Dynamic Immunization and Transaction Costs

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Summary

The aim of this paper is to analyse the effects of transaction costs on immunization strategies. We show, using linear programming techniques, that the optimal dynamic strategy against interest rate risk depends on the level of transaction costs. Moreover, it can be proved that the optimal solution can consist of choosing, initially, a portfolio with a duration smaller than the investor's planning period instead of selecting an immunized portfolio; the optimal solution takes advantage of the dynamic behaviour of portfolio duration.

Immunisation dynamique et coûts de transaction

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Résumé

Le but de cet exposé est d'analyser les effets des coûts de transaction sur les stratégies d'immunisation. Nous montrons, à l'aide de techniques de programmation linéaire, que la stratégie dynamique optimale contre le risque du taux d'intérêt dépend du niveau des coûts de transaction. Il peut de plus être prouvé que la solution optimale peut consister à choisir, initialement, un portefeuille d'une durée moins importante que la période de planification de l'investisseur au lieu de retenir un portefeuille immunisé ; la solution optimale tire profit du comportement dynamique de la durée du portefeuille.

1. INTRODUCTION

One of the main results concerning the development of portfolio strategies against interest rate risk is the Dynamic Global Immunization Theorem enunciated by Khang in 1983:

> "Consider an investor who has a planning period of length m. Suppose the forward interest rates structure shifts up or down by a stochastic shift parameter at any time during the investor's planning period. If the investor follows the Strategy in such a world, then the investor's wealth at the end of his or her planning period will be no less than the amount anticipated on the basis of the forward interes rates structure observed initially (or at time 0). Furthermore, the investor's wealth at time m will actually be greater than the amount anticipated initially, if at least one shock takes place during the planning period" (Khang, 1983).

The Strategy, as Khang calls it, consists of a continuous portfolio rebalancing in order to keep portfolio duration equal to the length of the remaining planning period.

But Khang's theorem was based on a set of hypothesis including:

(1) the forward interest rates structure $g(t)$, $t \geq 0$, change to $g^*(t,\lambda)$ where λ is a stochastic shift parameter. To be concrete he assumes that,

$$g^*(t,\lambda) = g(t) + \lambda$$

(2) absence of transaction costs.

The first hypothesis avoids the problem of the risk of misestimation of the term structure behaviour called by Fong y Vasicek (1983) "Immunization risk"[1] and the second the high costs that a continuous portfolio rebalancing process may incur.

In order to give an answer to these two problems we develope a dynamic portfolio selection model, in a discrete context, that will provide a framework for the introduction of transaction costs as well as the risk associated with unexpected twists of the term structure of interest rates.

First, we state a basic portfolio selection model to obtain the optimal strategy for an investor who wishes to guarantee a minimum return over a given planning period. Later on, this model is enlarged to introduce immunization risk and transaction

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[1] Bierwag called it "stochastic process risk", and defines "stochastic process" as "the way in which the term structure shifted from period to period", adding afterwards that "it is conceivable that an investor could assume an incorrect stochastic process and, as a consequence, the percieved durations would be different from the actual ones. The investor ... losses from misestimation (or misguesstimation) of the correct process can be substantial".(Bierwag, 1987).

costs.

Finally, we show with an example the effects of these two modifications on the optimal strategy and show that under hypothesis (1) and (2) our model provides the results enunciated by Khang's theorem.

2. THE BASIC MODEL

The dynamic model we propose here is based on an earlier static model (Meneu and Navarro, 1991) which uses the fact that immunization is a maximin strategy and linear programming techniques.

The initial hypothesis of the static model are:

1. The term structure of interest rates (TSIR) is flat.
2. TSIR changes consist of parallel movements of the whole term structure, i. e., short and long term interests rate changes are equal.
3. There is a bound for interest rate fluctuations so that a maximun and a minimun interest rates are considered. This range of variability of interest rates is important as far as the potential loss or gain derived from interest rates changes, and so interest rate risk, depends on the size of interest rates movements.

4. Financial markets are competitive: individual investors decisions don't affect interest rates which are given exogenously.

5. Immunization risk is not considered by inversors.

6. Absence of transaction costs.

7. Perfect divisibility of financial assets.

As we have just mentioned, immunization can be described as a maximin stategy in a game against Nature (Bierwag and Khang, 1979) where the investor's target is to guarantee a minimum return over the horizon planning period (HPP) or equivalently, a minimun value at the end of the HPP. This guaranteed final portfolio value depends on the evolution of interest rates and the portfolio composition. This problem can be modeled as follows:

Let's assume an investor who wants to invest an amount of I_0 dollars in a market where fixed-income assets are traded. Different portfolio allocations among these assets can be considered as the "strategies" followed by the investor.

We also assume that just after the purchase of the selected portfolio, interest rates may change from its current value (denoted by r_c) to any of the following possibles values:

$$r_1, \ldots, r_c, \ldots, r_{m-1}, r_m. \quad (r_1 < \ldots < r_c < \ldots < r_m)$$

remaining unchanged until the end of the HPP.

These possible values of interest rates can be regarded as the strategies Nature can take.

The result of such a game can be summmarized by the following (n x m) payoff matrix:

INTEREST RATES

	r_1	. . .	r_j	. . .	r_m
Asset 1	v_{11}	. . .	v_{1j}	. . .	v_{1m}
. . .		. . . . .	. . . . .	. . . . .	
Asset i	v_{i1}	. . .	v_{ij}	. . .	v_{im}
. . .		. . . . .	. . . . .	. . . . .	
Asset n	v_{n1}	. . .	v_{nj}	. . .	v_{nm}

where v_{ij} denotes portfolio value at the end of the HPP if an amount of I_0 dollars is invested at the beginning of the HPP in asset i and just afterwards interest rate becames r_j, remaining unchanged until the end of the HPP.

Immunization is the maximin solution of the game described by that payoff matrix.

It can be proved that the maximin strategy of a game as the one just described can be obtained as the solution of a linear programe.(Dantzig, 1951).

Let x_i be the percentage of the initial portfolio value (I_0) invested in asset "i" (i=1, 2, ..., n). If interest rate shifts from r_c to r_j portfolio value at the end of HPP (t=T) is given by:

$$v_{1j}x_1+\ldots+v_{ij}x_i+\ldots+v_{nj}x_n$$

If we are seeking to guarantee a minimum portfolio value at t=T (denoted by V) it must be satisfied the following inequality

$$v_{1j}x_1+\ldots+v_{ij}x_i+\ldots+v_{nj}x_n \geq V \qquad \text{for all } j;$$

i.e. independently of portfolio composition, its value at t=T must be equal or bigger than the minimum value V. As investor wants V to be as big as posible, immunization can be approach through the following linear programe

$$\begin{aligned}
&\qquad\qquad Max \quad V \\
&S.t.: \\
&\sum_{i=1}^{n} v_{ij}x_i - V \geq 0 \qquad j=1,\ldots,m; \\
&\sum_{i=1}^{n} x_i = 1 \\
&x_i, V \geq 0 \qquad i=1,\ldots,n;
\end{aligned}$$

where

$$\sum_{i=1}^{n} x_i = 1$$

represents the budget constrain.

The programme solution x_i^* indicates the percentage of investment I_0 that has to be assigned to buy asset "i" and V^* the minimum portfolio value guaranteed at the end of the HPP.

In any case, it must be pointed out that the solution given by programme A is not exactly an immunized portfolio but an approximation to it. This is due to the fact that we only considered a finite number of scenarios (interest rate shifts). One way to obtain a more accurate result is to include as scenarios not only the biggest interest rate fluctuations but also, small enough shifts of interest rates.

3. THE DYNAMIC MODEL

The former selection model provides a portfolio that is immunized against interest rate risk only at the beginning of the HPP. Unfortunately, the dynamic behaviour of portfolio duration makes it imposible to keep the portfolio immunized during the whole planning period. Only if portfolio consist of zero-coupon bonds with maturity at t=T would be possible to keep potfolio immunized during the whole HPP without any rearrangement.

So, portfolio should be continuously restructured in order to keep its duration equal to the length of the HPP.

The selection model we describe next tries to obtain an optimal rebalancing path to keep portfolio free of interest rate risk. We will show that this selection model provides, so far, the solution suggested by Khang.

First we make a partition of the HPP into "k" subintervals of equal length

$$[t_0,t_1],\ [t_1,t_2],\ \ldots,\ [t_{k-1},t_k]$$

being t_0 the beginning of the HPP and t_k the end of the HPP and we assume that portfolio rebalancing is only allowed at the beginning of each subinterval.

At these points Nature can play the set of scenarios (interest rate changes) described in the former static model although the current level of interest rate is imposed to remain unchanged[2]. We must point out the fact that what does really matters in this model is not the level of interest rates but the effects of "changes" in interest rates on the final portfolio value.

A consequence of the model is that the amount available for investing at the beginning of the each period is suposed to grow at rate "r_c", i.e.

$$I_h = I_0(1+r_c)^{(t_h-t_0)}$$

[2] If we have not assumed a flat term structure it could be more sensible to assume that interest rates at the beginning of each subinterval are the "forward interest rates" derived from the initial term structure.

I_h being the amount of money available at the beginning of period $[t_h, t_{h+1}]$.

An additional problem that arises when trying to state this selection portfolio model is that we don't know at the beginning of the HPP the set of assets available in the future so, a hypothesis about future issues must be introduced to implement the model. In any case, it is worth pointing out that the aim of this portfolio selection model is to see how optimal portfolio is affected by market frictions[3].

In this model we assume that there are "q" different sort of assets with maturity at t_1, t_2, ...,t_k, ...,t_n, i. e., (q x n) different assets, the difference among these assets consisting of maturity dates and coupon rates. Then, we call asset (i,r) an asset with maturity in "i" periods time and with a coupon payment of C_r dollars per period $[t_{s-1}, t_s]$. In order to simplify the model we assume that principal and coupon payments are due at rebalancing dates (t_s, s=0, ..., k-1).

Now let $x_{s,i}^r$ be the number of units of asset (i,r) that are included in the optimal portfolio at t_s. Then $x_{s,i}^r - x^r_{s-1,i+1}$ is the number of assets (i,r) bought or sold at t_s (depending upon $x_{s,i}^r - x^r_{s-1,i+1}$ is positive or negative). Notice that at t_s, asset (i,r) is the same that asset (i+1,r) at t_{s-1}.

[3] However, it could be possible to foresee future assets if there exist issuing programmes as it usually happens in practice.

Let $b_{s,i}^r$ and $s_{s,i}^r$ be the number of assets (i,r) bought and sold at t_s respectively and $p_{s,i}^r$ the value at t_s of asset (i,r) priced at interest rate r_c. Then, the following set of constrains must be satisfied:

(a) $x_{0,i}^r = b_{0,i}^r$ $\qquad i = 1, \ldots, n; \; r = 1, \ldots, q;$

(b) $x_{si}^r - x^r_{s-1,i+1} - b_{s,i}^r + s_{s,i}^r = 0; \qquad s = 1, \ldots, k\text{-}1; \; i = 1, \ldots, n\text{-}s; \; r = 1, \ldots, q;$

(c) $x^r_{s-1,1} = s_{s,0}^r$ $\qquad s = 1, \ldots, k; \; r = 1, \ldots, q;$

(d) $x^r_{k-1,i} = s^r_{k,i-1}$ $\qquad i = 2, \ldots, n\text{-}k+1; \; r = 1, \ldots, q;$

The first set of constrains (a) indicates the number of assets bought at the beginning of the HPP. The second group (b), the purchase and sales at each t_s. In the third group of constrains (c) $s_{s,0}^r$ represents the number of assets with maturity at t_s which must be equal to the number of assets with one period to maturity at t_{s-1}; these set of constrains are included in the model to distinguish those assets that are sold at t_s from those with maturity at t_s that will have a different treatment. Finally, the last group of constrains (d) indicates that at $t=t_k$ (the end of the HPP) all assets must be sold.

The initial budget constrain is now replaced by:

$$\sum_{r=1}^{q}\sum_{i=1}^{n} b_{0,i}^{r}\, p_{0,i}^{r} = I_0$$

where I_0 is the available amount of money at the beginning of the HPP.

But the budget constrain must be satisfied not only at t_0 but during the whole HPP, so we have to add the following set of budget constrains:

$$\sum_{i=1}^{n-s}\sum_{r=1}^{q} p_{s,i}^{r} b_{s,i}^{r} - \sum_{i=1}^{n-s}\sum_{r=1}^{q} p_{s,i}^{r} s_{s,i}^{r} - \sum_{r=1}^{q} p_{s,0}^{r} s_{s,0}^{r} - \sum_{i=0}^{n-s+1}\sum_{r=1}^{q} C_r x_{s-1,i+1}^{r} = 0; \quad s=1,\dots,k-1;$$

$$\sum_{i=1}^{n-k}\sum_{r=1}^{q} s_{k,i}^{r}\, p_{k,i}^{r} + \sum_{r=1}^{q} s_{k,0}^{r}\, p_{k,0}^{r} + \sum_{i=0}^{n-s+1}\sum_{r=1}^{q} C_r\, x_{k-1,i+1}^{r} = V_k$$

where

C_r is the coupon payment generated by one unit of asset of sort "r";

$p_{s,0}^{r}$ is the nominal value of asset of sort "r" with maturity at t_s, and

$p_{s,0}^{r} s_{s,0}^{r}$ are the principal repayments due at t_s.

What this set of constraints is trying to represent is the fact that the amount of money invested at each t_s in new purchases ($p_{s,i}^{r} b_{s,i}^{r}$) must come from sales ($p_{s,i}^{r} s_{s,i}^{r}$), coupon payments ($C_r x_{s,i}^{r}$) and principal repayments ($p_{s,0}^{r} s_{s,0}^{r}$).

As in the static model we assume that the investor's aim is to maximise, at each t_s, the guaranteed final portfolio value. Then, if we denote by V_s the minimum final portfolio value guaranteed at each t_s, the following set of constrains must be satisfied:

$$\sum_{i=1}^{n-s} \sum_{r=1}^{q} x_{s,i}^{r} \, v_{s,i}^{r}(j) \geq V_s \qquad j=1,\dots,m; \quad s=0,1,\dots,k-1;$$

where $v_{s,i}^{r}(j)$ is, now, the final value of an investment of $p_{s,i}^{r}$ dollars in asset (i,r) at t_s if interest rates change, just afterwards, from r_c to r_j remaining unchanged until the end of the HPP.

As far as investor wants to maximise these minimum guaranteed portfolio final values the objective function is:

$$Max \sum_{s=0}^{k} V_s$$

So the whole model is:

$$Max \sum_{s=0}^{k} V_s$$

$S.t.$:

$$\sum_{i=1}^{n-s}\sum_{r=1}^{q} x_{s,i}^{r}\, v_{s,i}^{r}(j) \geq V_s \qquad j=1,\ldots,m; \quad s=0,1,\ldots,k-1;$$

$$x_{0,i}^{r} = b_{0,i}^{r} \qquad i = 1, \ldots, n; \quad r = 1, \ldots, q;$$

$$x_{si}^{r} - x_{s-1,i+1}^{r} - b_{s,i}^{r} + s_{s,i}^{r} = 0; \qquad s = 1, \ldots, k-1; \quad i = 1, \ldots, n-s; \quad r = 1, \ldots, q;$$

$$x_{s-1,1}^{r} = s_{s,0}^{r} \qquad s = 1, \ldots, k; \quad r = 1, \ldots, q;$$

$$x_{k-1,i}^{r} = s_{k,i-1}^{r} \qquad i = 2, \ldots, n-k+1; \quad r = 1, \ldots, q;$$

$$\sum_{r=1}^{q}\sum_{i=1}^{n} b_{0,i}^{r}\, p_{0,i}^{r} = I_0$$

$$\sum_{i=1}^{n-s}\sum_{r=1}^{q} p_{s,i}^{r} b_{s,i}^{r} - \sum_{i=1}^{n-s}\sum_{r=1}^{q} p_{s,i}^{r} s_{s,i}^{r} - \sum_{r=1}^{q} p_{s,0}^{r} s_{s,0}^{r} - \sum_{i=0}^{n-s+1}\sum_{r=1}^{q} C_r x_{s-1,i+1}^{r} = 0 \qquad s=1,\ldots,k-1;$$

$$\sum_{i=1}^{n-k}\sum_{r=1}^{q} s_{k,i}^{r}\, p_{k,i}^{r} + \sum_{r=1}^{q} s_{k,0}^{r}\, p_{k,0}^{r} + \sum_{i=0}^{n-s+1}\sum_{r=1}^{q} C_r\, x_{k-1,i+1}^{r} = V_k$$

$$x_{si}^{r},\ b_{s,i}^{r},\ s_{s,i}^{r},\ V_s \geq 0$$

Now we illustrate the former model with a very easy example:

Let's assume an investor with a horizon planning period (HPP) of 18 months and 4 different fixed income assets available in the market; their characteristics are:

TABLE 1.- ASSET CHARACTERISTICS.

	Term to maturity (years).	Annual coupon (paid half-yearly).	Duration (years).
Asset 1	0.5	10%	0.5
Asset 2	1	10%	0.9761905
Asset 3	1.5	10%	1.429705
Asset 4	2	10%	1.861624

We assume he has one million dollars to invest among these assets and that the current interest rate compounded semiannually is 10%. Interest rate may move up or down by 100 basic points to 9% or 11%.

In this case, there is only one sort of assets with four different maturity dates, i.e., $q = 1$ and $n = 4$. (So we will drop superscript "r" from the variables notation).

The programme is:

Max V0+V1+V2+V3

S. to:

(1) 114.6626X01+115.1851X02+115.6851X03+116.1636X04-1V0>=0

(2) 115.7625X01+115.7625X02+115.7625X03+115.7625X04-1V0>=0

(3) 116.8676X01+116.3401X02+115.8401X03+115.3662X04-1V0>=0

(4) 109.725X11+110.225X12+110.7035X13-1V1>=0

(5) 110.25X11+110.25X12+110.25X13-1V1>=0

(6) 110.775X11+110.275X12+109.8011X13-1V1>=0

(7) 105X21+105.4785X22-1V2>=0

(8) 105X21+105X22-1V2>=0

(9) 105X21+104.5261X22-1V2>=0

(10) 1X01-1B01=0

(11) 1X02-1B02=0

(12) 1X03-1B03=0

(13) 1X04-1B04=0

(14) -1X02+1X11-1B11+1S11=0

(15) -1X03+1X12-1B12+1S12=0

(16) -1X04+1X13-1B13+1S13=0

(17) 1X01-1S10=0

(18) -1X12+1X21-1B21+1S21=0

(19) -1X13+1X22-1B22+1S22=0

(20) 1X11-1S20=0

(21) 1X22-1S31=0

(22) 1X21-1S30=0

(23) 100B01+100B02+100B03+100B04=1000000

(24) -5X01-5X02-5X03-5X04+100B11+100B12+100B13-100S11-100S12-100S13-100S10=0

(25) -5X11-5X12-5X13+100B21+100B22-100S21-100S22-100S20=0

(26) 5X21+5X22-1V3+100S31+100S30=O

(27) Xsi, Bsi, Ssi, Vi >=0 FOR ALL s, i.

The results are:

TABLE 2.- OPTIMAL PORTFOLIO. (Number of assets).

	$x_{s,1}$	$x_{s,2}$	$x_{s,3}$	$x_{s,4}$	Portf. Duration
s = 0	2639.5356	0	0	7360.4648	1.5022
s = 1	0	9951.29	548.71051	-------	0.9998
s = 2	11025.001	0	-------	-------	0.5

We can see the result is consistent with Khang theorem: optimal portfolio duration is always aqual to the remaining HPP (The small difference between these two variables is due to the fact of considering a finite number of scenarios about interest rate changes[4].

[4] A more accurate solution can be obtained if small interest rate changes are introduced in the payoff matrix. For example if scenarios about interest rates are $r_1=9\%$, $r_2=9.9\%$, $r_3=10\%$, $r_4=10.1\%$ and $r_5=11\%$ the duration of optimal portfolio at each t_s are 1.5006, 1.0000 and 0.5000 years respectively.

4. INTRODUCTION OF IMMUNIZATION RISK

In the portfolio selection model we have just developed we made the hypothesis of a flat term structure of interest rates assuming also that its movements are parallel, i.e. all interest rates must move jointly by the same amount.

But as it is well known, the nature of the dynamic of interest rates is by far much more complex. In fact, when designing strategies against interest rate risk, assumptions as a flat term structure and pararell shifts can be very dangerous if there is a twist of the term structure.

In order to minimise the risk derived from an unexpected behaviour of the term structure Fong and Vasicek proved that this risk may be reduced if the payments stream generated by a portfolio is concentrated around the end of the HPP.

As a trivial example we have the case of a portfolio consisting of zero coupon bonds with maturity date at the end of the HPP. By definition it would be an immunized portfolio and, at the same time, due to the total concentration of payments at the end of the HPP it is free of immunization risk.

Then, the problem lies in choosing, among the infinite number of immunized portfolios, the one that generates the most concentrated payments stream around the end of the HPP.

The way of doing so is penalizing the objetive function as follows.

For each asset, we calculate the dispersion measure $d^r_{s,i}$ proposed by Fong y Vasicek (Fong and Vasicek, 1983):

$$d^r_{s,i} = \frac{\sum_{j=1}^{l^r_{s,i}} (t_j - t_k)^2 \; C_j (1 + r_c)^{-(t_j - t_s)}}{\sum_{j=1}^{l^r_{s,i}} C_j (1 + r_c)^{-(t_j - t_s)}}$$

where

$l^r_{s,i}$ is the number of payments until maturity generated by asset (i, r) from t_s onwards.

C_j is the amount of the j-th payment generated by asset (i, r) from t_s onwards.

$(t_j - t_k)$ is the difference (days, months, years, ...) between the due date of the j-th payment generated by asset (i, r) and the end of the HPP.

$d^r_{s,i}$ is the choosen dispersion measure of the payments stream generated by asset (i, r) from t_s onwards.

This dispersion measure is translated into money units by the following expresion

$$A\ d_{s,i}^{r} \qquad\qquad A > 0;$$

where the value of coefficient A depends on the risk aversion of each individual investor, introducing a subjective element in the model.

As we want to minimise portfolio dispersion at every t_s the total penalization will be given by

$$A \sum_{s=0}^{k-1} \sum_{i=1}^{n-s} \sum_{r=1}^{q} d_{s,i}^{r}\ p_{s,i}^{r}\ x_{s,i}^{r}$$

This term should be introduced in the model by reducing the objective function value which becomes:

$$Max \sum_{s=0}^{k} V_s - A \sum_{s=0}^{k-1} \sum_{i=1}^{n-s} \sum_{r=1}^{q} d_{s,i}^r \, p_{s,i}^r \, x_{s,i}^r$$

If we apply this procedure to the former example we have:

TABLE 3.- ASSET DISPERSIONS.

$d_{0,1} = 1$	$d_{1,1} = 0.25$	$d_{2,1} = 0$
$d_{0,2} = 0.285714$	$d_{1,2} = 0.011904$	$d_{2,2} = 0.2380952$
$d_{0,3} = 0.058956$	$d_{1,3} = 0.238662$	----------
$d_{0,4} = 0.274916$	----------	----------

If we make A = 1, the optimal portfolios are:

TABLE 4.- OPTIMAL PORTFOLIO. (Number of assets).

	$x_{s,1}$	$x_{s,2}$	$x_{s,3}$	$x_{s,4}$	Portf. Duration
s = 0	0	0	8382.429	1617.571	1.499571
s = 1	0	9951.796	548.2039	-------	0.99989
s = 2	11025.00	0	-------	-------	0.5

The solution obtained is, as we expected, the immunized portfolio of minimum dispersion: this result is still consistent with Khang's theorem.

It could be possible to obtain different optimal portfolios depending on the value of A. These results can be easily obtained through the parametric analysis of the coefficients of the objective function. In any case, the solutions are always immunized portfolios except when extremely high values of coefficient A are considered.

5. INTRODUCTION OF TRANSACTION COSTS

In this model we assume that transactions costs incurred at each portfolio rebalancing are a percentage "α" of the volume traded (in dollars) at each date.

We also assume that principal and coupon payments don't generate any transaction costs although it could be easy to apply any other assumption in the model.

The way we introduce transaction costs in the model consist of reducing the amount available for new investments in the budget constrains. Another way of seeing this procedure is assuming that purchase prices are increased by a percentage "α" and sale prices are reduced by the same proportion. So budget constrains are modified as follows

$$\sum_{r=1}^{q}\sum_{i=1}^{n} b_{0,i}^{r}\, p_{0,i}^{r}(1+\alpha) = I_0$$

$$\sum_{i=1}^{n-s}\sum_{r=1}^{q}(1+\alpha)p_{s,i}^{r}b_{s,i}^{r} - \sum_{i=1}^{n-s}\sum_{r=1}^{q}(1-\alpha)p_{s,i}^{r}\, s_{s,i}^{r} - \sum_{r=1}^{q} p_{s,0}^{r}s_{s,0}^{r} - \sum_{i=0}^{n-s+1}\sum_{r=1}^{q} C_r\, x_{s-1,i+1}^{r} = 0$$

$$s=1,\dots,k-1;$$

$$\sum_{i=1}^{n-k}\sum_{r=1}^{q} s_{k,i}^{r}\, p_{k,i}^{r}(1-\alpha) + \sum_{r=1}^{q} s_{k,0}^{r}\, p_{k,0}^{r} + \sum_{i=0}^{n-s+1}\sum_{r=1}^{q} C_r\, x_{k-1,i+1}^{r} = V_k$$

After applying these changes to the portfolio selection model, it is possible to see how the result depends on the value of coefficient α, i.e. the level of transaction costs. Moreover, and this is in our opinion the main result of applying this portfolio selection model, the optimal portfolio need not to be an immunized portfolio any longer even for very low values of α.

Unfortunately it is not possible to apply the parametric analysis for obtaining problem solutions depending on the value of α because coefficient of basic variables of some constrains are involved.

In any case, and giving values to α some of the results obtained are:

TABLE 5.- OPTIMAL PORTFOLIO. (Number of assets)(α=0.05%).

	$x_{s,1}$	$x_{s,2}$	$x_{s,3}$	$x_{s,4}$	Portf. Duration
s = 0	0	0	9446.579	548.4232	1.453404
s = 1	0	9946.579	548.4232	-------	0.99989
s = 2	11018.42	0	-------	-------	0.5

TABLE 6.- OPTIMAL PORTFOLIO. (Number of assets)(α=0.20%).

	$x_{s,1}$	$x_{s,2}$	$x_{s,3}$	$x_{s,4}$	Portf. Duration
s = 0	0	0	9980.041	0	1.429705
s = 1	0	10478.05	0	0	0.976191
s = 2	11000.9	0	0	0	0.5

As we can see, even when a very low rate of transaction costs is included in the model (0.05%) the optimal portfolio path starts with an initial portfolio with a duration less than the HPP, i.e., a portfolio that is not immunized (table 5). When transaction costs are increased to 0.2% optimal portfolio is only inmunized during the last subinterval $[t_2, t_3]$ (table 6). We can also see that when transaction costs are high enough (in this case 0.2%) the optimal strategy consist of buying an unique asset and keeping it until the end of HPP.

We can observe in tables 5 and 6 that investment is now more concentrated in the asset with 18 months to maturity compared with the optimal portfolio path when transaction costs are not considered. This is to avoid the cost that sales and purchases imply when an immuinizing strategy is followed.

6. CONCLUSIONS.

We have seen that the portfolio selection model developed along this paper provides results that agree whith the Dynamic Global Immunization Theorem when the same hypothesis hold.

However, we have proved with an example that when transaction costs are included in the problem, immunized portfolios don't need to be the optimal strategy against interest rate risk.

The analysis of the results suggests two things. First, we have that immunization is the optimal strategy only if the possible losses derived from interest rate changes are greater than the certain losses derived from transaction costs.

Secondly, it must be taken into account that it is possible to take advantage of the dynamic behaviour of portfolio duration; if initial portfolio duration is lower than the HPP,the pass of time itself makes duration to be closer to the remainig of the HPP, process that can be helped by reinvesting coupon and principal payments appropiately.

So when transaction costs exist the optimal strategy may consist of a portfolio with a duration smaller than the HPP, instead of equal as immunization theorems suggest.

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