Semi-Markov Modelization for the Financial Management of Pension Funds

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Summary

The aim of this paper is to construct a stochastic model that is suitable for the management of pension funds both from static and dynamic point of view. To face the problem, we use particular non-homogeneous semi-Markov processes. The approach regards only the outlays and the incomes that are connected with the evolution dynamics of the funds members. This phenomenon is really complex and in the paper many of the aspects are not yet developed, but, in our opinion, the main aspects of the problem are faced and solved giving so the possibility to face all the other problems in future works.

La modélisation semi-Markov
pour la gestion financière des fonds de retraite

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Résumé

Le but de la présente étude est d'élaborer un modèle stochastique utilisable pour la gestion des fonds de retraite tant d'un point de vue statique que d'un point de vue dynamique. Pour aborder le problème, nous utilisons des processus particuliers non homogènes semi-Markov. La formule ne tient compte que des dépenses et des recettes qui sont liées aux caractéristiques dynamiques évolutives des membres du fonds. Ce phénomène est extrêmement complexe et un grand nombre de ces aspects n'ont pas encore été développés dans le cadre de la présente étude, mais, à notre avis, les principaux éléments du problème ont été résolus, ce qui permettra d'aborder tous les autres problèmes dans des travaux ultérieurs.

1. Introduction.

The problem of dynamic management of pension funds can be faced by means of simulation models using stochastic processes, or by the elaboration of theoretical stochastic models.

From the simulation point of view there are many contributions of Italian authors developing different ways to this approach (see for example Tomassetti 1973, 1979, 1991, Crisma 1981, Bacinello 1988, Volpe, Manca 1988). Furthemore this aspect was developed not only from theoretical point of view but also from the practical one.

The models based on stochastic processes were developed in other papers see for example (Balcer, Sahin 1983), (Janssen, Volpe 1984), (De Dominicis, Manca 1985). But only recently there was a tentative theoretical paper (De Dominicis, Manca, Granata 1991) that gave a stochastic model able to consider many of the aspects of this complex problem. This approach uses a generalization of Discrete Time Non Homogeneous Semi-Markov Reward Processes (DTNHSMRP) but it was never applied.

This paper starts from the results obtained in (De Dominicis, Manca, Granata 1991), improves in substantial aspects the previous paper, tries to explain how the model solves the problem, presents an algorithm that is suitable for the applications. This algorithm is a generalization of the one presented on (Carravetta, De Dominicis, Manca 1989) for Non Homogeneous Semi-Markov Process (NHSMP) (Janssen, De Dominicis 1983).

2. The dynamic management of private pension funds.

Before describing the stochastic structure we would like to introduce the problem we want to face.

To follow the dynamic evolution of the funds it is necessary to calculate the value of incomes and expenditures that the funds has to face. Furthermore, it is necessary to compute the present value of all the financial flows that are involved.

We assume that we have to follow the evolution of the population that are members of the considered pension funds. The model improves the one proposed by De Dominicis, Manca, Granata (1991); of course the funds will be followed on a bounded time interval.

The model envisages m states. The first m-5 states consider all possible worker states, i.e. the different job positions within the firm.

Other four states regards the different kinds of pensions that the funds can have and the m^{th} state is the absorbing state, i.e. the state of leaving the funds in which there are no more charges. More precisely, state (m-4) represents a funds member that gets a disability pension; state (m-3) represents a funds member that has a pension and retired from the work position because of reaching the elderly age; state (m-2) represents people that retired from the work position before reaching the retiring age but with the right to get a partial amount of pension; state(m-1) represents the relatives of a funds member that get a pension because of the death of this member.

So, we have the following states:

1 : worker state of degree 1

⋮

m-5 : worker state of degree m-5

m-4 : disability pensioner state

m-3 : elderly retired pensioner state

m-2 : early retired pensioner state

m-1 : survivor pensioner state

m : absorbing state.

The state dynamic is given in figure 1.

Of course it is also possible to subdivide or to aggregate some of these states if necessary.

It is evident that the financial flows will be positive for the funds in the case of working member and negative if the member is pensioned.

Furthermore the flow value depends from his seniority, i.e. member duration in the funds, and from the degree in which he is (or was), i.e. state 1,..,m-5.

It is evident that the dynamics in the state space is specified by the rules of the pension funds defined by the concerned institution. But in general we can say that a worker changes state in the case he is promoted to an other degree, or when he leaves the work. Let us mention that if a member leaves the work, then, he will necessary go to the state m-4 if he becomes disabled, m-3 if he retires on a superannuation pension, m-2 if he has the right to get a pension but has not yet the pension age, m-1 if he dies and leaves a family. At last, he will go to the absorbing state in the case he abandons the funds without any kind of pension.

From states m-4, m-3, m-2, in the case of state change, it is possible to reach the survivor state m-1, if the pensioner dies and leaves a family or the absorbing state m in the case of death without family. At last, the survivors can only go to the absorbing state. In this case there are two reasons for going to this state: the death and the loose of the right to get a pension.

3. DTNHSMRP and semi-Markov processes.

Let $((X_n, T_n), n \in N)$ be the two-dimensional random sequences defined on a probability space (Ω, F, P) with values in $J \times N$ where J is the "state space", i.e.

$$J = \{1, \ldots, m\}. \tag{3.1}$$

The r.v. X_n $(n \geq 0)$ represents the pension funds state evolution of a member and T_n his age (in years) at transition n. Of course, we have:

$$0 \leq T_0 \leq T_1 \leq \ldots \leq T_n \leq \ldots \tag{3.2}.$$

where T_0 represents the initial "age" when the member comes in the funds.

Our basic assumption (see Janssen & Volpe 1984) is to suppose that the $X - T$ process is a discrete non homogeneous Markov additive process (Janssen & De Dominicis 1983) of discrete matrix kernel **b**. This means that:

$$b_{ij(st)} = P[X_{n+1} = j, T_{n+1} = t / X_n = i, T_n = s]. \tag{3.3}$$

This implies that if:

$$Q_{ij(st)} = P[X_{n+1} = j, T_{n+1} \leq t / X_n = i, T_n = s] \tag{3.4}$$

then:

$$b_{ij(st)} = \begin{cases} 0 & s \geq t \\ Q_{ij(st)} - Q_{ij(st-1)} & s < t \end{cases} \tag{3.5}$$

$$Q_{ij(st)} = \sum_{h=s}^{t} b_{ij(sh)}. \tag{3.6}$$

In a similar way if:

$$p_{ij(s)} = P[X_{n+1} = j / X_n = i, T_n = s] \tag{3.7}$$

then:

$$p_{ij(s)} = Q_{ij(s\infty)} \tag{3.8}$$

and finally if:

$$S_{i(st)} = P[T_{n+1} \leq t / X_n = i, T_n = s] \tag{3.9}$$

then:

$$S_{i(st)} = \sum_{j=1}^{m} Q_{ij(st)}. \quad (3.10)$$

Let us remark that non-homogeneity is only retained for the age and not for the number of the state changes.

As it is known we have:

$$Q_{ij(st)} = p_{ij(s)} G_{ij(st)}. \quad (3.11)$$

where the $p_{ij(s)}$ for $i, j \in \{1, \ldots, m\}, s \in N$ are the transition probabilities of the imbedded non-homogeneous Markov chain and $G_{ij(st)}$ are the increasing functions of the sojourn time in the state i.

Of course $Q_{ij(ss)} = 0$ and $S_{i(s\infty)} = 1, i, j \in \{1, \ldots, m\}$. Furthermore we always have: $0 \le s \le t$.

We will now introduce a non decreasing stochastic sequence:

$$H = (H_n, n \ge 0) \quad (3.12)$$

where H_n represents the seniority of the member at the time of n^{th} transition (see De Dominicis, Manca, Granata 1991). Of course, the H-process is entirely defined by the initial random variable H_0, $H_0 \ge 0$. By definition we have that:

$$H_{n+1} = H_n + T_{n+1} - T_n \quad (3.13)$$

if $X_n = i$ at time $T_n = s$ and $X_{n+1} = j$ at time $T_{n+1} = t$.

Once we have introduced the $(H_n)_{n \in N}$ process, we can define the following probabilities:

$${}^{r}G_{ij(st)} = P[T_{n+1} \le t, H_{n+1} \le r + t - s / X_n = i, X_{n+1} = j, T_n = s, H_n = r] \quad (3.14)$$

$${}^{r}b_{ij(st)} = P[X_{n+1} = j, T_{n+1} = t, H_{n+1} = r + t - s / X_n = i, T_n = s, H_n = r] \quad (3.15)$$

$${}^{r}Q_{ij(st)} = P[X_{n+1} = j, T_{n+1} \le t, H_{n+1} \le r + t - s / X_n = i, T_n = s, H_n = r] \quad (3.16)$$

$${}^{r}Q_{ij(st)} = \sum_{h=s}^{t} {}^{r}b_{ij(sh)}. \quad (3.17)$$

Of course the following relations hold:

$$ {}^{r}Q_{ij(st)} = {}^{r}p_{ij(s)}\ {}^{r}G_{ij(st)} \tag{3.18} $$

where ${}^{r}p_{ij(s)}$ and ${}^{r}G_{ij(st)}$ have the same meaning that in (3.11) with the difference that now r represents, in the pension case, the seniority at age s. In general, it may represent the cumulated sojourn time in the system up to s.

Also in this case we have:

$$ {}^{r}S_{i(st)} = \sum_{h=s}^{t}\sum_{j=1}^{m} {}^{r}b_{ij(sh)} = \sum_{j=1}^{m} {}^{r}Q_{ij(st)} \tag{3.19} $$

and:

$$ {}^{r}Q_{ij(ss)} = 0,\ {}^{r}S_{i(s\infty)} = 1,\ i,j \in \{1,\dots,m\};\ 0 \le r \le s \le t, \tag{3.20} $$

where ${}^{r}S_{i(st)}$ is the probability that the discrete time non-homogeneous Markov renewal process is in some state of the set J at time t, having left the state i reached at age s with a seniority equal to r.

It is now possible to associate a semi-Markov process $\{{}^{r}Z_t : r,t \in N\}$ to the discrete time non-homogeneous Markov renewal process $\{(X_n, T_n, H_n) : n \in N\}$, by putting ${}^{r}Z_t = i$ for $T_n \le t < T_{n+1}$, given that $X_n = i$ and $H_n = r$. We can define the following probabilities:

$$ {}^{r}P_{ij(st)} = P\left[{}^{r+t-s}Z_t = j \mid {}^{r}Z_s = i \right] \tag{3.21} $$

that are the probabilities that the semi-Markov process is in the state j at time t having left the state i that it reached at time s with a cumulated sojourn time at the last transition before is equal to r.

Using the first transition method we obtain:

$$ {}^{r}P_{ij(st)} = \delta_{ij}(1 - {}^{r}S_{i(st)}) + \sum_{\theta=s}^{t}\sum_{h=1}^{m} {}^{r}b_{ih(s\theta)}\ {}^{r+\theta-s}P_{hj(\theta t)} \tag{3.22} $$

Where:

$$ {}^{r}b_{ij(st)} = \begin{cases} 0 & s \ge t \\ {}^{r}Q_{ij(st)} - {}^{r}Q_{ij(st-1)} & s < t \end{cases} \tag{3.23} $$

4. Reward structure.

To apply this model to the pension funds problem we need to consider a reward structure connected with the semi-Markov process.

Let us define now:

${}^{r}V_{i(st)}$: the discounted expected value at a fixed epoch of the reward in the time interval $[s,t[$, given that there is an entrance in state i at the age s and that the seniority (cumulated sojourn time), up the time s, is r.

${}^{r}r_i$: the amount paid per time period in the state i with a "seniority" r.

z : the fixed rate of interest.

$a_{\overline{h}|z}$: the present value of an unitary h-period annuity i.e.:

$$a_{\overline{h}|z} = \sum_{k=1}^{h} v^k; \qquad v = (1+z)^{-1} \tag{4.1}$$

The evolution equations are the following ones:

$$\begin{aligned} {}^{r}V_{i(st)} = (1 - {}^{r}S_{i(st)})\ {}^{r}r_i a_{\overline{t-s}|z} + \sum_{\theta=s}^{t}\sum_{h=1}^{m} {}^{r}b_{ih(s\theta)}\ {}^{r}r_i a_{\overline{\theta-s}|z} + \\ + \sum_{\theta=s}^{t}\sum_{h=1}^{m} {}^{r}b_{ih(s\theta)}\ {}^{r+\theta-s}V_{h(\theta t)}(1+z)^{s-\theta} \end{aligned} \tag{4.2}$$

$$i = 1 \ldots m; \quad 0 \le r \le s \le t; \quad r,s,t \in N$$

Now we would like to give the financial meaning of the equations (4.2) in the case of pension funds.

As we told in the previous paragraph the ${}^{r}V_{i(st)}$ are the discounted expected values of the rewards that have to be paid from s to t when a member is arrived in the state i at age s with a seniority r. These formulas are compound from the following three parts:

$$(1 - {}^{r}S_{i(st)})\ {}^{r}r_i a_{\overline{t-s}|z} \tag{4.3}$$

$$\sum_{\theta=s}^{t}\sum_{h=1}^{m} {}^{r}b_{ih(s\theta)}\ {}^{r}r_i a_{\overline{\theta-s}|z} \tag{4.4}$$

$$\sum_{\theta=s}^{t}\sum_{h=1}^{m} {}^{r}b_{ih(s\theta)}\ {}^{r+\theta-s}V_{h(\theta t)}(1+z)^{s-\theta} \tag{4.5}$$

In relation (4.3), the term $1 - {}^{r}S_{i(st)}$ represents the probability to remain in the state i once that a member arrived there at age s with a seniority r. So this member has to pay (or get) at each period of time the reward ${}^{r}r_i$ and (4.3) represents the expected present value of this.

Relation (4.4) gives the expected present value of the rewards that a member arrived in i at age s with a seniority r paid in this state before leaving it.

At last, relation (4.5) represents the expected present value of the rewards that a member that arrived in the state i at age s and changed his situation at age θ has to pay in the other states. These values are paid at time $\theta - s$, so we need to discount them.

5. The statistical distribution of the funds population at time t.

Solving (4.2) we can get the expected present value for members in the situation described previously. In a pension funds, there are many people and each of them has a given situation at time 0 in the funds. But to study the funds development, we need to compute the mean number of members in different situations. Let us define:

${}^{r}N_{(s)}$: the number of members at time 0 with seniority r and age s,

${}^{r}N_{i(s)}$: the number of members of age s with seniority r that at time 0 are in the state i,

${}^{r+s-t}N_{(st)}$: the number of members initially of age s and seniority r in the system, still present at age $t > s$ at time $t - s$,

${}^{r+s-t}N_{j(st)}$: the number of members in the state j at age t and at time $t - s$ that at age s had a seniority r,

${}^{r+s-t}\overline{N}_{j(st)}$: the mean number of the ${}^{r+s-t}N_{j(st)}$,

${}^{r+s-t}N'_{i(st)}$: the mean number of members of age s and seniority r that were in the state i at time 0 and are in some state at age t at time $t - s$.

Let ${}^{r+t-s}F_{(st)}$ the probabilities for a member in state i with seniority r at time t to go to the absorbing state m at time $t-s$. These values can be obtained in the following way:

$$ {}^{r+t-s}F_{(st)} = \frac{\sum_{i=1}^{m-1} {}^{r}P_{im(st)}\, {}^{r}N_{i(s)}}{{}^{r}N_{(s)}} \tag{5.1} $$

where:

$$ {}^{r}N_{(s)} = \sum_{i=1}^{m-1} {}^{r}N_{i(s)} \tag{5.2} $$

The distribution of the ${}^{r+t-s}N_{(st)}$ of individuals initially of age s and seniority r in the system at age $t > s$ and time $t - s$ is given by:

$$P\left[\,{}^{r+t-s}N_{(st)} = g\right] = \binom{{}^{r}N_{(s)}}{g}\left(1 - {}^{r+t-s}F_{(st)}\right)^{g}\left({}^{r+t-s}F_{(st)}\right)^{{}^{r}N_{(s)}-g} \tag{5.3}$$

$$g = 0, \ldots, {}^{r}N_{(s)}.$$

Using the multinomial distribution for

$$\left({}^{r+t-s}N_{1(st)} = k_1, \ldots, {}^{r+t-s}N_{m(st)} = k_m\right),$$

the vector of the numbers of individuals present in each state of the system at age t and at time $t-s$, we have:

$$P\left[\,{}^{r+t-s}N_{1(st)} = k_1, \ldots, {}^{r+t-s}N_{m(st)} = k_m\right] =$$

$$= \frac{\left(\sum_{i=1}^{m} {}^{r}N_{i(s)}\right)!}{k_1! \ldots k_m!}\left(\left({}^{r+t-s}q_{1(st)}\right)^{k_1} \ldots \left({}^{r+t-s}q_{m(st)}\right)^{k_m}\right) \tag{5.4}$$

where:

$${}^{r+t-s}q_{j(st)} = \frac{\sum_{i=1}^{m} {}^{r}P_{ij(st)}\,{}^{r}N_{i(s)}}{{}^{r}N_{(s)}} \qquad j = 1 \ldots m \tag{5.5}$$

Finally we have (Fisz 1963 p. 164):

$${}^{r}\overline{N}_{j(st)} = E\left({}^{r}N_{j(st)}\right) = {}^{r}N_{(s)}\;{}^{r+t-s}q_{j(st)}. \tag{5.6}$$

Relations (5.4) and (5.5) derive from formulas proved in (Carravetta, De Dominicis, Manca 1981).

Furthermore we also have the following relations:

$${}^{r}N'_{i(st)} = \sum_{j=1}^{m} {}^{r}b_{ij(st)}\;{}^{r}N_{i(s)} \tag{5.7}$$

If we know the initial distribution at time 0 of the people in the states, we can get recursively the mean number of the members in the states in the following years.

6. The solution of evolution equations in the transient case.

The equations (3.22) and (4.2) in matrix form can be written in the following ways:

$$ {}^{r}P_{(st)} = \left(I - {}^{r}D_{(st)}\right) + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)} \; {}^{r+\theta-s}P_{(\theta t)} \tag{6.1}$$

$$ {}^{r}V_{(st)} - \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}(1+z)^{s-\theta} \; {}^{r+\theta-s}V_{(\theta t)} = $$

$$ = \left(I - {}^{r}D_{(st)}\right) {}^{r}Ra_{\overline{t-s}|\, z} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)} \; {}^{r}Ra_{\overline{\theta-s}|\, z} \tag{6.2}$$

where ${}^{r}P_{(st)}$, ${}^{r}D_{(st)}$, ${}^{r}B_{(st)}$ are square $m \times m$ matrices, ${}^{r}V_{(st)}$ an m vector and:

$$ 0 \le r \le s \le t, \quad r, s, t \in N \tag{6.3}$$

$$ {}^{r}d_{ij(st)} = \begin{cases} 0 & i \neq j \\ {}^{r}S_{i(st)} & i = j \end{cases} \tag{6.4}$$

$$ {}^{r}R = \begin{bmatrix} {}^{r}r_1 \\ {}^{r}r_2 \\ {}^{r}r_3 \\ \vdots \\ {}^{r}r_m \end{bmatrix} \tag{6.5}$$

Let us remark that in relations (6.1) and (6.2) the sum starts from $s+1$ because we have that the matrix ${}^{r}B_{(ss)} = [0] \;\; \forall s$.

If we want to solve the two evolution equations in the transient case, and if we consider the same time horizon for both r.v. H_n and T_n, i.e we consider k periods, we have:

$$ 0 \le r \le s \le t \le k \tag{6.6}$$

instead of relation (6.3) and then the formulas (6.1) and (6.2) hold.

In both equations (6.1) and (6.2), because of the particular structure of the systems, no matrix inversion is necessary to get the solutions. In fact, in the case of (6.1), we have:

$$ {}^{k}P_{(kk)} = I, \tag{6.7}$$

$$ {}^{k-1}P_{(kk)} = I, \tag{6.8} $$

$$ {}^{k-1}P_{(k-1k-1)} = I, \tag{6.9} $$

$$ {}^{k-1}P_{(k-1k)} = {}^{k-1}B_{(k-1k)}\ {}^{k}P_{(kk)} + (I - {}^{k-1}D_{(k-1k)}). \tag{6.10} $$

The last relation gives the value of ${}^{k-1}P_{(k-1k)}$.
Then as:

$$ {}^{k-2}P_{(kk)} = I, \tag{6.11} $$

$$ {}^{k-2}P_{(k-1k-1)} = I, \tag{6.12} $$

$$ {}^{k-2}P_{(k-1k)} = {}^{k-2}B_{(k-1k)}\ {}^{k-1}P_{(kk)} + (I - {}^{k-2}D_{(k-1k)}), \tag{6.13} $$

$$ {}^{k-2}P_{(k-2k-2)} = I, \tag{6.14} $$

we find the following "backward" values:

$$ {}^{k-2}P_{(k-2k-1)} = {}^{k-2}B_{(k-2k-1)}\ {}^{k-1}P_{(k-1k-1)} + (I - {}^{k-2}D_{(k-2k-1)}), \tag{6.15} $$

$$ \begin{aligned} {}^{k-2}P_{(k-2k)} = {}& {}^{k-2}B_{(k-2k-1)}\ {}^{k-1}P_{(k-1k)} + \\ & {}^{k-2}B_{(k-2k)}\ {}^{k}P_{(kk)}(I - {}^{k-2}D_{(k-2k)}). \end{aligned} \tag{6.16} $$

Going on with this special case of "backward substitution" we finally obtain:

$$ {}^{0}P_{(00)} = I \tag{6.17} $$

$$ {}^{0}P_{(01)} = {}^{0}B_{(01)}\ {}^{0}P_{(11)} + (I - {}^{0}D_{(01)}) \tag{6.18} $$

$$ {}^{0}P_{(02)} = {}^{0}B_{(01)}\ {}^{1}P_{(12)} + {}^{0}B_{(02)}\ {}^{2}P_{(22)}(I - {}^{0}D_{(02)}) \tag{6.19} $$

$$ {}^{0}P_{(0k)} = \sum_{\theta=1}^{k} {}^{0}B_{(0\theta)}\ {}^{\theta}P_{(\theta k)} + (I - {}^{0}D_{(0k)}). \tag{6.20} $$

The equations (6.2) have essentially the same structure, the only differences are that there are some multiplicative present value factors and that instead of finding matrices, as in the previous case, solving (6.2) we get vectors.

The formulas we wrote before are useful to solve the equations in the general case but, if we want to face the problem of pension application, there are some differences. In fact the age of the funds member must have at least eighty years of range and the seniority has at most a fifty years range. For this reason the previous formulas can't work in this application. Considering that seniority couldn't be greater than a fixed number of years, let us define:

k_a : maximum considered age spread, this means that if 20 is the first working age and 100 is the maximum age of life we have $k_a = 80$,

k_s:maximum achievable seniority,

and:

$$ {}^{r}U_{(st)} = \left(I - {}^{r}D_{(st)}\right)\ {}^{r}Ra_{t-s\ ,\ z} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{r}Ra_{\theta-s\ ,\ z}. \tag{6.21} $$

Solving the system (6.2), we get the following equations:

$$ {}^{k_s}V_{(k_a k_a)} = {}^{k_s}R, \tag{6.22} $$

$$ {}^{k_s}V_{(k_{a-1} k_{a-1})} = {}^{k_s}R, \tag{6.23} $$

$$ {}^{k_s}V_{(k_{a-1} k_a)} = {}^{k_s}B_{(k_{a-1} k_{a-1})}(1+z)^{-1}\ {}^{k_s}V_{(k_a k_a)} + {}^{k_s}U_{(k_{a-1} k_a)}, \tag{6.24} $$

and so on. In general, equations (6.1) and (6.2) change respectively in the following way:

$$ {}^{r}P_{(st)} = \begin{cases} \left(I - {}^{r}D_{(st)}\right) + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}P_{(\theta t)} & t \le k_s \\ \left(I - {}^{k_s}D_{(st)}\right) + \sum_{\theta=s+1}^{k_s} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}P_{(\theta t)} + \\ \quad + \sum_{\theta=k_s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{k_s}P_{(\theta t)} & t > k_s \end{cases} \tag{6.25} $$

$$
{}^{r}V_{(st)} = \begin{cases} {}^{r}U_{(st)} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}V_{(\theta t)}(1+z)^{s-\theta} & t \leq k_s \\ {}^{k_s}U_{(st)} + \sum_{\theta=s+1}^{k_s} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}V_{(\theta t)}(1+z)^{s-\theta} + \\ \quad + \sum_{\theta=k_s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{k_s}V_{(\theta t)}(1+z)^{s-\theta} & t > k_s \end{cases} \qquad (6.26)
$$

Also in this case there is no matrix inversion and the system can be solved by means of a particular "backward substitution".

7. Final results and a short algorithm.

The presented model allows to manage a pension funds in which the considered "rewards" changes because of the state and the seniority in the state. In fact, if we consider as active reward all the amount of the salary paid to the workers, let us define:

E : estate of the funds,

A : total salary present value,

H : total outlays present value,

M : equilibrium rate (i.e. the percentage of the salary that it is necessary for the equilibrium of the funds).

To get the value of H and A we need to solve twice the (6.26), the first time putting equal to 0 all the outlay rewards and equal to the total amount of the salary the positive (for the funds) rewards. The second time it is necessary to put equal to 0 all the positive rewards, giving the right values to the outlays. If we define:

${}^{r}r_i^{'}$ = salary total amount of a member in the state i with seniority r, $i = 1, \ldots, m-5$, then formulas of the previous paragraph become:

$$
{}^{r}R^{'} = \begin{bmatrix} {}^{r}r_1^{'} \\ \vdots \\ {}^{r}r_{m-5}^{'} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7.1)
$$

$$ {}^{r}R'' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ {}^{r}r_{m-4} \\ {}^{r}r_{m-3} \\ {}^{r}r_{m-2} \\ {}^{r}r_{m-1} \\ {}^{r}r_{m} \end{bmatrix} \tag{7.2} $$

$$ {}^{r}U'_{(st)} = \left(I - {}^{r}D_{(st)}\right) {}^{r}R' a_{t-s,\, z} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\, {}^{r}R' a_{\theta-s,\, z} \tag{7.3} $$

$$ {}^{r}U''_{(st)} = \left(I - {}^{r}D_{(st)}\right) {}^{r}R'' a_{t-s,\, z} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\, {}^{r}R'' a_{\theta-s,\, z} \tag{7.4} $$

$$ {}^{r}V'_{(st)} = \begin{cases} {}^{r}U'_{(st)} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}V'_{(\theta t)}(1+z)^{s-\theta} & t \le k_s \\ {}^{k_s}U'_{(st)} + \sum_{\theta=s+1}^{k_s} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}V'_{(\theta t)}(1+z)^{s-\theta} + \\ \quad + \sum_{\theta=k_s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{k_s}V'_{(\theta t)}(1+z)^{s-\theta} & t > k_s \end{cases} \tag{7.5} $$

$$ {}^{r}V''_{(st)} = \begin{cases} {}^{r}U''_{(st)} + \sum_{\theta=s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}V''_{(\theta t)}(1+z)^{s-\theta} & t \le k_s \\ {}^{k_s}U''_{(st)} + \sum_{\theta=s+1}^{k_s} {}^{r}B_{(s\theta)}\ {}^{r+\theta-s}V''_{(\theta t)}(1+z)^{s-\theta} + \\ \quad + \sum_{\theta=k_s+1}^{t} {}^{r}B_{(s\theta)}\ {}^{k_s}V''_{(\theta t)}(1+z)^{s-\theta} & t > k_s \end{cases} \tag{7.6} $$

Using these new formulas we have:

$$ A = \sum_{t=0}^{k_a} \sum_{s=0}^{t} \sum_{r=0}^{min(s,k_s)} \sum_{i=1}^{m} {}^{r}V'_{i(st)}\ {}^{r}N'_{i(st)}, \tag{7.7} $$

$$ H = \sum_{t=0}^{k_a} \sum_{s=0}^{t} \sum_{r=0}^{min(s,k_s)} \sum_{i=1}^{m} {}^{r}V''_{i(st)}\ {}^{r}N'_{i(st)}, \tag{7.8} $$

and finally:

$$ M = \frac{H - E}{A}. \tag{7.9} $$

By means of (7.7), (7.8) and (7.9) we get the static study of the funds, in the sense of knowledge of total outlay present values (technical reserve) and the equilibrium rate.

Once we get M it is also possible to follow the dynamical development of the pension funds. Now let:

$$ {}^r r_j = {}^r r_j' M \quad j = 1, \ldots, m-5, \tag{7.10}$$

then, we have to solve the (6.26).

If I_k and O_k represent respectively the total input and total output for year k, we have:

$$ I_k = \sum_{s=0}^{k_s - k} \sum_{r=0}^{s} \sum_{j=1}^{m-5} {}^r\overline{N}_{j(s,s+k)} \, {}^r r_j \tag{7.11}$$

$$ O_k = \sum_{s=0}^{k_a - k} \sum_{r=0}^{min(s,k_s)} \sum_{j=m-4}^{m} {}^r\overline{N}_{j(s,s+k)} \, {}^r r_j \tag{7.12}$$

By means of (7.11) and (7.12) we get respectively the annual entrances and outlays of the funds and we can follow its dynamical development.

Finally, we would like to show the main steps of the algorithm to be used for the model development.

SMPM (Semi-Markov Pension Model) algorithm

1) INPUT the ${}^rN_{i(s)}$, ${}^rp_{ij(s)}$, ${}^rG_{ij(st)}$, ${}^rR'$, ${}^rR''$

2) SOLVE the equations (6.25) and get the ${}^rp_{ij(st)}$

3) FIND the ${}^r\overline{N}_{j(s,t)}$ and ${}^rN'_{j(s,t)}$

4) SOLVE (7.5) and get ${}^rV'_{i(st)}$

5) SOLVE (7.6) and get ${}^rV''_{i(st)}$

6) FIND M

7) CONSTRUCT the equilibrium entrance rewards

8) SOLVE (6.26) and get ${}^rV_{i(st)}$

9) FIND (7.11) and (7.12) and get the dynamical funds development.

10) STOP

It is to remark that in this model some relevant aspects involved in the study of pension funds are not taken in account, for example:

i) the estimation of data,

ii) the selection of decision variables,

iii) the possible use of the ALM techniques (J.Janssen 1993) for managing pension funds from the point of view of estate funds,

iv) the changes of the rewards because of time running. In fact the inflation and the salary agreements change the active and passive funds rewards,

v) the possibility to consider new entrance members in the funds.

We hope to face all these aspects in the near future, but we believe that this paper could be considered as an important step in the way to create a stochastic model that can manage the complex phenomenon of pension funds without using only simulation models based for example on the Montecarlo method.

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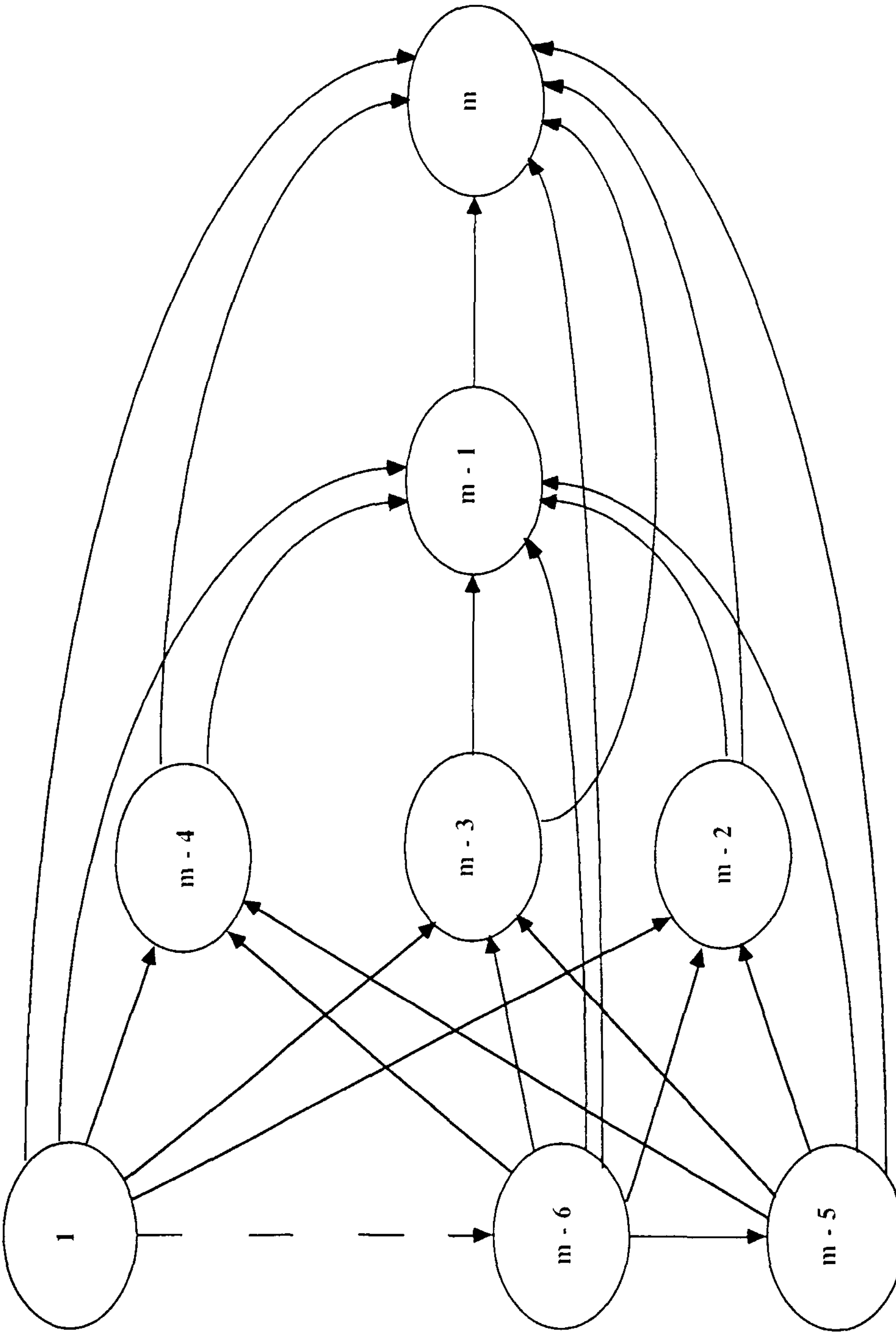

Figure 1