetic average options quickly and exactly for values of T up to about 50, quickly and approximately for larger values of T. Apart from possibly slower convergence, the exact methods are not affected by volatility, and the approximate methods seem to display only modest decreases in accuracy for high volatilities. The approximations are theoretically closer to the true option value than those advocated by Hull and White [1993]. Time weighted averages and averages using a number of reset points different from the number of time intervals can be accommodated without any essential change. (In fact, using fewer reset points is an advantage since it means the frequency distributions can be described in less detail.) The paper's methods can be applied recursively to describe the temporal evolution of payoff distributions and hence value American options.

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Appendix I: A Generating Function for the Density of V.

For any fixed T, the sums of path indices can be defined as a random variable V. While the probability density function of V can be found recursively (see Neave [1993]), it is also instructive to examine its generating function. The distribution of V is characterized by:

Proposition: If the process (2.1) continues for T periods, the number of paths with sum of exponents V is equal to the number of ways the indices of the coefficients of the generating function

$$h_T(s) = \prod_{t=1}^{T} [a_t + a_{-t} s] \qquad \text{(A1.1)}$$

sum to V.

Proof: Let U_t, a binomial random variable, represent (the exponents of) the possible outcomes at time t. It is easily verified that:

$$V = TU_1 + (T-1)U_2 + \ldots + U_T .$$

Then since V is a sum of binomial random variables it has a generating function of the form (4.1); cf. Feller [1957, ch. 11]. ■

Interpretation: The coefficients a_t and a_{-t} are interpreted respectively as p and q, and their products give the path probabilities. For example, if $T = 3$ the product $a_1 a_2 a_{-3} s$ describes a path whose sum is zero and whose probability is $p^2 q$. The power of s, (and the number of negative signs in any product of coefficients) indicates the number of downward moves in the path to which it refers.

Appendix II: Calculating the Conditional Frequency Distributions

This appendix shows how the frequency distribution for a mixed bundle can be used to calculate a conditional frequency distribution for the subset of paths whose arithmetic averages all exceed the call strike price. It is organized as follows. First, a recursive method for calculating all paths ending at a given price is developed. Then, this method is adapted to show how a frequency distribution can be calculated for a subset of paths attaining only some nodes. Next, use bounding arguments to eliminate any nodes through which paths cannot pass without having an average at least as low as the strike price. The resulting conditional distribution contains all paths whose average exceeds[15] the exercise price, along with additional paths whose average is less than the exercise price. These additional paths are then eliminated by direct enumeration to obtain the desired, exact conditional frequency distribution. The enumerations are limited by using geometric averages to eliminate most of the candidate paths.

Consider the bundle of 56 paths beginning at (0, 0) and ending at (8, 2). The frequency distribution for these paths can be obtained either using the generating function described in Appendix I or recursively. Since a recursive approach can also be used to determine the numbers and frequency distributions of only some paths beginning and ending at given points, Table A.1 shows how it can be organized.

Table A.1 Preliminary, backward induction calculation.

	0	1	2	3	4	5	6	7	8
5						1			
4					4		1		
3				10		3		1	
2			20		6		2		1
1		35		10		3		1	
0	56		15		4		1		
-1		21		5		1			
-2			6		1				
-3				1					

[15]Alternatively, one can first form a restricted distribution which contains only some of paths whose average exceeds the exercise price, then calculate which additional paths need to be added.

As the array indicates, each node value, proceeding backwards, is the sum of the entries at the adjacent higher and lower nodes for the next period. (If there is only one adjacent node, the number is the same. The backward calculation is convenient for generating the frequency distribution, as shown next.) For example, the number of paths through (4, 4) is 4 = 1 + 3, the number of paths through (5, 5) and (5, 3) respectively. There is exactly one path (4, -2): it is only possible to proceed from (4, -2) to (5, -1) and still end at (8, 2).

Table A.2 shows how the data of Table A.1 can be used to obtain the frequency distribution. (The column distributions are hypergeometric if all possible paths from zero to a given node are being tabulated.)

Table A.2 Forward induction to determine the hypergeometric distribution:

	0	1	2	3	4	5	6	7	8
5						1			
4					4		6		
3				10		15		21	
2			20		24		30		56
1		35		30		30		35	
0	56		30		24		20		
-1		21		15		10			
-2			6		4				
-3				1					

Considered in a forward direction, the 56 paths of the example distribute according to the ratios of the entries in the previous column. For example, from (1, 1) 20 of the 35 paths go to (2, 2); the remaining 15 go to (2, 15). Fifteen of the 21 paths from (1, -1), also go to (2, 0), and the remaining 6 go to (2, -2). From Table A.1, we see that of any 10 paths emanating from (3, 1), 6 go to (4, 2) and 4 to (4, 0). Thus in Table A.2, of the 30 paths through node (3, 1), 18 will go to (4, 2) and 12 to (4, 0). (This result can be checked either by direct enumeration or by comparison with the hypergeometric distribution developed in Section 3.)

The same methods can be used to find restricted distributions. For example, inspection shows that the path passing through (3, -3) has an arithmetic average less than unity. Thus, to search for paths which have an arithmetic average of at least unity, it is possible to begin with a restricted array as in Table A. 3. (In this example the restriction has no practical effect, but for larger values of T similar considerations can reduce the number of necessary path evaluations substantially.)

Table A.3 Backward induction to determine a restricted number of paths ending at 2:

	0	1	2	3	4	5	6	7	8
5						1			
4					4		1		
3				10		3		1	
2			20		6		2		1
1		35		10		3		1	
0	55		15		4		1		
-1		20		5		1			
-2			5		1				
-3				0					

The restricted distribution is found in the same way as before, but using the restricted number of paths as in Table A.4:

Table A.4 Forward induction to determine the restricted distribution:

	0	1	2	3	4	5	6	7	8
5						1			
4					4		6		
3				10		15		21	
2			20		24		30		55
1		35		30		30		34	
0	55		30		24		19		
-1		20		15		9			
-2			5		3				
-3				0					

We now continue, using the restricted array, to evaluate paths through one or more of the remaining nodes with a negative price index. Clearly no path which has only positive price indices can have an arithmetic average of less than unity; hence all such paths must be retained to compute the contribution of paths ending at (8, 2). However, paths whose arithmetic average is less than unity need to be removed.

To determine which paths require to be removed, note that any path ending at (8, 2) must have five upward and three downward moves. The search is thus reduced to examining paths with five upward moves and with one or more negative vertices other than (3, -3). Moreover, since any path with a non-negative sum will have a geometric average greater than unity, its arithmetic average will also be greater than unity. Thus we need only consider paths whose geometric average is less than unity; i.e. paths whose sum of indices is negative.

The properties of the generating function developed in Appendix I can be used to establish the following method of representation and associated search routine. An upward move at time 1 adds 8 to the path sum, a downward move at time 2 subtracts 7 from it, and so on. In a manner similar to representing numbers binomially, is simplest to record upward moves using positive values and downward moves as zero. This procedure creates index sums ranging downward from 36 to 0 in steps of 1. These values are related to the actual path sums by the formula $V = -36 + 2W$, where V is the desired path sum and W is the quantity obtained following the procedure in this paragraph.

Table A.5 Using the generating function to find specific paths

8	7	6	5	4	3	2	1	W	V
8				4	3	2	1	18	0
	7		5		3	2	1	18	0
		6	5	4		2	1	18	0
	7			4	3	2	1	17	-2
		6	5		3	2	1	17	-2
		6		4	3	2	1	16	-4

The paths determined from this procedure are shown in Table A.6. (The paths with a sum of zero are used only to demonstrate that no other paths can be eliminated.)

Table A.6 The paths found in Table A.5.

0	1	2	3	4	5	6	7	8	V	Arith. Avg.
0	1	0	-1	-2	-1	0	1	2	0	1.008919
0	-1	0	-1	0	-1	0	1	2	0	1.006114
0	-1	-2	-1	0	1	0	1	2	0	1.008919
0	-1	0	-1	-2	-1	0	1	2	-2	.983200
0	-1	-2	-1	0	-1	0	1	2	-2	.983200
0	-1	-2	-1	-2	-1	0	1	2	-4	.960286

At this point, recall that we eliminated one of the original 56 paths using a bounding argument. The present calculations show that another 3 can be eliminated by our evaluations, leaving the frequency distribution of vertices for 52 remaining paths as shown in Section 4.

Appendix III: Values of path arithmetic averages in a bundle

The following graph displays, for the example of section 4, the arithmetic averages for the 56 paths ending at the time-price node (8, 2), plotted in decreasing order. The graph is typical of the pattern of averages for a given time-price node, and can be approximated using a cubic or power function. For large values of T, such approximations can be used to eliminate many of the calculations described in Appendix 2.