Immunization Measures for Life Contingencies

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Abstract

The traditional definitions of duration and M^2 for a certain cash flow are extended to cover contingent cash flows, such as life insurance and annuity products. Essentially, we define a kind of "expected duration" and "expected M^2", where the expectation is taken with respect to the (probability) measure of the contingent cash flow. As examples, we derive the duration and M^2 for some common life insurance and annuity cash flows in terms of classical actuarial symbols. We discuss some properties of expected duration and M^2 and relate them to the traditional duration and M^2 definitions. Numerical examples are given.

Key words: duration, M^2, immunization, life insurance, annuities.

Mesures d'immunisation conditionnelle

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Résumé

Les définitions classiques de la durée et de la masse monétaire au sens de M^2 pour un certain cash flow sont élargies pour couvrir les cash flows conditionnels, tels que les assurances sur la vie et les rentes. Nous définissons essentiellement une sorte de « durée anticipée » et de « la masse monétaire anticipée au sens de M^2 », dans lesquels l'anticipation est considérée par rapport à la mesure (de probabilité) du cash flow conditionnel. Nous en dérivons, sous forme d'exemples, la durée et la masse monétaire au sens de M^2 pour certains cash flows dérivés d'assurances sur la vie et de rentes communément offertes en termes de symboles actuariels classiques. Nous traitons de certaines propriétés de durée et de la masse monétaire au sens de M^2 anticipées et les comparons aux définitions classiques de durée et de la masse monétaire au sens de M^2. Des exemples numériques sont donnés.

Mots-clé : durée, masse monétaire au sens de M^2, immunisation, assurance sur la vie, rentes.

1 Introduction

In the past few years, a whole investment consulting business has grown up around immunization concepts. The main reason underlying the boom of this line of business is the increasing fluctuation of interest rates in financial markets. When interest rate changes, any financial institution or pension fund incurs a risk called "balance sheet risk" if its assets have different sensitivity to the change of interest rate than its liabilities. However, there exist difficulties with the application of classical immunization concepts to the insurance and pension fund management industries. Most immunization strategies are based on some immunization measures, such as duration, M^2 and convexity. These concepts are usually defined for certain cash flows [1, 2, 3, 5, 10, 12, 13, 16, 19, 20, 25]. Few insurance and pension liabilities are characterized by any certain cash flows [21].

In this paper, we show that the traditional definition of duration and M^2 can be extended to cover contingent cash flows, where the time of payment depends upon the occurrence or non-occurrence of a certain event or a state of the nature. Most life insurance and annuity products fall into this category of cash flows since the cash flows provided by these products depend on the event of "death" or "survival". Essentially, we define a kind of "expected duration" and "expected M^2" where the expectation is taken with respect to the survivorship distribution. We also give some concise formulas for the expected duration and M^2 for some common life insurance and annuity products by using actuarial symbols. Mathematical results show that these duration and M^2 statistics can be used as immunization measures. However, we have to emphasize here that this result is only correct in the sense of a "mean" (or average) since we average out the effect of mortality by taking expectations. The immunization measures vary with the nature of the assumed changes in future interest rates. We calculate the immunization measures for life contingencies under different stochastic models of the term structure of interest rates.

The rest of the paper is arranged as follows: Section 2 sets the stage of our studies by reviewing the traditional definitions of duration and M^2 for certain cash flows and their properties when the yield curve is flat and when it is subject to parallel shift. The extension of duration and M^2 for contingent cash flows is given in Section 3. Mathematical results in Section 4 show that these duration and M^2 statistics can be used as immunization measures. Section 5 discusses the immunization measures for life contingencies under different stochastic models of the term structures of interest rates. We incorporate the following three models for interest rates into the calculation of immunization measures for life contingencies:

1. The AR(1) process as in Panjer and Bellhouse [14, 15];

2. The Vasicek model [22]; and

3. The Cox, Ingersoll and Ross model [8].

Section 6 gives some numerical examples of the immunization measures of life contingencies under both deterministic and stochastic models of term structure of interest rates. The last section discusses some practical aspects when the extended notions of duration and M^2 are used.

2 Duration and M^2 for Certain Cash Flows

Any financial assets and liabilities can be characterized as streams of cash flow. For example, a bond confers on its owners a finite stream of coupon payments and a principal repayment. An n year term insurance provides a \$1 benefit at death if the insured dies within n years. The cash flow can be certain or more generally contingent upon the occurrence of some events. For a certain cash flow, we mean that both its amount and time of payment are predetermined, such as in the case of bonds and mortgages(if we ignore the default and prepayment risk). When either the amount or the time of payment, or both, are not certain, it is called a contingent cash flow.

Consider a certain cash flow $b_1, b_2, \ldots, b_n$ due at the times $t_1, t_2, \ldots, t_n$ respectively, where the amounts b_i and the times t_i are predetermined. Suppose that $\delta(t)$ is the current(at time zero) forward rate at time t, $0 \le t \le \infty$. The current value of the cash flow can be written as

$$\begin{aligned} P_0 &= \sum_{i=1}^{n} b_i e^{-\int_0^{t_i} \delta(s)ds} \\ &= \sum_{i=1}^{n} b_i d_{t_i}, \end{aligned} \tag{1}$$

where $d_{t_i} = e^{-\int_0^{t_i} \delta(s)ds}$ is the discount function.

For this cash flow, the duration can be defined as

$$\begin{aligned} D &= \frac{\sum_{i=1}^{n} t_i b_i d_{t_i}}{P_0} \\ &= \sum_{i=1}^{n} t_i w_i, \end{aligned} \tag{2}$$

where $w_i = \frac{b_i d_{t_i}}{P_0}$ and $\sum_{i=1}^{n} w_i = 1$. From this definition, we can obviously define the duration as the weighted average of the time periods in which the cash flows occur, where the weights, w_i, are the present values of the cash payments relative to the present price of the income stream. If the current forward rate $\delta(s)$ is a constant δ (the yield curve is flat), and the change of the yield curve is parallel, then duration D can also be defined as

$$D = -\frac{1}{P}\frac{dP}{d\delta}. \tag{3}$$

Hence, duration is a measure of price elasticity with respect to instantaneous rate of interest [11].

These two definitions are equivalent when $\delta(t)$ is a constant δ. For a constant interest rate and non-negative cash flows, some properties of duration D can be summarized as follows [3]:

1. The duration of the cash flow is equal to n if and only if the cash flow is a single payment at time n;
2. The duration of the cash flow is bounded between zero and the time to maturity;
3. The duration of the cash flow is inversely related to its yield to maturity;
4. The duration of the portfolio of m cash flows is equal to the weighted average of durations of individual cash flow, the weights being the proportions of money invested on each security.

A measure, M^2, of dispersion of cash flow dates, is usually defined as

$$M^2 = \sum_{i=1}^{n}(t_i - D)^2 w_i = \sum_{i=1}^{n} t_i^2 w_i - D^2. \tag{4}$$

This measure is closely related to convexity. We choose to use M^2 as a convenient second order measure since it is analogous to the concept of variance in statistics using the weights as probabilities.

3 Duration and M^2 for Contingent Cash Flows

We have summarized several definitions of duration for a general certain cash flow in the last section. These definitions are valid only for certain cash flow; i.e., the amount and the time of each payment are known. However, often the occurrence and the size of financial payments are uncertain. For example, most insurance and annuity products provide contingent payments based on the random event of death or survival. In this section, we extend the previous definitions to cover contingent payments where the time of payment depends on a random factor, but where the amount of payment is predetermined if the random event occurs. Most life insurance and annuity products fall into this category of cash flows.

We adopt closely the notation used in *Actuarial Mathematics* (AM) [4]. Actually, chapter 4 of AM gives a general valuation model for these kinds of cash flows. As stated by the authors:

> In fact, the general model is useful in any situation where the size and time of a financial impact can be expressed solely in terms of the time of the random event. [1]

[1]See *Actuarial Mathematics*, p. 85.

Suppose a cash payment, $b_T \geq 0$, occurs at a random time T, which has a density function $g(t)$. The present value of the payment, b_t, is $Z_T = b_T d_T$, which is also a random variable since both b_T and d_T depend on the random variable T. If we know the payment function b_t, discount function d_t, and the density function $g(t)$, we can calculate the expectation of the present value of the payment simply as

$$P = E(Z_T) = E(b_T d_T) = \int_0^\infty z_t g(t)dt = \int_0^\infty b_t d_t g(t)dt. \tag{5}$$

We can define the duration for this uncertain cash flow as the expected mean time of payment as follows:

$$\begin{aligned} D &= \frac{E(Tz_T)}{P} \\ &= \frac{\int_0^\infty tz_t g(t)dt}{\int_0^\infty z_t g(t)dt} \\ &= E(Tw_T) \end{aligned} \tag{6}$$

where $w_t = \frac{z_t}{P}$. Obviously, $\int_0^\infty w_t dt = 1$.

For constant interest rates, the duration can also be defined as $D = -\frac{1}{P}\frac{dP}{d\delta} = -\frac{1+i}{P}\frac{dP}{di}$ as in the last section.

If the yield curve is flat, the discount function becomes $d_t = (1+i)^{-t} = v^t$. The present value P can be written as

$$P = E(z_T) = E(b_T d_T) = \int_0^\infty (1+i)^{-t} b_t g(t)dt.$$

If we take the derivative of P with respect to i, we have

$$\frac{dP}{di} = -\int_0^\infty tb_t(1+i)^{-t-1}g(t)dt = \frac{-1}{1+i}\int_0^\infty tb_t v^t g(t)dt$$

By the price sensitivity definition of duration, $D = -\frac{1+i}{P}\frac{dP}{di}$, D can be written as

$$\begin{aligned} D &= \frac{\int_0^\infty tb_t v^t g(t)dt}{P} \\ &= \frac{\int_0^\infty tz_t g(t)dt}{\int_0^\infty z_t g(t)dt}. \end{aligned}$$

Comparing this equation with the defining equation (6), we can see that they are exactly the same. Hence, we can also use equation

$$D = -\frac{1+i}{P}\frac{dP}{di} \tag{7}$$

as an alternative definition of duration for contingent payments if interest rates are constant.

Obviously, the expected duration defined by (7) also has the properties listed in Section 1.

Similarly, M^2 can be defined by a variance term as

$$\begin{aligned} M^2 &= E[(T-D)^2 w_T] \\ &= E(T^2 w_T) - D^2. \end{aligned} \tag{8}$$

When interest rates are constant, we can also show that

$$E(T^2 w_T) = \frac{1}{(1+i)^2 P}\frac{d^2 P}{di^2} - D. \tag{9}$$

By substituting equation (9) into equation (8), we have

$$M^2 = \frac{1}{(1+i)^2 P}\frac{d^2 P}{di^2} - D - D^2. \tag{10}$$

The duration $Dur(.)$ of a few life insurance and annuity cash flows can be obtained as follows in an obvious actuarial notation [23, 17]:

$$\begin{aligned}
Dur(\overline{A}^{1}_{x:\overline{n}|}) &= \frac{(\overline{IA})^{1}_{x:\overline{n}|}}{\overline{A}^{1}_{x:\overline{n}|}} \\
Dur(\overline{A}_x) &= \frac{(\overline{IA})_x}{\overline{A}_x} \\
Dur(A_{x:\frac{1}{\overline{n}|}}) &= n \\
Dur(\overline{A}_{x:\overline{n}|}) &= (\frac{\overline{A}^{1}_{x:\overline{n}|}}{\overline{A}_{x:\overline{n}|}})\frac{(\overline{IA})^{1}_{x:\overline{n}|}}{\overline{A}^{1}_{x:\overline{n}|}} + (\frac{A_{x:\frac{1}{\overline{n}|}}}{\overline{A}_{x:\overline{n}|}})n \\
&= (\frac{\overline{A}^{1}_{x:\overline{n}|}}{\overline{A}_{x:\overline{n}|}})Dur(\overline{A}^{1}_{x:\overline{n}|}) + (\frac{A_{x:\frac{1}{\overline{n}|}}}{\overline{A}_{x:\overline{n}|}})Dur(A_{x:\frac{1}{\overline{n}|}}) \\
Dur({}_{m|}\overline{A}_x) &= \frac{{}_{m|}(\overline{IA})_x}{{}_{m|}\overline{A}_x} \\
Dur(\overline{a}_x) &= \frac{(\overline{I}\overline{a})_x}{\overline{a}_x} \\
Dur(\overline{a}_{x:\overline{n}|}) &= \frac{(\overline{I}\overline{a})_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} \\
Dur({}_{n|}\overline{a}_x) &= \frac{(\overline{I}\overline{a})_x - (\overline{I}\overline{a})_{x:\overline{n}|}}{{}_{n|}\overline{a}_x} \\
&= n + Dur(\overline{a}_{x+n}).
\end{aligned}$$

These results can be obtained by using equation (7).

The dispersion of cash flow dates, M^2, can be calculated from equation (8) or (10). For example, for whole life insurance, M^2 can be expressed as

$$M^2(\overline{A}_x) = \frac{\int_0^\infty t^2 v^t_t p_x \mu_{x+t} dt}{\overline{A}_x} - (\frac{(\overline{IA})_x}{\overline{A}_x})^2. \qquad (11)$$

We cannot use actuarial symbols to express the numerator in the first item of the right hand of the above equation. But, it is easy to calculate this item in practice when we make use of tabular data for mortality.

Therefore, we can also calculate the duration and M^2 for traditional life insurance and annuity cash flows. The duration of the portfolio of these products also has the linear additive property of duration. Hence, the duration of the total liability of a life insurance company is the weighted average of individual policy durations.

4 Mathematical Justification of the Extension

In this section, we try to show that expected duration and M^2 in Section 3 can be used as immunization measures.

Consider a continuous cash flow $b_t \geq 0$, $0 \leq t \leq T$. Let $\delta(t)$ be the forward rate given at time $t = 0$ for the cash flow at time t. Then, the expected value of the cash flow at any date t is given by[2]

$$P(t) = \int_0^T b_s e^{\int_s^t \delta(\tau)d\tau} g(s)ds. \qquad (12)$$

The present value at time $t = 0$ of the cash flow $0 \leq t \leq T$ is

$$P(0) = \int_0^T b_s e^{-\int_0^s \delta(\tau)d\tau} g(s)ds,$$

and the expected future value of the cash flow at time $t = T$ is

$$P(T) = \int_0^T b_s e^{\int_s^T \delta(\tau)d\tau} g(s)ds.$$

Suppose that an instantaneous change in the yield curve occurs as represented by a change of the forward rate from $\delta(\tau)$ to $\delta(\tau) + \epsilon(\tau)$, $0 < \tau \leq T$. Then, the change of the value of the cash flow $P(t)$ is

$$\begin{aligned} \Delta P(t) &= \int_0^T b_s e^{\int_s^t [\delta(\tau)+\epsilon(\tau)]d\tau} g(s)ds - \int_0^T b_s e^{\int_s^t \delta(\tau)d\tau} g(s)ds \\ &= \int_0^T b_s e^{\int_s^t \delta(\tau)d\tau} g(s)[e^{\int_s^t \epsilon(\tau)d\tau} - 1]ds. \qquad (13) \end{aligned}$$

[2]This formulation can be used for life insurances and life annuities. The general valuation formula for life annuities can be written as $P(t) = \int_0^T b_s [e^{\int_s^t \delta(\tau)d\tau}]_t p_x ds$. The interpretation of $g(s)$ is "the probability of payment at time s" in the case of both life insurance and life annuity. Hence it may be more convenient to interpret $g(s)$ as a general measure.

Denote

$$f(s) = e^{\int_s^t \epsilon(\tau)d\tau} - 1,$$

and assume that $\epsilon(\tau)$ is twice differentiable for all τ. We expand $f(s)$ about the point $s = t$

$$f(s) = f(t) + f'(t)(s-t) + f''(\eta)\frac{(s-t)^2}{2!},$$

where η is between s and t. Since η depends on s as well as t, we write η as $\eta(s,t)$ or simply as $\eta(s)$ for fixed t.

$$\begin{aligned} f'(s) &= e^{\int_s^t \epsilon(\tau)d\tau}[-\epsilon(s)], \\ f''(s) &= e^{\int_s^t \epsilon(\tau)d\tau}[-\epsilon(s)]^2 - e^{\int_s^t \epsilon(\tau)d\tau}[\epsilon'(s)]. \end{aligned}$$

Hence,

$$f(t) = 0$$

$$f'(t) = -\epsilon(t)$$

and

$$f''(\eta) = e^{\int_\eta^t \epsilon(\tau)d\tau}[(\epsilon(\eta))^2 - \epsilon'(\eta)].$$

Then

$$f(s) = -\epsilon(t)(s-t) + \frac{e^{\int_\eta^t \epsilon(\tau)d\tau}[(\epsilon(\eta))^2 - \epsilon'(\eta)]}{2}(s-t)^2.$$

If we substitute the above equation into equation (13) and simplify, we get

$$\begin{aligned} \Delta P(t) &= -\epsilon(t)\int_0^T (s-t)b_s e^{\int_s^t \delta(\tau)d\tau} g(s)ds \\ &+ \frac{1}{2}\int_0^T \{e^{\int_\eta^t \epsilon(\tau)d\tau}[(\epsilon(\eta))^2 - \epsilon'(\eta)]\}\{(s-t)^2 b_s e^{\int_s^t \delta(\tau)d\tau} g(s)\}ds. \end{aligned} \tag{14}$$

Let us denote

$$h_1(s) = e^{\int_\eta^t \epsilon(\tau)d\tau}[(\epsilon(\eta))^2 - \epsilon'(\eta)],$$

and

$$h_2(s) = (s-t)^2 b_s e^{\int_s^t \delta(\tau)d\tau} g(s).$$

If we assume $f''(s)$ is continuous, both $h_1(s)$ and $h_2(s)$ are continuous functions, and $h_2(s) \geq 0$. By the first mean value theorem of integration [26], there exists a $\xi \in (0,T)$ such that

$$\int_0^T h_1(s)h_2(s)ds = h_1(\xi)\int_0^T h_2(s)ds.$$

Then, we can write equation (14) as follow

$$\begin{aligned} \Delta P(t) &= -\epsilon(t)\int_0^T (s-t)b_s e^{\int_s^t \delta(\tau)d\tau} g(s)ds \\ &+ \frac{1}{2}e^{\int_{\eta(\xi)}^t \epsilon(\tau)d\tau}[\epsilon^2(\eta(\xi)) - \epsilon'(\eta(\xi))]\int_0^T (s-t)^2 b_s e^{\int_s^t \delta(\tau)d\tau} g(s)ds. \end{aligned}$$

If we divide the above equation by $P(t)$, we then obtain the relative change of the value of the cash flow due to the change in the forward rate.

$$\begin{aligned}\frac{\Delta P(t)}{P(t)} &= \frac{\Delta P(t)}{P(0)e^{\int_0^t \delta(\tau)d\tau}} \\ &= \frac{-\epsilon(t)}{P(0)}\int_0^T (s-t)b_s e^{-\int_0^s \delta(\tau)d\tau} g(s)ds \\ &+ e^{-\int_{\eta(\xi)}^t \epsilon(\tau)d\tau}\frac{[\epsilon^2(\eta(\xi)) - \epsilon'(\eta(\xi))]}{2P(0)}\int_0^T (s-t)^2 b_s e^{-\int_0^s \delta(\tau)d\tau} g(s)ds. \quad (15)\end{aligned}$$

We can see from the above equation that if the first term is zero and the second term is positive, we can be sure that $\frac{\Delta P(t)}{P(t)}$ is also positive. That is, we require

$$t = \frac{\int_0^T s b_s e^{-\int_0^s \delta(\tau)d\tau} g(s)ds}{P(0)},$$

and

$$\epsilon^2(\eta(\xi)) - \epsilon'(\eta(\xi)) > 0. \quad (16)$$

Since

$$D = \frac{\int_0^T s b_s e^{-\int_0^s \delta(\tau)d\tau} g(s)ds}{P(0)}$$

and

$$M^2 = \frac{\int_0^T (s-D)^2 b_s e^{-\int_0^s \delta(\tau)d\tau} g(s)ds}{P(0)},$$

then

$$\frac{\Delta P(t)}{P(t)} = -\epsilon(t)[D-t] + c[M^2 + (D-t)^2], \quad (17)$$

where

$$c = e^{-\int_{\eta(\xi)}^t \epsilon(\tau)d\tau}\frac{[\epsilon^2(\eta(\xi)) - \epsilon'(\eta(\xi))]}{2}.$$

It should be noted that if the shock function $\epsilon(\eta(\xi))$ is a constant, $\epsilon^2(\eta(\xi)) - \epsilon'(\eta(\xi)) > 0$. This yields the Fisher-Weil immunization theorem [25].

Following Shiu [20], we do not assume that the shock function here is a constant, i.e., a parallel shift of the yield curve. Instead, we assume that the shock function is a function of time t. As long as condition (16) holds, immunization is achieved when the planning period is equal to the duration. Condition (16) means that the $f(x)$ is convex. Hence, we can conclude that if the forward rate changes once, such that the shock function satisfies the condition (16), the income stream is immunized to the change if the duration of the income stream calculated at the present term structure of the interest rate equals the planning period, t.

5 Immunization Measures under Stochastic Models of Interest Rates

The previous section uses the assumption of a deterministic additive change of interest rates. As pointed by several authors, this assumption gives rise to the arbitrage opportunities since arbitrage profits could be earned by having a long position of two discount bonds and a short position in a third with an intermediate maturity equal to the duration of the long position. Hence, this assumption of the change of interest rates is not compatible with equilibrium models of the term structure of interest rates.

In the past a few decades, different interest models have been proposed. For a concise summary of different stochastic models of interest rates, we refer to Sharp [18]. Empirical studies have not been able to show that one model describes the real data consistently better than alternative ones. The valuation of life contingencies under stochastic models of term structure of interest rate has been studied by several authors, such as Panjer and Bellhouse [14, 15], Waters [24] and Boyle [6]. Boyle [5] examined the question of immunization when the term structure of interest rates is stochastic. In the valuation, since both mortality rate and the interest rate are stochastic, so the present value of life contingent functions can be expressed as

$$P = E(Z) = E_2(E_1(Z|T)) = E_2(b_T d_T) = \int_0^\infty b_t d_t g(t)dt, \tag{18}$$

where E_1 is the expectation with respect to the stochastic interest rate for fixed t and E_2 is the expectation with respect to the mortality measure. The discount factor d_t is the price of a pure discount bond paying one dollar at time t. The variance of Z can be written as

$$V(Z) = E_2(V_1(Z|T)) + V_2(E_1(Z|T)). \tag{19}$$

From this expression, we see that the variance consists of two component. The term $E_2(V_1(Z|T))$ represents the variance of Z due to the stochastic price of the pure discount bonds. It is a weighted average of the variance of the pure discount bond price and the weights are the probability of death at time t. The term $V_2(E_1(Z|T))$ measures the variability due to mortality, where the variability of interest rate is averaged out by taking expectation with respect to it.

For simplicity, we adopt the following three models of term structure of interest rates for the calculation of immunization measures for life contingencies

- The Panjer and Bellhouse conditional AR(1) model [14, 15];
- The Vasicek model [22]; and
- The Cox, Ross and Ingersoll model [9].

The Panjer and Bellhouse AR(1) Model:

Panjer and Bellhouse [14, 15] used AR(1) and AR(2) models to describe the change of the forward rate and showed that they fit a number of interest rate series. Here, we examine the AR(1) process only.

Assume that the forward rate in year t is given by

$$r_t = \theta + \phi[r_{t-1} - \theta] + \epsilon_t, \tag{20}$$

where $\epsilon_t, t = 1, 2, \ldots$ is a white noise process with mean 0 and variance σ^2. This model implies that the forward rates in any year depend upon the forward rate in the previous year and some constant level. Given the initial observation r_0, the price of a pure discount bond of term t can be expressed as

$$d_t = e^{-[t\theta + \frac{(1-\phi^t)(r_0-\theta)}{(1-\phi)}] + \sigma^2[G(t)-H(t)]}, \tag{21}$$

where

$$G(x) = \frac{x}{2}\frac{1+\phi}{1-\phi} - \frac{1-\phi^x}{(1-\phi)^2},$$

$$H(x) = \frac{\phi^2}{1-\phi^2}[\frac{1-\phi^{2x}}{2} + \frac{\phi}{1-\phi}(1-\phi^x)(1-\phi)^{x-1}].$$

Duration for a pure discount bond maturing at time t can be calculated as the elasticity of the present value with respect to the initial observation $f(0)$ of the conditional AR(1) process.

$$D(t) = -\frac{1}{d_t}\frac{\partial d_t}{\partial r_0} = \frac{1-\phi^t}{1-\phi}. \tag{22}$$

We can see that the duration of a pure discount bond under this AR(1) model is always less than t, and approaches to $\frac{1}{1-\phi}$ as t increases.

The Vasicek Model:

Vasicek [22] assumed that the spot rate follows a diffusion process expressed by the following equation

$$dr = f(t,r)dt + \rho(t,r)dz_t, \tag{23}$$

where z_t is a Wiener process with incremental variance dt. The functions $f(t,r)$ and $\rho^2(t,r)$ are, respectively, the drift and variance of the diffusion process. Suppose that the present value of a pure discount bond $P(t,s,r)$ of term s depends only on the spot rate r and the term s. Then $P(t,s,r)$ can also be expressed by a diffusion equation after applying Itô's lemma to $P(t,s,r)$.

$$\begin{aligned} dP &= [\frac{\partial P}{\partial t} + f\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}]dt + \rho\frac{\partial P}{\partial r}dz \\ &= P\mu dt + P\sigma dz \end{aligned} \tag{24}$$

where

$$\begin{aligned} \mu &= \frac{1}{P}\frac{\partial P}{\partial t} + f\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}, \\ \sigma &= \rho\frac{\partial P}{\partial r}/P. \end{aligned} \tag{25}$$

This equation holds for bonds of all maturities s. By using an arbitrage argument similar to that used to derive the Black Scholes options pricing formula, Vasicek showed that the ratio $\frac{\mu(t,s,r)-r}{\sigma(t,s,r)}$ should not depend on s. In particular, this ratio is zero if investors have liquidity neutrality which results in $\mu(t,s,r) = r$. Then, we obtain a partial differential equation for the price of the pure discount bond from the definition of μ by equation (26) as follows:

$$\frac{\partial P}{\partial t} + f\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2} - rP = 0. \tag{26}$$

Equation (26) plus the boundary condition $P(t,t,r) = 1$ gives a complete description of pure discount bond price $P(t,s,r)$. If r follows the Ornstein-Uhlenbeck process,

$$dr = \alpha(\gamma - r)dt + \rho dz, \tag{27}$$

the pure discount bond price is given by

$$d_s = P(t,s,r) = \exp[F(\alpha,t,s)(V(\alpha,\rho,\gamma)-r)-(s-t)V(\alpha,\rho,\gamma)-\frac{\rho^2}{4\alpha^3}F(\alpha,t,s)^2] \tag{28}$$

where

$$\begin{aligned} F(\alpha,t,s) &= \frac{1}{\alpha}(1-e^{-\alpha(s-t)}), \\ V(\alpha,\rho,\gamma) &= \gamma - \frac{1}{2}\frac{\rho^2}{\alpha^2}. \end{aligned}$$

The duration of the pure discount bond, calculated as an elasticity of price with respect to r, is given by

$$D_s = -\frac{1}{d_s}\frac{\partial d_s}{\partial r} = F(\alpha,t,s). \tag{29}$$

The CIR Model:

Cox, Ingersoll and Ross [8] assumed that the spot rate is described by the following diffusion process

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz. \tag{30}$$

Under the assumptions of this spot rate and liquidity neutrality, the pure discount bond price is given by

$$d_s = P(t,s,r) = A(t,s,\gamma,\theta,\sigma^2)e^{-B(t,s,\gamma,\theta,\sigma^2)r} \tag{31}$$

where

$$\begin{aligned} A(t,s,\gamma,\theta,\sigma^2) &= \left(\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(s-t)]/2}}{(\gamma+\kappa)(e^{\gamma(s-t)-1}+2\gamma}\right)^{\frac{2\gamma\theta}{\sigma^2}}, \\ B(t,s,\gamma,\theta,\sigma^2) &= \frac{2(e^{\gamma(s-t)}-1)}{(\gamma+\kappa)(e^{\gamma(s-t}-1)+2\gamma}, \\ \gamma &= \sqrt{\kappa^2+2\sigma^2}. \end{aligned}$$

The duration for a pure discount bond under this model of term structure of interest rate is given by

$$D_s = -\frac{1}{d_s}\frac{\partial d_s}{\partial r} = B(t,s,\gamma,\theta,\sigma^2). \tag{32}$$

From the diffusion equation (24), we see that the change in the price of pure discount bond when spot rate changes is proportional to $\frac{\partial P}{\partial r}\frac{1}{P}$. Therefore, $\frac{\partial P}{\partial r}\frac{1}{P}$ is a proper measure of basis risk for the pure discount bond. For a certain cash flow b_1, $b_2, \ldots, b_n$ due at the times $t_1, t_2, \ldots, t_n$, the present value is (using B for the cash flow)

$$P(t,B,r) = \sum_{i=1}^{n} b_i P(t,t_i,r),$$

with differential

$$\begin{aligned} dP(t,B,r) &= \sum_{i=1}^{n} b_i dP(t,t_i,r) \\ &= \left(\sum_{i=1}^{n} b_i P(t,t_i,r)\mu(t,t_i,r)\right)dt + \left(\sum_{i=1}^{n} b_i P(t,t_i,r)\sigma(t,t_i,r)\right)dz \\ &= \mu(t,B)P(t,B,r)dt + \sigma(t,B)P(t,B,r)dz, \end{aligned}$$

where

$$\mu(t,B) = \frac{\sum_{i=1}^{n} b_i P(t,t_i,r)\mu(t,t_i,r)}{P(t,B,r)} \tag{33}$$

and (under liquidity neutrality, $\mu(t,B) = r$)

$$\sigma(t,B) = \rho\frac{dP(t,B,r)}{P(t,B,r)}. \tag{34}$$

Hence, $\frac{\partial P}{\partial r}\frac{1}{P}$ is also a measure of basis risk for a cash flow.

Stochastic duration D, defined as the term of a zero coupon bond which has the same basis risk as the cash flow [9]

$$\begin{aligned}\frac{\partial P(t,D,r)}{\partial r}\frac{1}{P(t,D,r)} &= \frac{\partial P(t,B,r)}{\partial r}\frac{1}{P(t,B,r)} \\ &= \frac{\sum_{i=1}^{n} b_i \frac{\partial P(t,t_i,r)}{\partial r}}{\sum_{i=1}^{n} b_i P(t,t_i,r)}.\end{aligned}$$

If we denote

$$\Phi(t,s,r) = \frac{\partial P(t,s,r)}{\partial r}\frac{1}{P(t,s,r)}, \tag{35}$$

then the duration, D_B, for the cash flow $B = (b_1, b_2, \ldots, b_n)$ is

$$D_B = \Phi^{-1}[\frac{\sum_{i=1}^{n} b_i \frac{\partial P(t,t_i,r)}{\partial r}\Phi(t,t_i,r)}{\sum_{i=1}^{n} b_i P(t,t_i,r)}], \tag{36}$$

where Φ^{-1} is the inverse function with respect to s. For the contingent cash flow, this formula can be modified as

$$D_B = \Phi^{-1}[\frac{\sum_{i=1}^{n} b_i g(t_i) \frac{\partial P(t,t_i,r)}{\partial r}\Phi(t,t_i,r)}{\sum_{i=1}^{n} b_i g(t_i) P(t,t_i,r)}]. \tag{37}$$

For the Panjer and Bellhouse conditional AR(1),

$$\Phi^{-1}(x) = -\frac{\log[1-(1-\phi)x]}{\log(\phi)}, \tag{38}$$

for the Vasicek model,

$$\Phi^{-1}(x) = -\frac{1}{\alpha}\log(1-\alpha x) \tag{39}$$

and for the CIR model,

$$\Phi^{-1}(x) = -\frac{1}{\lambda}\log[\frac{2+x(\lambda-\kappa)}{2-x(\lambda-\kappa)}]. \tag{40}$$

6 Numerical Examples

To compare the duration of life contingencies under different assumptions of the term structure of interest rates, we construct Table 1 and Table 2. Table 1 gives the present value and duration for a pure discount bond of various terms; Table 2 gives the present value and duration of term life insurance benefit cash flows. The Canadian Institute of Actuaries 1982-1988 Male Mortality Table with a 15-year select period is used in the calculation of the present values and the durations of the life insurance cash flows.

The parameters in different interest rate models are chosen so that the present value of the pure discount bond have broad agreement.

Table 2 show that the duration increases with the length of the policy, but at a decreasing rate. For duration of life insurance with the same term, duration increases with the age at early ages and decreases at later ages. The reason is that both interest rate and mortality rate affect the calculation of the duration, the effect of interest rate on the duration dominates the effect of mortality rate at early ages and vice versa at later ages.

Under different models of the term structure of interest rates, the durations of life insurances are similar for short term contracts, but differ substantially for long term ones. Durations under the stochastic models are smaller than under the constant interest model. For constant interest rates, any small change in the spot rate results in a shift of the yield curve of almost equal amount at all maturities; therefore, it has a more profound effect on the long term bond than the short one. However, the conditional AR(1), Vasicek and CIR models exhibit mean reversion where spot rate adjusts itself to its long mean value constantly. Therefore, the change of spot rate has a more profound effect on the short term contract than on the long term one [5]. From Table 2, we see that the duration for long term contract is almost indistinguishable under the conditional AR(1), Vasicek and CIR models while it is quite different under constant assumption. However, this argument is gradually weakened as the mortality rate increases along the age.

7 Managerial Aspects of the Use of the Extension

Traditional immunization fails if at least one of the assumptions underlying the model fails to hold. For example, Fisher and Weil planning period immunization usually fails because the assumption of parallel shifts fails. As in Shiu [20], our result is not based on the assumption of parallel shift. In summary, our immunization result is based on the following two assumptions:

1. The shock function satisfies condition (16);

2. The mortality assumption is appropriate.

Assumption 2 is implied in our result, since our result is only correct in the "mean" sense. Taking n-year term insurance as an example, we know that its expected duration is usually less than n. However, if a person dies exactly at age $x+n$, the single death payment is made at age $x+n$. This results in the actual duration of the single payment being much longer than the expected duration. But for a large portfolio, the expected duration should provide a good estimate of the duration calculated from the cash flow of the liabilities incurred. Moreover, we can use M^2 to estimate the extent to which the actual duration can possibly deviate from the expected duration. For

Table 1: **Present Value and Duration of a Payment of \$100 Due in n Years**

Years	Constant[a]		Panjer's AR(1)[b]		Vasicek[c]		CIR Model[d]	
1	95.238	1.0	96.070	1.0000	95.034	.95163	95.033	.95119
2	90.703	2.0	92.206	1.9000	90.166	1.8127	90.160	1.8096
3	86.384	3.0	88.421	2.7100	85.433	2.5918	85.416	2.5823
4	82.270	4.0	84.728	3.4390	80.859	3.2968	80.825	3.2764
5	78.353	5.0	81.134	4.0951	76.461	3.9347	76.403	3.8986
6	74.622	6.0	77.645	4.6856	72.248	4.5119	72.162	4.4553
7	71.068	7.0	74.267	5.2170	68.227	5.0341	68.107	4.9527
8	67.684	8.0	71.002	5.6953	64.398	5.5067	64.241	5.3965
9	64.461	9.0	67.852	6.1258	60.758	5.9343	60.563	5.7919
10	61.391	10.0	64.817	6.5132	57.306	6.3212	57.070	6.1439
15	48.102	15.0	51.333	7.9411	42.635	7.7687	42.211	7.3942
20	37.689	20.0	40.451	8.7842	31.635	8.6466	31.080	8.0775
25	29.530	25.0	31.784	9.2821	23.449	9.1792	22.834	8.4470
30	23.138	30.0	24.932	9.5761	17.375	9.5021	16.757	8.6457
35	18.129	35.0	19.537	9.7497	12.873	9.6980	12.291	8.7523
40	14.205	40.0	15.301	9.8522	9.5368	9.8168	9.0120	8.8093
45	11.130	45.0	11.979	9.9127	7.0651	9.8889	6.6069	8.8398
50	8.7204	50.0	9.3764	9.9485	5.2340	9.9326	4.8433	8.8561
55	6.8326	55.0	7.3384	9.9696	3.8774	9.9591	3.5502	8.8649
60	5.3536	60.0	5.7429	9.9820	2.8725	9.9752	2.6024	8.8695
65	4.1946	65.0	4.4942	9.9894	2.1280	9.9850	1.9075	8.8720
70	3.2866	70.0	3.5169	9.9937	1.5764	9.9909	1.3982	8.8733
75	2.5752	75.0	2.7520	9.9963	1.1679	9.9945	1.0249	8.8741
80	2.0177	80.0	2.1535	9.9978	.86517	9.9966	.75124	8.8744
85	1.5809	85.0	1.6852	9.9987	.64093	9.9980	.55065	8.8746
90	1.2387	90.0	1.3187	9.9992	.47482	9.9988	.40363	8.8747
95	.97055	95.0	1.0319	9.9996	.35175	9.9993	.29585	8.8748
100	.76045	100.0	.80744	9.9997	.26058	9.9995	.21686	8.8748

Parameters values for each model:

a. $r = 0.05$.

b. $\theta = 0.05$, $\phi = 0.90$, $\delta_0 = 0.04$, $\sigma = .01$.

c. $\rho = \sqrt{.0002}$, $\gamma = 0.07$, $\alpha = 0.1$.

d. $\sigma = \sqrt{0.002857}$, $\mu = 0.07$, $\kappa = 0.1$.

Table 2: **Present Value and Duration of a Life Insurance of \$100, Using CIA 1982 Male Mortality Table**

Ages	Term	Constant		Panjer's AR(1)		Vasicek		CIR Model	
20	5	.36137	2.8037	.36915	2.7144	.35740	2.6962	.35731	2.6893
	10	.62318	4.9654	.64363	4.5776	.60653	4.4989	.60582	4.4551
	15	.85772	7.1536	.89298	6.2621	.81894	6.0866	.81666	5.9684
	20	1.0859	9.4351	1.1374	7.8259	1.0147	7.5215	1.0096	7.2926
	40	2.7679	23.325	2.9491	15.584	2.2481	14.302	2.1944	13.108
	60	6.3034	38.836	6.7495	23.100	4.3512	20.621	4.1379	18.165
	85	8.0052	44.789	8.5720	25.340	5.1980	22.378	4.8945	19.514
40	5	.45932	3.2648	.47064	3.1763	.45311	3.1581	.45296	3.1506
	10	1.2260	6.3402	1.2751	5.9673	1.1811	5.8897	1.1790	5.8389
	15	2.4057	9.6856	2.5297	8.8390	2.2473	8.6656	2.2371	8.5182
	20	4.0515	13.103	4.2929	11.623	3.6580	11.327	3.6273	11.030
	40	13.673	25.682	14.655	20.266	10.818	19.305	10.522	18.112
	60	18.267	30.973	19.598	22.810	13.682	21.466	13.193	19.868
	65	18.304	31.035	19.637	22.828	13.701	21.479	13.210	19.878
60	5	3.1897	3.3576	3.2703	3.2730	3.1450	3.2555	3.1439	3.2482
	10	8.6843	6.3907	9.0351	6.0351	8.3638	5.9611	8.3490	5.9121
	15	16.054	9.4643	16.872	8.6543	15.030	8.4891	14.965	8.3505
	20	24.016	12.283	25.400	10.874	21.864	10.596	21.701	10.330
	30	35.339	16.318	37.583	13.565	30.871	13.062	30.475	12.544
	40	38.304	17.668	40.779	14.241	33.007	13.641	32.519	13.032
	45	38.416	17.740	40.900	14.268	33.081	13.662	32.588	13.049
80	5	32.568	2.8948	33.289	2.8041	32.194	2.7857	32.185	2.7785
	10	54.749	4.8777	56.531	4.5218	53.332	4.4495	53.273	4.4096
	15	65.712	6.1716	68.178	5.4947	63.295	5.3617	63.167	5.2786
	20	69.086	6.7204	71.789	5.8411	66.208	5.6733	66.041	5.5653
	25	69.630	6.8408	72.374	5.9041	66.654	5.7273	66.477	5.6134

Parameters values for each model:

a. $r = 0.05$.

b. $\theta = 0.05$, $\phi = 0.90$, $\delta_0 = 0.04$, $\sigma = .01$.

c. $\rho = \sqrt{.0002}$, $\gamma = 0.07$, $\alpha = 0.1$.

d. $\sigma = \sqrt{0.002857}$, $\mu = 0.07$, $\kappa = 0.1$.

this purpose, we prefer to use M^2 instead of convexity as a second order immunization measure.

Since the expected duration of a portfolio of various contracts is equal to the weighted average of the durations of the individual contracts, we just need to know how to calculate the duration for each contract. Some identities in actuarial mathematics have some interesting effects on duration analysis. For example, since $\overline{A}_x + \delta\overline{a}_{\overline{x}} = 1$, a portfolio having a whole life insurance and a life annuity paying δ has a zero duration if the interest rate changes once, in a parallel shift. Hence, adding or removing this portfolio(or a multiple of it) to the liability does not alter the duration of the total liability at all. It can be proven that the expected duration is a decreasing function of the force of mortality if the interest rate is constant [17]. So the expected duration is also a decreasing function of the age of insured, since the force of mortality increases as the age increases (at least for higher ages). In practice, we rarely use the age of insureds to underwrite the risk since we charge different premiums for different ages. But from the asset/liability management point of view, we can use age as an indicator for the increase or decrease of the liability duration. Hence, A/L management is an integrated process which has to take consideration of both sides of the balance sheet.

We need to exercise caution if we want to apply stochastic duration in practice. First, we have to choose a model of term structure of interest rates. There are a number of interest rate models in finance literature; however, empirical studies have not been able to show that one model fit the data consistently better than alternative ones. This poses a question: which interest rate model should we choose? Second, even though we make our choice of an interest rate model, how do we estimate those parameters in the model? Since most interest rate models are based on diffusion process, interested readers are recommended to refer Basawa and Rao[7]. Third, in theory at least, duration analysis and liability valuation should be done on a consistent basis. Clearly, accounting rules regarding valuation restrict the balance sheet asset and liability values. To the extent that these rules are inconsistent with the "true" underlying model, immunization of balance sheet surplus may not be attained when theoretical immunization occurs.

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