Maximising Long Term Return

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Summary

This paper is motivated by the attempt to answer the following question: "For an investor with an extremely long time horizon, what is the optimum portfolio, and can one improve upon the traditional answer of '100% in equities'?"

In the process of answering the above question, the paper provides a much needed clarification of the meaning of the phrase "expected long term return" from the viewpoint of long term investors such as pension funds, insurance companies and charities. "Mean" returns turn out to be "meaningless" for practical purposes. Particular care is needed in Asset Liability Modelling work, where presentation of outcomes in terms of means and standard deviations is very misleading. The author calculates the "expected long term return" for various portfolios, including those containing equities, bonds and European equity options, for the geometric Brownian stochastic model. In particular, the author demonstrates that the long term returns to be expected from two strategies currently popular in the retail market, namely guaranteed capital and enhanced income strategies (via put purchases and call sales respectively) are likely to disappoint those who subscribe to them.

In addition, the author demonstrates what may be termed "portfolio synergy" effects arising from rebalancing, namely that the "expected long term return" for a portfolio does **not** equal the linear combination of "expected long term returns" for the portfolio constituents. This is true whether the portfolio is rebalanced periodically or continuously.

Finally, the author attempts to answer the original question, subject to various constraints as to whether short asset positions are allowed, and concludes that "100% in equities" **can** frequently be improved upon.

Maximisation des rendements à long terme

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Résumé

Le but de cet exposé est de tenter de répondre à la question suivante : « Pour un investisseur disposant d'une perspective à très long terme, quel est le meilleur portefeuille possible et peut-on améliorer la proposition classique de "100 % en titres de participation" ? »

En se proposant de répondre à cette question, l'exposé tente de clarifier l'expression « rendement escompté à long terme » de la perspective des investisseurs à long terme tels que caisses de retraite, compagnies d'assurance et oeuvres de charité. Pratiquement parlant, un rendement « moyen » ne veut rien dire. Il faudra procéder avec précaution à l'élaboration de Modèles de gestion des actifs et des passifs dans lesquels la présentation des résultats en termes d'écarts moyens et standard peut être très trompeuse. L'auteur calcule le « rendement escompté à long terme » pour divers portefeuilles, notamment ceux contenant des titres de participations, des obligations et des options européennes sur actions, pour le modèle géométrique stochastique brownien. L'auteur démontre en particulier que les rendements escomptés à long terme et fondés sur deux stratégies prisées à l'heure actuelle sur le marché du détail, à savoir les stratégies de garantie du capital et de maximisation du revenu (faisant intervenir respectivement des options de vente et d'achat) s'avèreront sans doute décevants pour ceux qui y souscrivent.

L'auteur démontre de plus ce qui peut être appelé les effets de « la synergie de portefeuille » résultant d'un ré-équilibrage, à savoir que le « rendement escompté à long terme » pour un portefeuille n'est pas égal à la combinaison linéaire des « rendements escomptés à long terme » pour les divers éléments constitutifs de ce portefeuille. Ceci est vrai, que le portefeuille soit ré-équilibré de façon périodique ou de façon continue.

Enfin, l'auteur tente de répondre à la question initiale, sous réserve de diverses contraintes à savoir si des positions à découvert pour les actifs sont permises ou non, et conclut que la proposition « 100 % en titre de participation » laisse souvent à désirer.

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Introduction

The motivation behind this paper arose from the paper I wrote for the 3rd AFIR Congress (see reference [1]) when I demonstrated that fantastic *mean returns* were available on equity call options (exceeding 20% real per annum), as compared to the typical 5% to 6% real pa that it seems reasonable to expect from equities over the long term. (This fact is no doubt well known to financial economists, but is one which many actuaries and other advisers are unaware of. The high mean returns essentially arise from the geared nature of the equivalent call option hedging portfolio.)

Now consider a very immature pension fund, for which assets might need to be invested for more than 30 years before benefits started to exceed contribution income. The traditional advice given to such a client is to invest the assets almost entirely in a portfolio of diversified equities. The view is that such a portfolio can be expected to maximise returns over the long term.

If however such an investor (and even more so an investor with an even longer time horizon) were able, by bearing acceptable levels of extra risk (which the longer time horizon should facilitate) if necessary, to capture even one tenth of the "excess returns" indicated by the above mean returns on call options, then very significant improvements in benefits or reductions in contribution costs could be achieved. In addition, since the higher mean returns from call options arise essentially from gearing (in the sense of taking short cash positions which are then invested in equities), perhaps other "geared portfolios" exist, which, while satisfying whatever constraints that the pension fund is subject to with regard to borrowing and short sales, also lead to significant increases in the returns that may be expected over the long term.

First, however, some precision in the use of the phrase "expected return" when applied to the long term is required.

What is a sensible definition of expected return for a long term investor?

Assume that a long term investor (for example a pension fund, an insurance company or a charity) invests in a portfolio. Let (one plus the return) achieved by holding this portfolio over a discrete time period $t=0$ to $t=1$ be the random variable $\mathbf{R_1}$ ie \$1 invested at time 0 grows to \$$R_1$ at time 1. Similarly, call (one plus the return) achieved by holding the portfolio from time t=i-1 to time t=i $\mathbf{R_i}$. In fact, for simplicity, I shall use **return** below to mean (one plus the return), for example I would describe a deposit account yielding interest at 8% as having a return of 1.08, and R_i may be taken as describing the return achieved on the portfolio during the ith time period.

Then the *actual return* achieved by holding the portfolio (with rebalancing at the start of each unit time period) over a long time period T is equal to $R_1.R_2....R_T$, the product of the R_i terms during each unit time period up to time T. The annualised return achieved is $R_a=(R_1.R_2....R_T)^{1/T}$ and more significantly, the log annualised return is $L_a = (\Sigma\ L_i)/T$, where the summation is from i=1 to i=T and $L_i = \log_e R_i$.

Providing returns in successive periods are independent, (and in particular in the case of the **geometric Brownian model** that underlies both traditional "modern portfolio theory" and the Black Scholes option pricing model):

log R_t is distributed as $N (\{ \mu - \frac{1}{2} \sigma^2 \}t, \sigma^2 t)$ [where here R_t represents the return over the time period 0 to t]

then the log asset return distributions will be sufficiently well behaved as to satisfy the Central Limit theorem. Over the long term, as T gets large, the log annualised return L_a will tend to $\mu - \frac{1}{2} \sigma^2$, the mean of the single period log return of the portfolio.

In other words, providing returns in successive periods are independent, **the annualised return that the investor will achieve over the long term will get increasingly close to exp θ, where θ is the mean of the log return distribution.** If returns are serially correlated, then convergence may still frequently occur in practice, at a faster rate if returns are negatively correlated and at a slower rate if returns are positively correlated.

In connection with whether the extremely high *mean returns* from call options can be captured, of significance here is the fact that, since there is a discrete non zero probability that the option will expire worthless (of the order of 1/3 for an at the money one year equity call option and of the order of 2/3 for a put option - see reference [1] for details), the *mean log return* for options is minus infinity, ie the long term expected return from holding them is in fact zero, ie total loss of capital! This indicates that there may be considerable difficulties in capturing the high *mean* returns, which in such a case would turn out to have little practical relevance.

Appropriate "risk/return" measures for use in Asset Liability Modelling work, and in particular the "meaninglessness" of the mean

Some consultants still fall into the trap of assuming that when clients trade off return against risk, mean single period returns and standard deviation of returns provide appropriate measures or proxies for return and risk respectively. This may be true for short time periods (say one year or less), but can be seriously flawed when applied to longer term periods because of the following factors:

- Over the long term, returns will converge to the exponential of the *log mean,* which in general is significantly different from the mean return. For example, I have already mentioned that while equity European call options have mean returns exceeding 1.25 (or 25% pa), their long term return is actually 0 (or total ruin!).
- The distributions involved will frequently be significantly skewed (either negatively or positively), and therefore even if the mean was a reliable central estimate (which it isn't), assuming that risk can be encapsulated in the single figure of the standard deviation can lead to serious understatement of the true risk.

In practice, many consultants have recognised that pure asset returns ignore the presence of the investor's liabilities and therefore focus on the *distribution* of funding levels (the ratio of assets to liabilities) and other measures which take explicit account of the liabilities. However, even these distributions are often far from symmetrical and therefore require more sophisticated approaches than pure mean variance optimisation.

In general, therefore, neither means nor standard deviations of outcomes (whether those outcomes be measured as returns or as measures which reflect the presence of liabilities such as funding ratios or contribution rates) are appropriate for helping long term investor clients to make risk/return preference choices between different portfolios. Instead, my colleagues and I have successfully used the following measures in our asset liability modelling work:

- The "return" concept is often best expressed via either the median or the central forecast outcome. The central forecast is the outcome that is obtained from simulation when all standard deviation parameters from the stochastic model are set to zero. This will in general not equate to the median.
- The "risk" concept is often best expressed via either a suitable (eg 5th) percentile outcome or the estimated probability of ruin/insolvency. It must be emphasised here that in general, these

measures cannot simply be calculated from the mean and standard deviation of the distribution, but instead must frequently be obtained by simulation.

Estimated Long Term Returns from Various Portfolios : Using the Geometric Brownian model

The geometric Brownian (or random walk) model for share prices is much used in financial theory so it is appropriate to consider its implications for the long term return to be achieved from various portfolios. I shall assume for the sake of simplicity that only three primary assets are available, shares, nominal bonds and cash. In addition however, single period European call and put options are available on both bonds and shares for a wide range of exercise prices. (The reader will readily be able to extend the results below to obtain practical results for further assets if he or she desires.)

The model: parameters

Assume both share and bond price indices are rebased to 1 at time 0. Parameters that I would regard as reasonable for the long term use of the geometric Brownian model are as follows:

Economic equilibrium considerations and historical analysis of market returns that we have carried out (Watsons Global Asset Study) in the major investment markets (including the US, Australia, Hong Kong, Canada, Japan, the UK and various continental European countries) indicate that it is reasonable to expect *long term real returns* from domestic government bonds (in local currency, in any of the above markets) to be of the order of 3% pa. Similarly, *long term real returns* from the major domestic equity indices may be expected to exhibit a risk premium of the order of 3% pa over bonds, leading to a *long term expected real return* on domestic equities of 6%.

At this point it is important to be precise and remember exactly how historical returns and standard deviations are calculated. The most rigourous and commonly used method is to calculate geometric annualised returns (equal to R_a in this notation of this paper, which will tend over the long term to exp θ, where θ is the mean of the log distribution and equals $\mu - \frac{1}{2}\sigma^2$ in the geometric Brownian case) and standard deviations of the logged returns (σ in the notation of this paper).

If we further assume that domestic inflation in the particular country of interest (for example the UK) has a mean log of 0.05 (ie 5% pa), then what we mean by the use of the phrase "*long term expected real return*" above may be expressed as $\mu_{bonds} - \frac{1}{2}\sigma_{bonds}^2 = 0.08$ and $\mu_{equity} - \frac{1}{2}\sigma_{equity}^2 = 0.11$ (where I have used round

In addition, I shall assume standard deviations of logged returns as follows:

$\sigma_{bonds} = 0.10$ (10% pa) and $\sigma_{equity} = 0.20$, and assume that the correlation of log returns of equities and bonds is 0.50.

In addition, the risk premium between short term bonds (cash) and long term bonds is assumed to be zero, for simplicity. This could be taken to reflect the widely held view that there is no strong empirical evidence to justify a *term structure premium* (or reward for investing long in government bonds).

We would regard the above parameters (for log mean returns, standard deviations and correlations) as reasonable for a long term version of the geometric Brownian model. In particular, I have assumed here an equity risk premium (over both bonds and cash) of some 3% pa, which although lower than that observed historically in many countries, is a reasonable estimate of what is sustainable over the long term from economic growth and in an environment where investors allow for the risk of inflation in their pricing of bonds. (For example, as I write this, UK index-linked government bonds - which are therefore exempt from inflation risk- currently yield just over 3%).

Development of the model

The share price S at time 1 is lognormally distributed, with log mean 0.11 and log standard deviation 0.20 (ie log S is normally distributed with mean 0.11 and s.d. 0.20). (The share price index S is considered to be a rolled-up index here and therefore includes the effect of dividend reinvestment in equities, with the 0.20 standard deviation reflecting the volatility not only of capital appreciation but of income. Please note that the parameters used here differ slightly from those used in my previous paper[1], in that - to reflect the more precise definition of *long term expected return* I have here set $\mu_{equity} = 0.13$ - in my previous paper it was 0.11 - the long term expected return in this paper.)

Similarly, the bond price B (also considered to be a rolled-up index) at time 1 is lognormally distributed, with log mean 0.08 and log standard deviation 0.10 (ie log B is normally distributed with mean 0.08 and s.d. 0.10).

Cash, of course, gives a constant log return of 0.08.

Both share and bond prices are assumed to be serially independent (ie neither series is correlated with itself over time) but log S and log B are assumed to display correlation of 0.5 with each other.

In addition to the above assets, it is assumed that single period European put and call options are available with any desired exercise price on both of bond and equity (rolled-up) price indices, at prices given by the Black Scholes formulae for assets with the above log mean and log volatility parameters. (The options are assumed to apply to the rolled up price indices for simplicity so that dividends may be ignored, however the principles and the calculations below readily extend to more complicated examples.)

The very long term return from any portfolio was shown above to be equal to exp θ where θ = the mean of the log return over a single period. Therefore, in order to calculate the long term return that may be expected from any portfolio (including those with short option positions), it is merely necessary either to calculate its mean log return theoretically or to estimate this parameter sufficiently accurately via a large enough number of simulations. For reasonable accuracy, 10,000 simulations are desirable and that is what I have used here. While theoretical calculation is possible in certain simple cases, statistical estimation by simulation works for all portfolios and is therefore used in the results shown below.

Results: Initial Portfolios of Interest

I first show the results for several basic portfolios that I felt to be of interest, namely 100% in each of the asset classes, and also the following:

- 2 equity protected put portfolios (with exercise prices set at 100% and 90% of the equity price index respectively). For these portfolios (which are similar to various retail funds which are currently popular because they are designed to give equity exposure with a capital guarantee), an amount equal to the Black Scholes price of the (one year) European put option is invested in the equity put option and the remaining assets are invested in equities.
- 2 enhanced income equity exposure portfolios (with call option exercise prices set at 100% and 110% of the equity index respectively). For these portfolios (which are similar to some other retail funds which are currently popular because they are believed - even in the weekend finance sections of the "serious" newspapers such as the Sunday Times!- to offer equity returns with extra income), an amount equal to (1 plus the Black Scholes price of the one year European call option) is invested in equities, financed via the original capital plus the proceeds of a short sale of the call option. (Such "covered writing" of calls is permissible for most pension funds, even when they are not allowed to hold short *cash*, ie to borrow.)

The prices for the put options turn out to be 4.417% and 1.154% of the initial capital respectively. The prices of the calls turn out to be 12.106% and 8.842% respectively.

For each portfolio, I show the mean of the log return (θ , as a measure of the long term return to be expected), the *mean return* (for comparison, to indicate how misleading this statistic can be), the standard deviation of the logged return (σ, as a measure of how long an investor would have to wait in order to capture the long term return), and for illustration percentile annualised *log returns* for investment both over a one year period over a 20 year period.

Portfolio	**Equity**	**Bonds**	**Call**	**Put**	**Protected Put 1**	**Protected Put 2**
	% by market value (at the start of each year)					
Equities	100.0				95.6	98.3
Bonds		100.0				
Equity Call (X=1.00)			100.0			
Equity Call (X=1.10)						
Equity Put (X=1.00)				100.0	4.4	1.7
Equity Put (X=0.90)						
Cash						
Long term return θ(%)	11.02	8.04	$-\infty$	$-\infty$	10.30	10.73
Mean return -1 (%)	13.92	8.93	42.15	-25.46	12.18	13.13
S.D. σ (%)	20.02	10.05	$4.*10^5$	$4.*10^5$	15.08	17.67
One year percentiles						
95th (%)	43.8	24.4	151.5	148.5	39.3	42.1
50th (median)	10.9	8.0	-4.8	Bust	6.4	9.2
25th	-2.5	1.2	Bust	Bust	-3.8	-4.2
5th	-21.7	-8.6	Bust	Bust	-4.4	-12.3
20 year percentiles						
95th (%)	18.4	11.7	Bust	Bust	16.1	17.4
50th=median	11.0	8.0	Bust	Bust	10.1	10.6
25th	7.9	6.4	Bust	Bust	7.9	8.0
5th	3.6	4.4	Bust	Bust	5.0	4.4

Note: the comment "Bust" implies exactly that, ie a total loss of capital has occurred!

Portfolio	**Cash**	**Enhanced Yield 1**	**Enhanced Yield 2**
	% by market value (at the start of each year)		
Equities	100.0	112.1	107.3
Bonds			
Equity Call (X=1.00)		-12.1	
Equity Call (X=1.10)			-7.3
Equity Put (X=1.00)			
Equity Put (X=0.90)			
Cash			
Long term return (%)	8.00	9.61	9.96
Mean return -1 (%)	8.33	10.50	11.18
S.D. σ (%)	0.00	8.95	11.61
One year percentiles			
95th	8.00	17.2	19.2
50th=median	8.00	12.6	16.6
25th	8.00	9.0	4.6
5th	8.00	-10.3	-14.7
20 year percentiles			
95th	8.00	12.6	14.0
50th=median	8.00	9.7	10.1
25th	8.00	8.3	8.2
5th	8.00	6.0	5.4

Comments on the above results

The long term return values show that statistical estimation error, although not entirely absent with 10,000 simulations (compare the observed 11.02% for 100% equity with the theoretical 11.00%), is small. Notice that none of these portfolios can be expected to produce a higher return over the long term than the 100% equity portfolio. In particular, those who subscribe to the "enhanced income" (equity plus short call) portfolios are going to be sadly disappointed since in the examples given they are likely to lose at least 1% pa relative to the 100% equity portfolio. The protected put portfolios achieve a significant reduction in 5th percentile risk, but at the significant cost of long term reductions in return of about 0.7% and ¼% pa respectively.

The 20 year percentiles show that convergence to the long term returns is occurring, slowly but inevitably.

The question now arises as to whether the above portfolios, in particular the 100% equity, can be improved upon, either in the sense of maximising long term return, or in the risk/return tradeoff as exemplified by the percentiles.

"Optimal" portfolios

I then used computer optimisation:

Firstly, to maximise μ under the following sets of constraints respectively:

- no short positions allowed, full asset range (this portfolio is called Max Return 1 below)
- short positions allowed only in the options, full asset range (Max Return 2)
- short positions in cash allowed, only two assets: equities and cash (Max Return 3)

Secondly, to maximise expected exponential utility (with parameter 5 to obtain a "Low Risk" portfolio, and with parameter 1 to obtain a "High Return" portfolio) .

The results for these portfolios (together with the 100% equity portfolio for comparison) are shown below:

Portfolio	**Equity**	**Max Ret. 1**	**Max Ret. 2**	**Max Ret.3**	**High Return**	**Low Risk**
	% by market value (at the start of each year)					
Equities	100.0	92.3		124.2	0.3	0.1
Bonds					0.1	
Equity Call (X=100)		6.0	18.1		14.3	2.9
Bond Call (X=100)			-8.2		-7.3	-1.4
Equity Put (X=100)			-6.3		-8.0	-1.6
Bond Put (X=100)		1.7	1.1		1.4	0.3
Cash			95.3	-24.2	99.2	99.7
Long term return (%)	11.02	11.24	11.48	11.15	11.41	9.18
Mean return -1	13.92	15.37	16.02	15.27	15.36	9.75
S.D.	20.02	24.47	26.15	24.88	24.66	4.93
One year percentiles						
95th	43.8	53.3	53.4	50.8	47.0	17.2
50th (median)	10.9	9.6	11.8	11.6	13.8	9.1
25th	-2.5	-7.9	-5.7	-5.1	-1.5	6.1
5th	-21.7	-25.4	-32.4	-30.5	-33.6	0.8
20 year percentiles						
95th	18.4	20.4	21.1	20.2	20.3	11.0
50th=median	11.0	11.1	11.5	11.1	11.5	9.1
25th	7.9	7.5	7.5	7.3	7.7	8.4
5th	3.6	2.3	1.8	1.9	2.0	7.3

Comments on the "Optimal" Portfolio Results

The optimisation process has shown that, depending on the extent to which short option positions are allowed, it **is** possible to enhance long term expected returns by up to about ½% pa, which although at first sight disappointing compared to the differentials of over 20% pa between the *mean* returns on call options and equities, would still allow a pension fund with a long time horizon to implement significant cost savings or benefit improvements. Of course, I have ignored practical matters here such as transaction costs and the probable reluctance of lay trustees to invest in such unusual portfolios. (It must not be forgotten however that any portfolio will suffer transaction costs to a degree, due to reinvestment/disinvestment and any rebalancing.)

Of course there is a "price" paid for this extra return and that is that the log volatility increases, which implies that the investor will need to wait longer in order to achieve the long term return with any desired level of confidence. However, the 20 year 50th percentile figures show that there is a more than 50% probability that the Maximum Return 2 and the "High Return" portfolios will significantly (in the sense of by more than say 0.2% pa - of course the Maximum Return 1 and 3 portfolios also outperform, but by a smaller amount) outperform a 100% equity portfolio. **The 1 year median figures also show that the "High Return" portfolio (and also the Maximum Returns portfolios 2 and 3) has more than a 50% probability of outperforming the 100% equity portfolio even over one year, which is very encouraging.**

(For the sake of completeness I should mention that in order to double check the results I also tested the portfolios with 100,000 simulations which gave almost identical results.)

The "Low Risk" portfolio offers a long term return exceeding the cash rate by almost 1.2% pa while having an exceedingly low volatility of some 5% pa. This portfolio may be found attractive as an alternative to bonds.

I conclude that the results are very encouraging and further research should be carried out, in particular to:

- determine whether extension to other asset classes and the availability of futures and currency options enables one to enhance the long term return further
- try and understand the exact source of the extra return (perhaps the detailed nature of the gearing involved).

Optimisation in the literature

The last point raised above leads me to confess that the nature of my job as a full time consultant means that my research time is highly geared (in the sense that it is borrowed, usually via short sleep positions) and therefore I have only been able to make a very cursory examination of the available literature on this subject.

Merton[2] 's "Continuous Time Finance" is a fascinating and extremely elegant volume on (amongst other topics) optimal consumption and portfolio selection under uncertainty, which provides closed form solutions in many cases, particularly in the case of HARA utility functions (of which the exponential utility function referred to above is a special case). Merton considers optimal portfolios in the continuous time case, which can be taken as more general than the discrete time case, since any policy which is available in the discrete time case (for example an annually rebalanced one) is also available in the continuous time framework.

Merton (pages 101 and following) shows how to solve the following optimal portfolio selection and consumption problem:

$$\max E \{ \int_0^T \exp(-\rho t) U[C(t)] dt + B[W(T),T] \}$$

where ρ reflects the investor's personal time preference/discount rate, U is a strictly concave utility function, T is the time horizon, B is a concave "bequest" function to represent the utility of terminal wealth W(T), and the optimisation is to find consumption functions C(t) and portfolio weights $w_i(t)$ which maximise expected utility, for m log normally distributed asset classes (with mu and sigma parameters μ_i and σ_i) and where wealth at time t is related to the portfolio weights and the consumption function as follows:

$dW(t)/dt = \mu^T \mathbf{w}$ (the vector dot product) times W(t) - C(t).

[It should be noted that the above formulation assumes that neither the utility functions U and B, nor consumption C nor the portfolio weights **w** depend explicitly on any variables except time t and wealth W(t). In practice, investors may be worried about inflation - so that U and B may depend on inflation as a random variable- and consumption may not be a free variable but prescribed (eg for a pension fund, C = net benefit outgo, which in general will be a function of t, price and wage inflation). Solutions to this more general and realistic problem will have to await a subsequent paper!]

Merton gives explicit solutions in the cases where U and B are power or exponential utility functions. However for the purposes of this paper, we are interested in maximisation of the expected log terminal wealth, ie U(C(t))=log C(t) and B(W(T),T) = log W(T). [To be precise we are also considering that no consumption is allowed - however, since it turns out that the portfolio weights are independent of the consumption decision, we can safely retain this term as applicable to the more general case.]

It turns out that the solution to the above dynamic optimisation problem is that, in order to optimise long term expected return, the following vector equations hold:

$\mathbf{w}(t)$= constant proportions = $\mathbf{h} + \mathbf{g}$ .... (I)

where $\mathbf{h}=\mathbf{V}^{-1}\mathbf{1} / (\mathbf{1}^T\mathbf{V}^{-1}\mathbf{1})$ is the minimum variance portfolio (V^{-1} is the inverse of the covariance matrix of the m assets, **1** is an m by 1-vector consisting of 1s)

and $\mathbf{g}= \mathbf{V}^{-1}\mu - (\mathbf{1}^T\mathbf{V}^{-1}\mu)\mathbf{V}^{-1}\mathbf{1}/(\mathbf{1}^T\mathbf{V}^{-1}\mathbf{1})$.

In the case where a risk free asset exists (without loss of generality the mth asset, with log mean r, leaving n=m-1 risky assets), the solution is (with the vectors now 1 by n and the covariance matrix V is n by n):

$$\mathbf{w}=\mathbf{V}^{-1}(\mu - r\mathbf{1}) \qquad \text{.......................................} \qquad \text{(II)}$$

and the proportion in the risk free asset = $1 - \mathbf{1}^T \mathbf{V}^{-1} (\mu - r\mathbf{1})$.

In either case, since the w_i are constant (and therefore do not depend either on t or on W(t)) the optimal portfolio is a continuously rebalanced one, which therefore has a lognormal distribution (see Wise[3] page 357), with parameters:

$\mathbf{w}^T\mu - \frac{1}{2}\mathbf{w}^T\mathbf{V}\mathbf{w}$, $\mathbf{w}^T\mathbf{V}\mathbf{w}$ (in the absence of a riskless asset) and

$\mathbf{w}^T(\mu - r\mathbf{1}) + r - \frac{1}{2}\mathbf{w}^T\mathbf{V}\mathbf{w}$, $\mathbf{w}^T\mathbf{V}\mathbf{w}$ (with a riskless asset).

In the practical example considered above, we have a risk free asset and two risky primary assets (the options are *derivative* assets and are equivalent in the continuous time model to hedged portfolios of the primary assets and cash and therefore it is not necessary or appropriate to include them) with the following parameters for μ : $\mu_1 = 0.13$ (for equities), $\mu_2 = 0.085$ (bonds), with covariance matrix V given by $V_{11} = 0.04$, $V_{12} = V_{21} = 0.5*0.2*0.1 = 0.01$ and $V_{22} = 0.01$.

The inverse covariance matrix is calculated to be $V_{11}^{-1} = 33.3333$, $V_{12}^{-1} = V_{21}^{-1} = -33.3333$, $V_{22}^{-1} = 133.3333$ and the optimal portfolio risky asset weights to yield the maximum long term return are calculated from equation (II) to be 1.5 allocated to equities, -1 allocated to bonds and 0.5 allocated to cash. In other words the portfolio is 200% invested, with the extra 100% financed via a short bond position. Such a portfolio has lognormal parameters 0.15 and 0.2646^2 and the log return to be expected over the long term from the portfolio is therefore 11.50% pa, a full ½% pa above the 100% equity portfolio. Computer simulation for the *annual rebalanced* version of this portfolio yields a long term log return of 11.48%, which suggests that the continuous time solution may be usefully considered in the annual rebalanced case.

It is also interesting to compare the Maximum Return (annually rebalanced) portfolio 3 that I obtained above (124% Equities, -24% cash, with a long term estimated log return of 11.15%) with the optimal continuous time long term portfolio when only cash and one risky asset, equities, are considered. In this case V^{-1} is a scalar, equal to 25, and the optimal allocation to equities is calculated to be 1.25, or 125%, with a -25% cash holding, and a long term log return of 11.125%.

The closeness of the result to the 124%, -24% portfolio obtained (numerically) in the annual rebalancing case is encouraging, and again suggests that the continuous case optimal results (which, unlike the annual rebalanced cases, can often be calculated algebraically) may probably be used to give close to optimal results in the (possibly more practical, due to the need to avoid frequent transaction costs) single period rebalancing case.

A Note on Short Selling Restrictions

The above analysis can also be carried out assuming no short sales, and the Maximum Return Portfolio 1 has shown that extra returns are available over the long term even when such restrictions are imposed.

In practice, of course, pension funds and other institutional investors are frequently subject to such restrictions, either by regulation or via the natural but self imposed desire to conform to normal practice. This paper suggests that in the case of long term investors, the rationale for such restrictions needs to be carefully reviewed, in the light of the substantial extra returns that can be expected to be foregone over the long term. It is also now possible for a pension fund to adopt any continuously rebalanced portfolio (including one with short positions) by simply purchasing an over the counter derivative product, with the financial intermediary on the other side of the bargain (who is in general not subject to the same restrictions) effectively holding the desired portfolio on the pension fund's behalf.

The principal risk that needs to be considered, is of course that the model (or parameters) under which one has calculated one's optimal portfolio turns out to be invalid! Nevertheless, a long term investor who is content with a long term expected return from a 100% equity portfolio (with say a 20% pa volatility), may well be even more content with a portfolio with a long term expected return ½% pa higher with a 25% pa volatility (but with the theoretical risk of negative assets in the event of a total collapse of world stock markets), or ¼% higher (for example with the previous portfolio supplemented by the purchase of a way out of the money tailored put option in order to remove the risk of negative assets). Of course, in the event of an 80% (or more) fall in world stockmarkets, the risk of negative assets is likely to be amongst the least of an investor's worries. Nevertheless, further research is needed to investigate the risks thoroughly: it is possible that excessive pursuit of extra long term returns by investors could actually precipitate an 80% stock market fall by driving asset prices to unsustainable levels!

Portfolio Synergy: The Effects of Rebalancing

A noteworthy point is that in general the annual returns in the approach that I have taken (of single period rebalancing) will *not be lognormally distributed,* in contrast to the continually rebalanced ones of Merton. This is because linear combinations of lognormals are not lognormal.

More importantly, there also exist what may be termed *portfolio synergy* effects arising from rebalancing. We have assumed above that the long term (log) return that may be expected from cash is 8%, whereas that from equities is 11%. If we now examine a portfolio which is invested equally in equities and in cash, then we find that the long term return from this portfolio is **not** the 9½% that our clients (and our colleagues, until we have pointed this out to them!) expect:

- in the case of annual rebalancing with an equity volatility of 20% pa, the long term log return is calculated from 10,000 simulations as 10.02%, in other words some ½% pa higher
- in the case of continuous rebalancing, the use of the formulae given above (or alternatively, see Wise, page 353 and take logs) tells us that the return is 10% pa exactly.

This feature, which at first sight seems peculiar, is directly related to the volatility of the risky asset class: if the standard deviation parameter is reduced from 0.20 to 0.15, then the extra return diminishes to some 0.28% in the continuous case. With higher volatilities, the "extra" return will exceed ½% pa.

[Note: In the continuously rebalanced case, the median return is equal to the long term return, implying that the optimal long term portfolio is also the one which maximises the median return over any time horizon. In contrast, in the case of buy and hold portfolios, the portfolio median converges to the linear combination of the asset class medians, but it can take a long time to do so. For example, the estimated median log return from 10,000 simulations of a 50/50 bond/equity portfolio with the component parts simply bought and held for a year ie with no rebalancing effect is 9.75% pa, whereas in the long run this return should converge to 9.50%.]

Such unanticipated synergy effects from portfolio rebalancing are very real factors that can be observed in practice. Indeed Wise gives historical data from portfolios of UK and international equities, UK bonds and cash over ten and twenty year periods which imply "extra" returns from monthly rebalancing varying from 0.44% pa to 1.01% pa. Of course, such returns might in practice be reduced by transaction costs, but when one considers that investment income and net cashflow investment/disinvestment requirements are usually available to meet rebalancing needs, it is possible that the extra transaction costs incurred will not significantly reduce the extra return available from rebalancing. This is a promising avenue for further research.

The synergy from rebalancing does of course raises several questions:

- how should we should set our assumptions, from the starting point of individual asset classes or from the viewpoint of mixed portfolios? For example, our default assumption may be that all major equity classes have the same expected return over the long term, of perhaps 5½ % real pa. A diversified equity portfolio may then turn out to have a long term return of 6% pa (eg with two equity assets, with volatilities 20% and 22%, and correlation 0.5, for a 50/50 portfolio).
- how can the consulting profession best help clients to understand these complex synergy effects?

Perhaps this is a further reason why the traditional "100% equity" portfolio may not provide the best answer to a very long term investor: we have seen that not only do other portfolios exist with higher expected long term returns, but such a portfolio takes very little advantage of the above synergy effect.

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References

[1] P J Lee ("Portfolio Selection In the Presence of Options", Transactions of the 3rd AFIR Congress, Rome 1993)

[2] Professor R C Merton ("Continuous Time Finance", particularly chapters 4, 5 and 6, Blackwell 1990)

[3] A J Wise ("The Investment Return from a Constantly Rebalanced Asset Mix", Transactions of the 3rd AFIR Congress, Rome 1993)