Modelling Equity Returns Using a Simple ARCH Model

Vincent Hua
Master of Economics Student
C\- School of Economic and Financial Studies
Macquarie University
Sydney NSW 2109
Australia

Telephone: 61 2 805 8572
Fax: 61 2 805 9481

Summary

Studies by Wilkie (1984, 1986, 1987) and Carter (1991), amongst others, have proposed models of equity returns for actuarial work. In particular, Carter proposes a random walk model for Australian equity returns.

The aim of this paper is to examine the use of a simple ARCH model for equity returns and to compare the performance of such a model with a model that is commonly used by actuaries - the random walk model of returns.

The model is fitted using Australian data for equity returns on the Australian All Ordinaries Index for the period January 1970 to December 1992. The relative performance of the model is assessed using simulation studies.

Acknowledgment

I would like to express my sincere thanks to Mr. Michael Sherris who encouraged me to summit this paper for the 4th AFIR International Colloquium and assisted me with various aspects of this paper from providing ideas to reading this final copy.

Modélisation des rendements des titres de participation à l'aide d'un modèle ARCH simple

Vincent Hua
Candidat à la Maîtrise d'économie
C\-Collège d'études économiques et financières
Université de Macquarie
Sydney NSW 2109
Australie

Téléphone : 61 2 805 8572
Fax : 61 2 805 9481

Résumé

Les études menées par Wilkie (1984, 1986, 1987) et Carter (1991), entre autres, ont proposé des modèles de rendements des titres de participation dans un but de travail actuariel. Carter propose, en particulier, un modèle cumulatif stochastique pour les rendements des titres de participation australiens.

Le but de cet exposé est d'examiner l'emploi d'un modèle ARCH simple pour les rendements des titres de participation et de comparer la performance d'un tel modèle avec celle d'un modèle communément utilisé par les actuaires, à savoir le modèle cumulatif stochastique des rendements.

Le modèle est élaboré à l'aide de données australiennes sur les rendements des titres de participation de l'Indice actuariel australien pour la période de janvier 1970 à décembre 1992. La performance relative du modèle est évaluée à l'aide d'études de simulation.

Remerciements

Je souhaite sincèrement remercier M. Michael Sherris qui m'a encouragé à soumettre le présent exposé pour le 4ème Réunion scientifique internationale AFIR et m'a aidé dans divers aspects de ce travail, qu'il s'agisse de fournir des idées ou d'en lire la version finale.

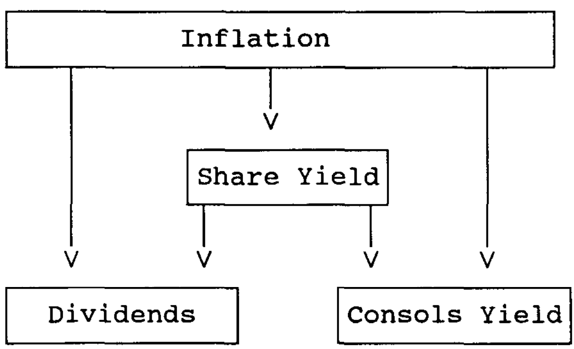
INTRODUCTION

Wilkie's Investment Model

Interest in stochastic investment models developed in the actuarial profession when a report was published by the UK Maturity Guarantees Working Party (1980). Since then, Professor A.D. Wilkie, a leading member of the group, has developed a comprehensive investment model which has been widely used in the U.K..

The complete Wilkie investment model includes inflation, share yields, dividends and consols yield. This paper is concerned only with modelling equity yields.

The emphasis of Wilkie's model is on long term investment returns. Therefore, it ignored all the variables that might be responsible for short term movements. Inflation is the only variable considered to be the driving force behind long term investment returns in Wilkie's share model. Graphically, this is represented as :



The arrows in the above diagram are strictly downwards. In other words, inflation is assumed to have significant influence on the financial markets, but the financial markets cannot exert influence on the level of inflation.

Wilkie's Share Yields Model

From the above diagram, it is obvious that inflation has to be modelled before the share yield can be modelled. Inflation, as measured by the retail prices index (RPI), is modelled by a first order auto regressive (AR(1)) process. Its equation is given by :

$$\nabla \ln Q(t) = QMU + QA(\nabla \ln Q(t-1) - QMU) + QSD.QZ(t)$$

where,

Q(t)	=	RPI at end year t
QZ(t)	=	N(0,1)
QMU	=	0.05
QA	=	0.60
QSD	=	0.05

Share yields are modelled as a function of the current inflation rate and the history of their past trends. It is given by the following equation :

$$\ln Y(t) = YW.\nabla \ln Q(t) + YN(t)$$

$$YN(t) = \ln(YMU) + YA(YN(t-1) - \ln(YMU)) + YSD.YZ(t)$$

where,

Y(t)	=	share yield at end year t
YZ(t)	=	N(0,1)
YMU	=	0.04
YA	=	0.60
YW	=	1.35
YSD	=	0.175

Background to Carter's Share Price Yields Model

Fitting of the Wilkie inflation and share yields model to Australian data was carried out by Carter (1991). The Wilkie share yields model was found to be unsuitable for Australia data for a number of reasons.

Carter's share price yields model is for the All Ordinaries Index. The parameters of the model were estimated from quarterly data of the All Ordinaries Index from the first quarter of 1968 to the second quarter of 1990 with an adjusted figure for the return for the October 1987 quarter.

Carter examined the relationship between inflation and share price yields. He then went on to look at the relationship between price yields and interest rates. His analysis of the data rejected any statistically significant relationship between the variables. He concluded that : "given the present or past price yields, a more accurate prediction of the next value, beyond pure randomness, is not possible" (Carter 1991 p.352-353).

Carter's Share Price Yields Model

Carter proposed a random walk model for Australian share price yields. The model is given by the following equation :

$$P_t = P_{t-1}\exp\{p_t\} \qquad \text{--- (1)}$$

where

p_t = force of share price yields over quarter t.

$\quad = c + {}^p\epsilon_t \qquad$ --- (2)

P_t = Share Price Index (All Ordinaries), SPI, at end of quarter t, time t.

c = 0.02

${}^p\epsilon_t = N(0, {}^ps^2)$

${}^ps$ = 0.1

Carter's model assumes a long term quarterly force of share price yields of 2 percent with a stochastic error term with mean 0 and standard deviation of 0.1. Further, he implicitly assumed that the SPI is log-normally distributed with conditional mean at any time t equal to $P_{t-1}\exp(c+\frac{1}{2}{}^ps^2)$ and conditional variance equal to $P_{t-1}^2\exp(2c+{}^ps^2)*(\exp({}^ps^2) - 1)$.

In order to understand the exact form of the model, substitute equation (2) into (1). The combined equation is :

$$P_t = P_{t-1}\exp\{c + {}^p\epsilon_t\} \qquad \text{--- (3)}$$

Substitute equation (3) for P_{t-1}, P_{t-2}, . . . P_0 leads to :

$$P_t = P_0\exp\{tc\}\exp(\Sigma{}^p\epsilon_t\} \qquad \text{--- (4)}$$

Assuming the ϵ's are independent, $\Sigma^p\epsilon_t$ is normally distributed with mean zero and variance $t^{*p}s^2$.

The variance of P_t according to (3) increases with time. To illustrate the point, one thousand simulations up to twenty-five years ahead were performed and they were summarised in TABLE I. Graph I to Graph V illustrate the simulated distribution of the level of the All Ordinaries Index using Carter's model.

TABLE I : Mean & Standard Deviation of The All Ordinaries Index. 1000 Simulations (Carter's Model)

	Period Of Simulation				
	5 Years	10 Years	15 Years	20 Year	25 Years
Mean	225.36	374.35	598.23	990.11	1616.78
Standard Deviation	106.36	268.46	530.09	1082.35	2037.44

Index at time zero = 100

The Carter model overlooks the fact that although share price yields seem to follow a random walk, their variance does not increase linearly with time. Therefore, a simple random walk model with a linear stochastic term should not be used to model share price yields. It is shown in the next part of this paper that there is strong evidence from the data that non-linearity exists.

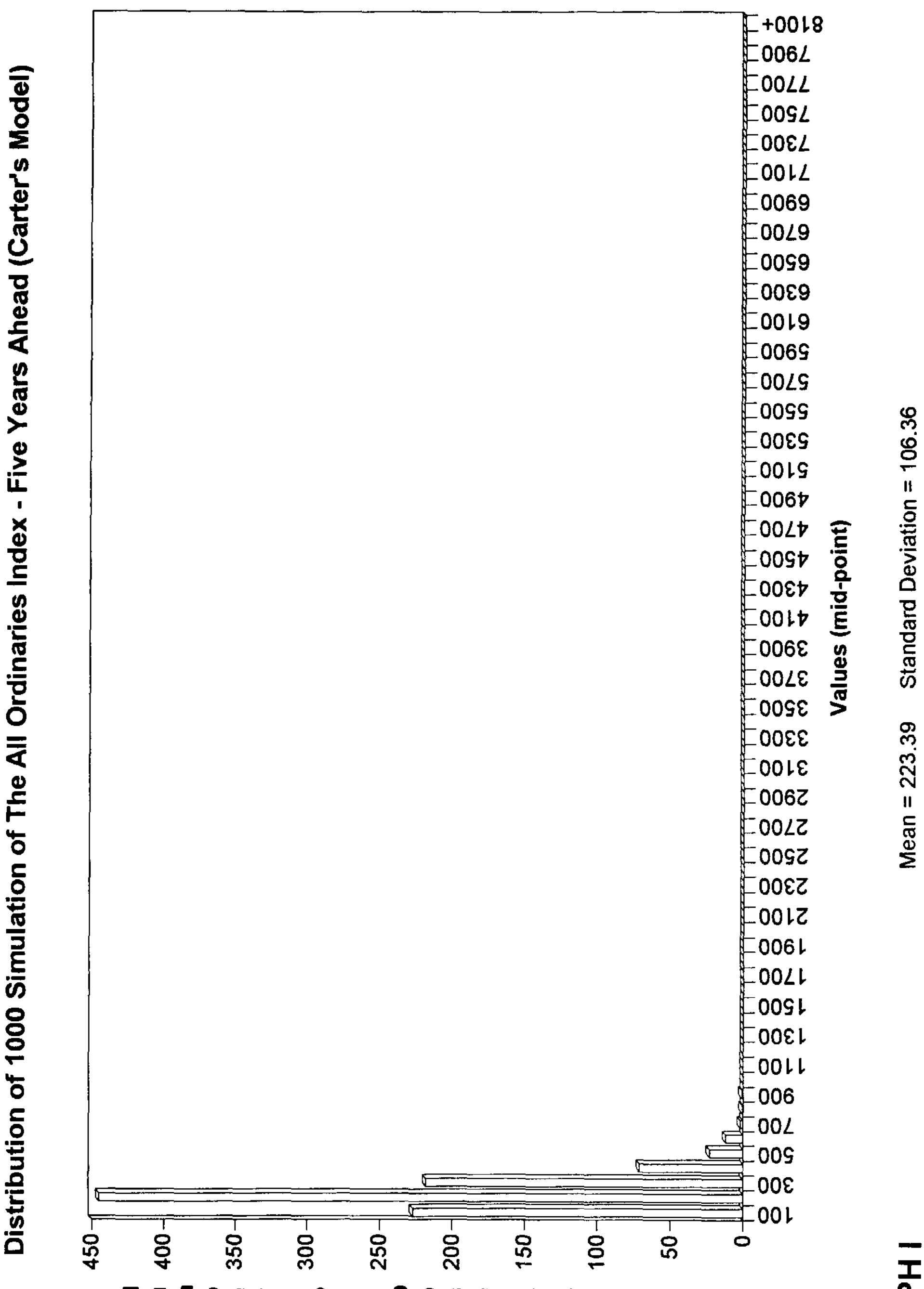
MODEL CRITERIA AND DATA ANALYSIS

Criteria Used To Choose A Model

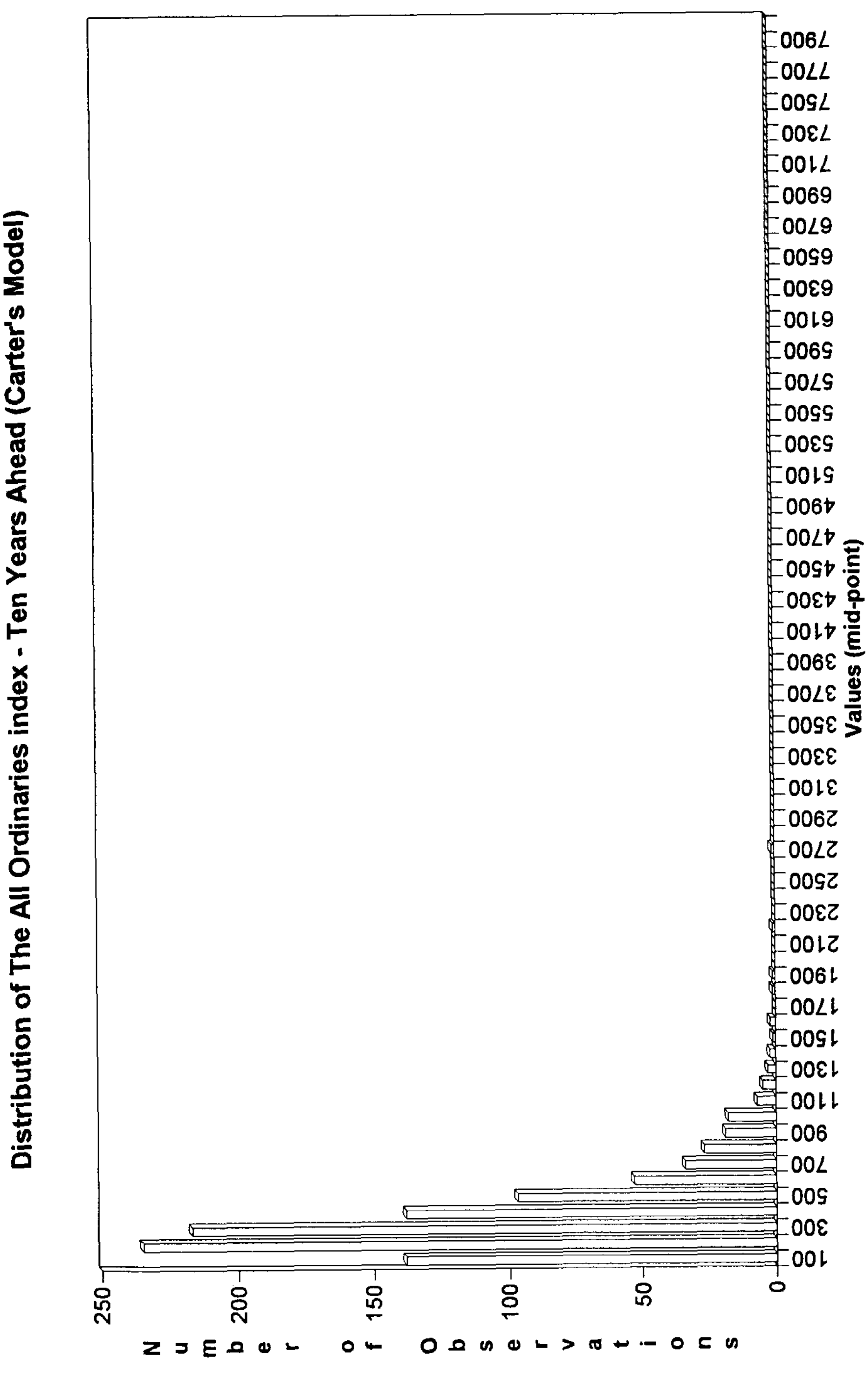
Any model will only be an approximation to the rules which convert relevant information and numerous beliefs and actions into market returns. Some approximations will be more accurate and helpful than others. An ideal model should satisfy five criteria.

1. Models should be consistent with past prices.

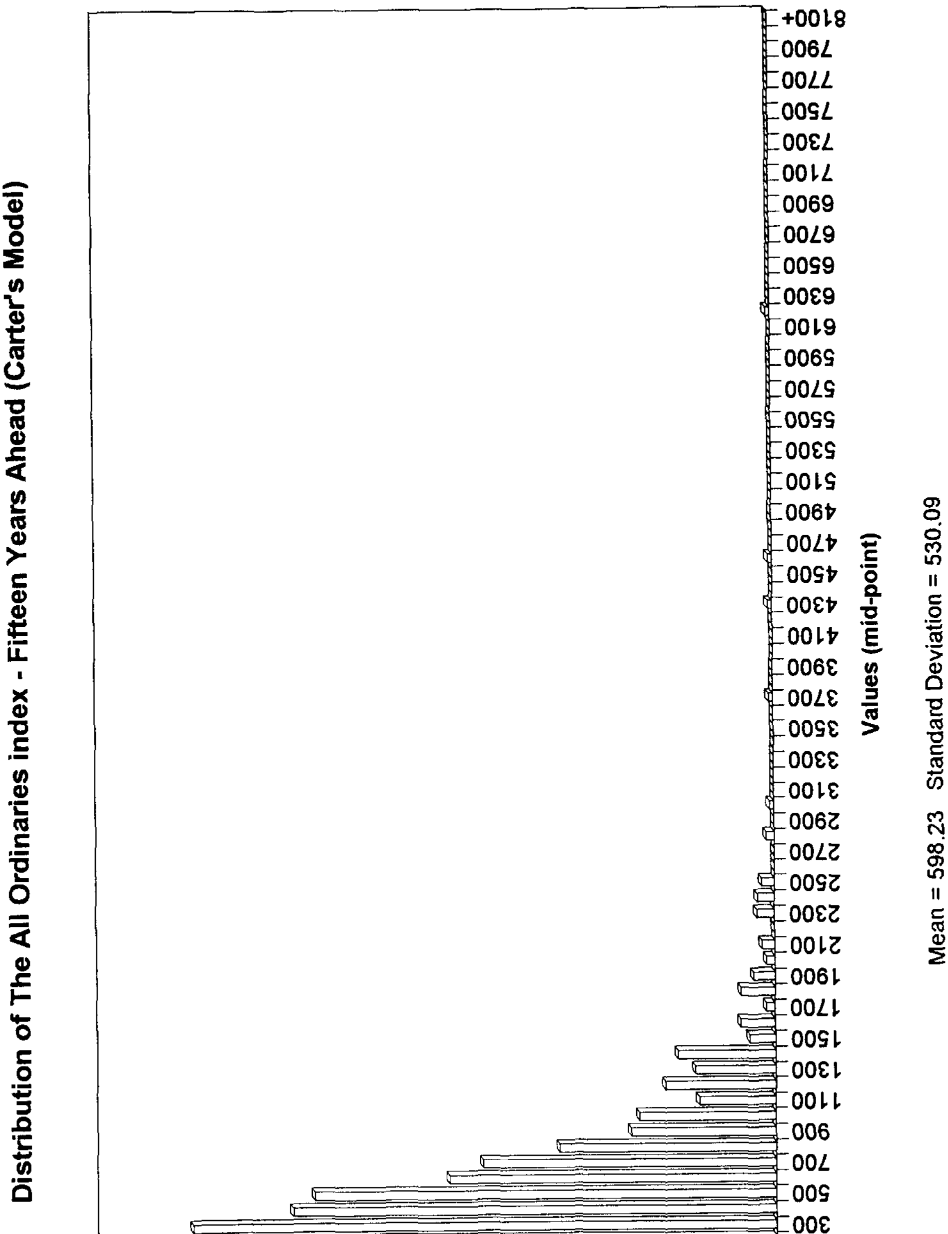
2. Hypotheses implied by a model ought to be amenable to rigorous testing.


Distribution of 1000 Simulation of The All Ordinaries Index - Five Years Ahead (Carter's Model)
Number of Observations
0
50
100
150
200
250
300
350
400
450
100
300
500
700
900
1100
1300
1500
1700
1900
2100
2300
2500
2700
2900
3100
3300
3500
3700
3900
4100
4300
4500
4700
4900
5100
5300
5500
5700
5900
6100
6300
6500
6700
6900
7100
7300
7500
7700
7900
8100+
Values (mid-point)
Mean = 223.39 Standard Deviation = 106.36

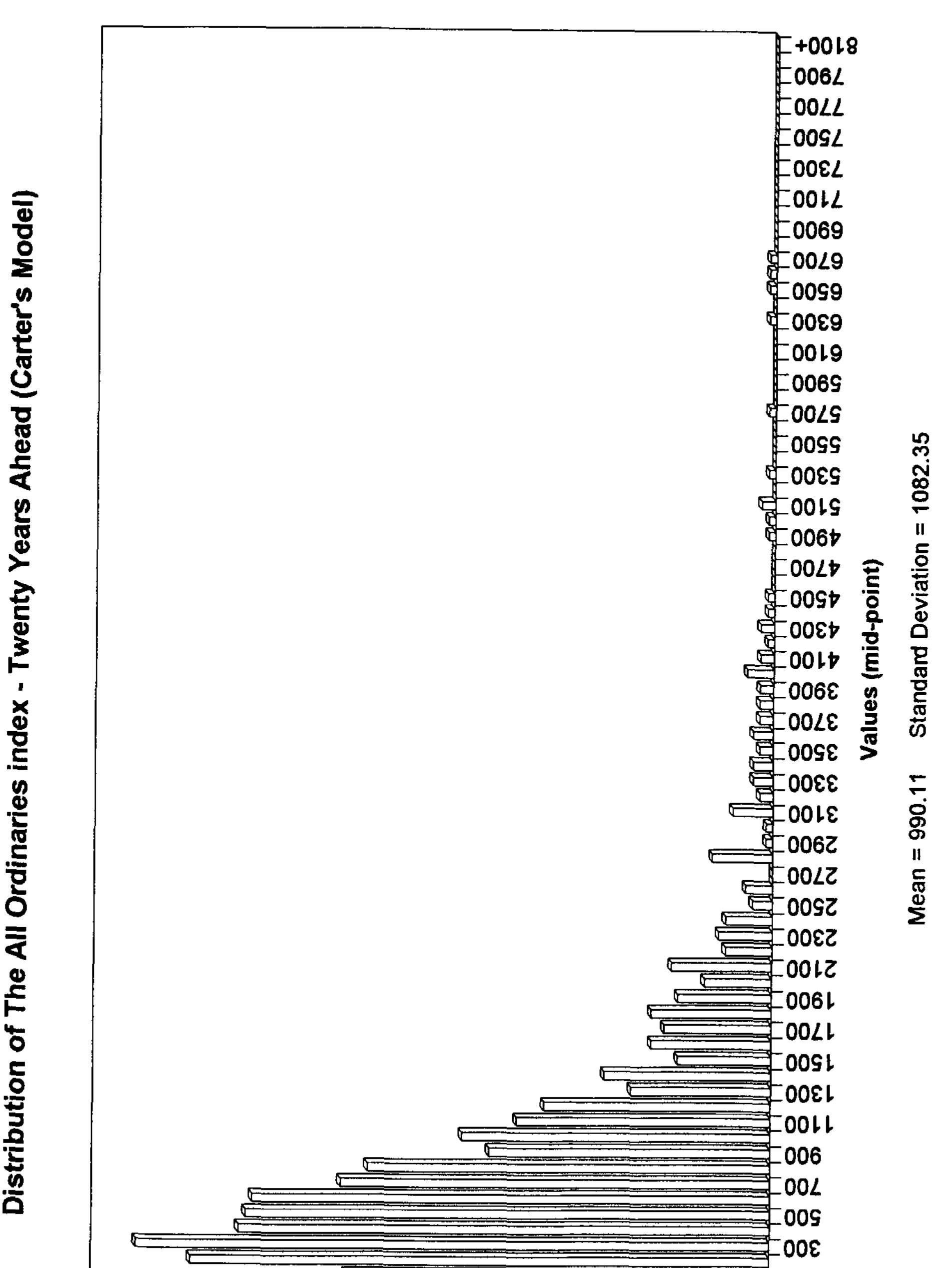
GRAPH I


Distribution of The All Ordinaries index - Ten Years Ahead (Carter's Model)
Number of Observations
250
200
150
100
50
0
100
300
500
700
900
1100
1300
1500
1700
1900
2100
2300
2500
2700
2900
3100
3300
3500
3700
3900
4100
4300
4500
4700
4900
5100
5300
5500
5700
5900
6100
6300
6500
6700
6900
7100
7300
7500
7700
7900
Values (mid-point)

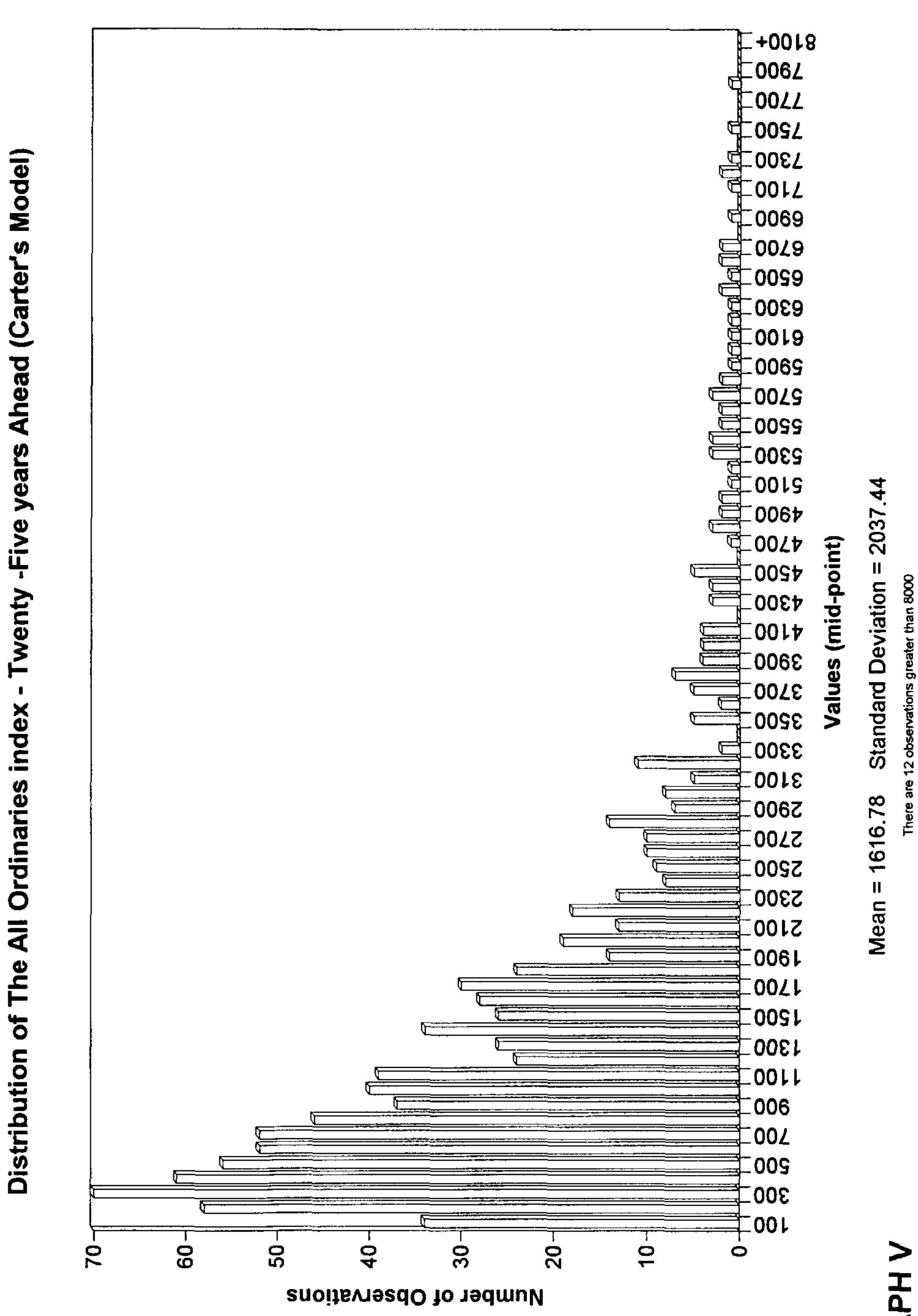
GRAPH II


Distribution of The All Ordinaries index - Fifteen Years Ahead (Carter's Model)
Number of Observations
160
140
120
100
80
60
40
20
0
100
300
500
700
900
1100
1300
1500
1700
1900
2100
2300
2500
2700
2900
3100
3300
3500
3700
3900
4100
4300
4500
4700
4900
5100
5300
5500
5700
5900
6100
6300
6500
6700
6900
7100
7300
7500
7700
7900
8100+
Values (mid-point)
Mean = 598.23 Standard Deviation = 530.09

GRAPH III


Distribution of The All Ordinaries index - Twenty Years Ahead (Carter's Model)
Number of Observations
0
10
20
30
40
50
60
70
80
90
100
100
300
500
700
900
1100
1300
1500
1700
1900
2100
2300
2500
2700
2900
3100
3300
3500
3700
3900
4100
4300
4500
4700
4900
5100
5300
5500
5700
5900
6100
6300
6500
6700
6900
7100
7300
7500
7700
7900
8100+
Values (mid-point)
Mean = 990.11 Standard Deviation = 1082.35

GRAPH IV


Distribution of The All Ordinaries index - Twenty -Five years Ahead (Carter's Model)
Number of Observations
70
60
50
40
30
20
10
0
100
300
500
700
900
1100
1300
1500
1700
1900
2100
2300
2500
2700
2900
3100
3300
3500
3700
3900
4100
4300
4500
4700
4900
5100
5300
5500
5700
5900
6100
6300
6500
6700
6900
7100
7300
7500
7700
7900
8100+
Values (mid-point)
Mean = 1616.78 Standard Deviation = 2037.44
There are 12 observations greater than 8000

GRAPH V

3. Models should be as simple as possible. Few parameters are preferable to many, the so-called principle of parsimony.

4. A model should provide forecasts of future returns and prices which are statistically optimal assuming the model is correct. It is even better if probability distributions for future prices can be calculated.

5. Finally, it is obviously beneficial if a model can be used to aid rational decision making.

Modelling Share Returns

Direct statistical analysis of share prices is difficult because consecutive prices are highly correlated and the variances of prices increase with time since inflation increases the expected values of most prices as time progresses. In other words, prices are not stationary.

Data Analysis of The Australian All Ordinaries Share Price Index Returns

The means, standard deviations, skewness, kurtosis and the number of outliers that are greater than two and three standard deviations are summarised in TABLE II below for the Australian All Ordinaries Share Price Index returns.

TABLE II : Summary Statistics for Monthly Returns of All Ordinaries Index

	Jan 80 -Dec 92		Jan 70 -Dec 92	
	RAW DATA	MODIFIED OCT 87	RAW DATA	MODIFIED OCT 87
MEAN	1.29	1.56	0.82	0.97
STD DEVIATION	6.41	5.38	5.89	5.29
SKEWNESS	-1.86	0.24	-1.33	0.01
KURTOSIS	15.28	3.54	12.55	3.57
OUTLIERS > 2 STD	17	16	17	16
OUTLIERS > 3 STD	3	2	3	2

There is a major problem with the inclusion of outliers in the data used to fit any model. (See Carter(1991)). The resultant model has too low a mean, too large a variance and is not representative of future outcomes. There are many ways this problem can be handled. In this study the monthly return containing the October 1987 Crash was made equal to the estimated mean. This action can be justified by the fact that there is only one major crash in the set of data which means that it cannot be reliably modelled. Secondly, the technique used to model crashes has to be different from the model for equity returns since the event is very unusual.

The average monthly return of the All Ordinaries Index for the period from January 1980 to December 1992 is 1.29% and for the period from January 1970 to December 1992 is 0.82%. This reflects the higher nominal returns during the 80's. Adjusting the October 1987 return by changing the return for that particular period to the average instead of using the actual data leads to marginally higher returns for both periods. The standard deviation of the monthly returns for the two periods are 6.41 and 5.89 respectively. The standard deviations are marginally lower by setting the October 1987 return to the mean.

From Table II, it is apparent that if we use the raw data the distribution of returns is not symmetric. However, by changing the October 1987 to the mean the distribution becomes much more symmetric (A perfectly symmetric distribution has skewness statistic of zero). Both time periods of data display these phenomena.

Parameters for the model in this paper were fitted using the data from January 1970 to December 1992. However, in order to examine the sensitivity of the estimated parameters to different data samples, three sets of data are used to estimate three different sets of parameters for different periods - January 1970 to December 1980, January 1980 to December 1992 and January 1970 to December 1992.

AutoCorrelation Function

The autocorrelation coefficients for returns, absolute returns and returns squared, for lags up to 12 months, are given in Table III. From Table III, it is apparent that the return for the current period is highly correlated with that of the last period. The first lag coefficient from TABLE III has a standard deviation of $n^{1/2} * \rho_{1,x}$ equal to 3.35, suggesting that the returns series are not realisations of a strict white noise process.

TABLE III : AUTOCORRELATION COEFFICIENTS

Lag	Returns	ABS. Returns	Returns^2
1	.2023	-.0090	.0119
2	.0218	.1207	.1912
3	.0607	.0875	.0917
4	.0765	.0149	.0195
5	-.0048	.0126	.0181
6	.0059	.0813	.0673
7	.0696	.0305	.0122
8	.1465	.0259	.0940
9	.0821	-.0122	.0001
10	.0170	-.0293	.0072
11	.0557	-.0733	-.0512
12	.0387	-.0424	-.0324

In general, nominal returns of financial assets are not stationary. Graph VI illustrates that although the mean returns of the All Ordinaries Index from January 1970 to December 1992 can be approximated by a constant value, the variance cannot. For instance, from 1970 to 1977 the variance of the returns was rather high but the variance was relatively low from 1978 to 1980. There is sufficient evidence to suggest that the variance of returns changes through time .

Although the unconditional distribution of return seems to be non-stationary, it is quite possible that the conditional returns can be modelled by a stationary process. In other words, we can model the standard deviation and the returns separately and use the value estimated for the standard deviation as an exogenous variable for the returns model.

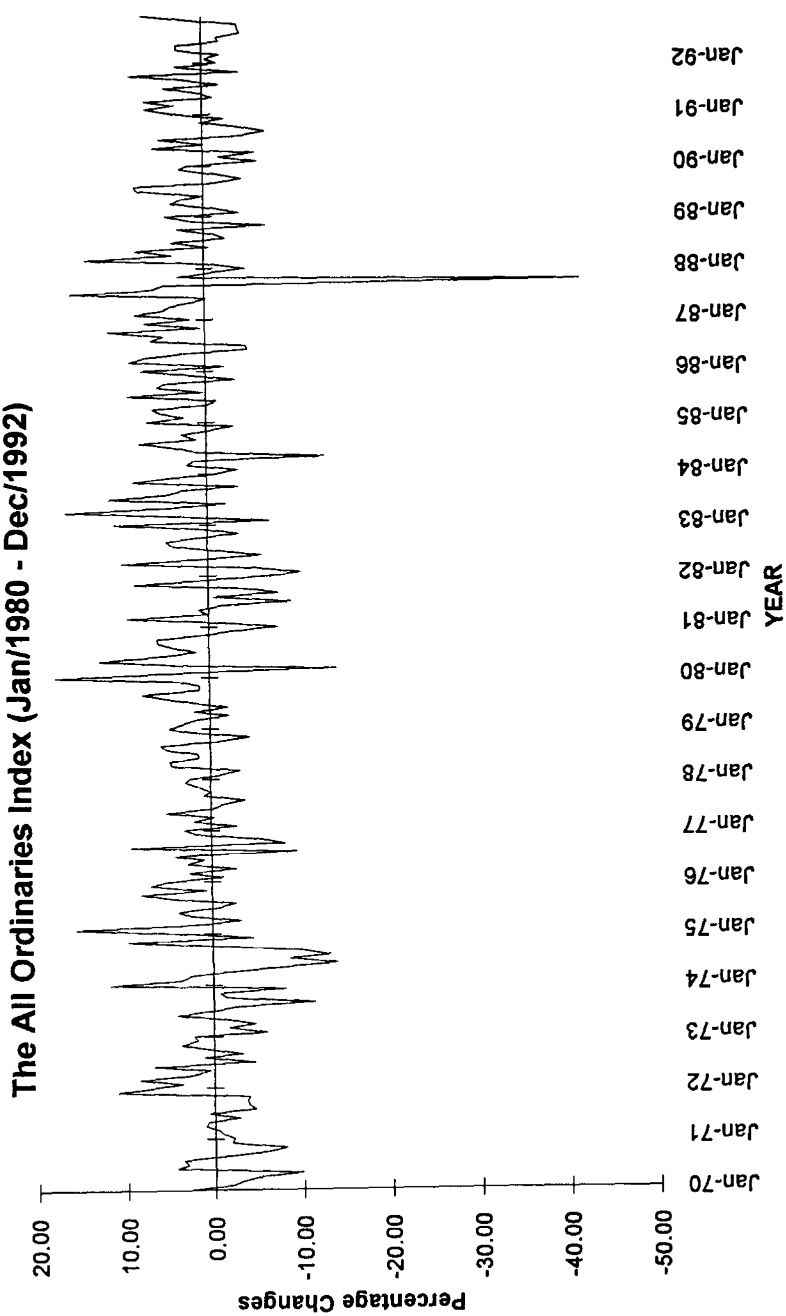
TIME VARYING PARAMETER MODELS

From the discussion above, it is clear that any investment return model should allow for the fact that variance and possibly the mean of returns changes over time. Further, returns tend to reverse to a long run equilibrium mean level as time passes.

The following model allows for such variation :

$$R_t = \mu_t + a(R_{t-1} - \mu_t) + \sigma_t \epsilon_t \qquad \text{-----} \qquad (4)$$

where R_t is the return over period t.


The All Ordinaries Index (Jan/1980 - Dec/1992)
Percentage Changes
20.00
10.00
0.00
-10.00
-20.00
-30.00
-40.00
-50.00
Jan-70
Jan-71
Jan-72
Jan-73
Jan-74
Jan-75
Jan-76
Jan-77
Jan-78
Jan-79
Jan-80
Jan-81
Jan-82
Jan-83
Jan-84
Jan-85
Jan-86
Jan-87
Jan-88
Jan-89
Jan-90
Jan-91
Jan-92
YEAR

Graph VI

$\epsilon_t \sim N(0,1)$
σ_t is the time varying standard deviation at time t.
μ_t is the time varying mean at time t.
a is a mean reversion parameter.

It should be clear that this model cannot be fitted using simple statistical techniques. A simple example will make this clear. Assume both μ_t and σ_t can be modelled as a simple function of their mean and standard deviation only. That is:

$$\mu_t = a_1\mu_\mu + \sigma_\mu Z_t \qquad \text{-----} \qquad (5)$$

where $Z_t \sim N(0,1)$
μ_μ is the mean of μ_t. It is time independent.
σ_μ is the standard deviation of μ_t. It is also time independent.
a_1 is a constant.

$$\sigma_t = a_2\mu_\sigma + \sigma_\sigma Z_t \qquad \text{-----} \qquad (6)$$

where $Z_t \sim N(0,1)$
μ_σ is the mean of σ_t. It is time independent.
σ_σ is the standard deviation of σ_t. It is also time independent.
a_1 is a constant.

We need to estimate seven parameters even with this model. The difficulty does not lie solely with the fact that there are too many parameters but also that these parameters have to be determined simultaneously since the function for the mean is conditional upon that of the standard deviation.

A Modified Model

Empirical data indicates that the variability of μ_t is much less than that of σ_t. I therefore simplify the model by assuming that μ_t is independent of time. This simplifies equation (4) above to :

$$R_t = \mu + a(R_{t-1} - \mu) + \sigma_t\epsilon_t \qquad \text{-----} \qquad (7)$$

where μ is now constant.

Empirical evidence from both Australia and overseas shows that the value of a (the mean reversion parameter) is very close to 1.00 (Carter(1991)). The value of a is assumed to be 1.0 for consistency with Carter. The implication of this assumption is that the return is now given by a random walk model with time varying random stochastic term. It is obvious that the exact form of the model cannot be specified at this stage since it depends on the process that generates σ_t. Nevertheless, the general form of the share yields is given by the following equation :

$$R_t = R_{t-1} + \sigma_t \epsilon_t \qquad \text{-----} \quad (8)$$

Model For σ_t

Constructive models for σ_t should have only a few parameters whilst permitting the mean and variance to be unconstrained and the feasible autocorrelations to cover a range of possible values consistent with the sample autocorrelation of the returns.

The next question is what distribution can we assume for σ_t? It cannot be normal since σ_t cannot have positive probability of a negative value. The lognormal family of distributions is the most convenient for obtaining straightforward mathematical results[1]. Therefore it is assumed that the unconditional log of σ_t is normally distributed with mean α and standard deviation β. That is

$$\log(\sigma_t) \sim N(\alpha, \beta^2) \text{ and } \beta > 0 \qquad \text{-----} \quad (9)$$

The unconditional mean and variance of σ_t are given by :

$$E[\delta_t] = e^{\alpha + \frac{1}{2}\beta^2} \qquad \text{---------} \quad (10)$$

$$Var[\delta_t] = e^{2\alpha+\beta^2}(e^{\beta^2} - 1) \qquad \text{--------} \quad (11)$$

σ_t is assumed to follow a first order autoregressive model. Its mathematical form is given as :

$$\log(\sigma_t) = \alpha + a_1(\log(\sigma_{t-1} - \alpha)) + Z_t \qquad \text{----} \ (12)$$

[1]It has also been found to give satisfactory results. See Clark(1973) and Tauchen and Pitts (1983)

where Z_t is stochastic random variable with mean zero and variance equal to $\beta^2(1-a^2_1)$ and it is assumed that Z_t and ϵ_t are uncorrelated. The variance formula is obtained by finding the variance of equation (12). Since the variance of $\log(\sigma_t)$ is equal to β^2, the formula for the variance of Z_t can easily be obtained.

α is the parameter of the log-normal distribution.

a_1 is a parameter which measures the speed of the change of the conditional variance. If a_1 equals one then the conditional variance follows a random walk.

The justification for choosing the above model is found in the autocorrelation coefficients presented in TABLE II which shows that they are mostly positive for the first few lags and this must also apply to σ_t. The simplest possible model having positive autocorrelation at several lags is an AR(1) process.

It can also be justified by observation of the data series itself. From Graph VI, it appears that the variance of returns follows a pattern. A month with high volatility usually follows a high volatility month.

The Lognormal, Non-linear, Autoregressive Share Price Yields Model

By assuming σ_t is given by equation (11), the share price yields model is basically a lognormal, first order autoregressive model. This type of model is generally known as a auto-regressive conditional heteroscedastic (ARCH) model. The model has three parameters, α, β, and a_1.

Numerical values of these parameters can be estimated from past data. Methods are outlined in Appendix I. Numerical values of these parameters are summarised in the next section. However, it should be noticed that it is not possible to find the conditional variance given all past observations, ie $\text{var}(X_t | x_{t-1}, x_{t-2}, \ldots)$, because the component series $\{\sigma_t\}$ and $\{\epsilon_t\}$ are not observable. If the past σ_t were known, then an explicit formula exists for the conditional variance of R_t.

TABLE IV summarises fitted values for all the parameters in the model. The parameters were estimated from three sets of data. The first set is monthly returns from January 1970 to December 1992. The second set is monthly returns from January 1970 to December 1980 and the third set is from January 1981 to December 1992.

TABLE IV : Estimates for Simple ARCH Model

Variable	Estimate		
	Jan 70-Dec 92	Jan 70-Dec 80	Jan 81-Dec 92
μ	0.00823	0.00335	0.012949
α	-3.01	-3.016	-2.996
β	0.258	0.320	0.087
a_1	0.79	0.69	0.50

μ is the mean return.

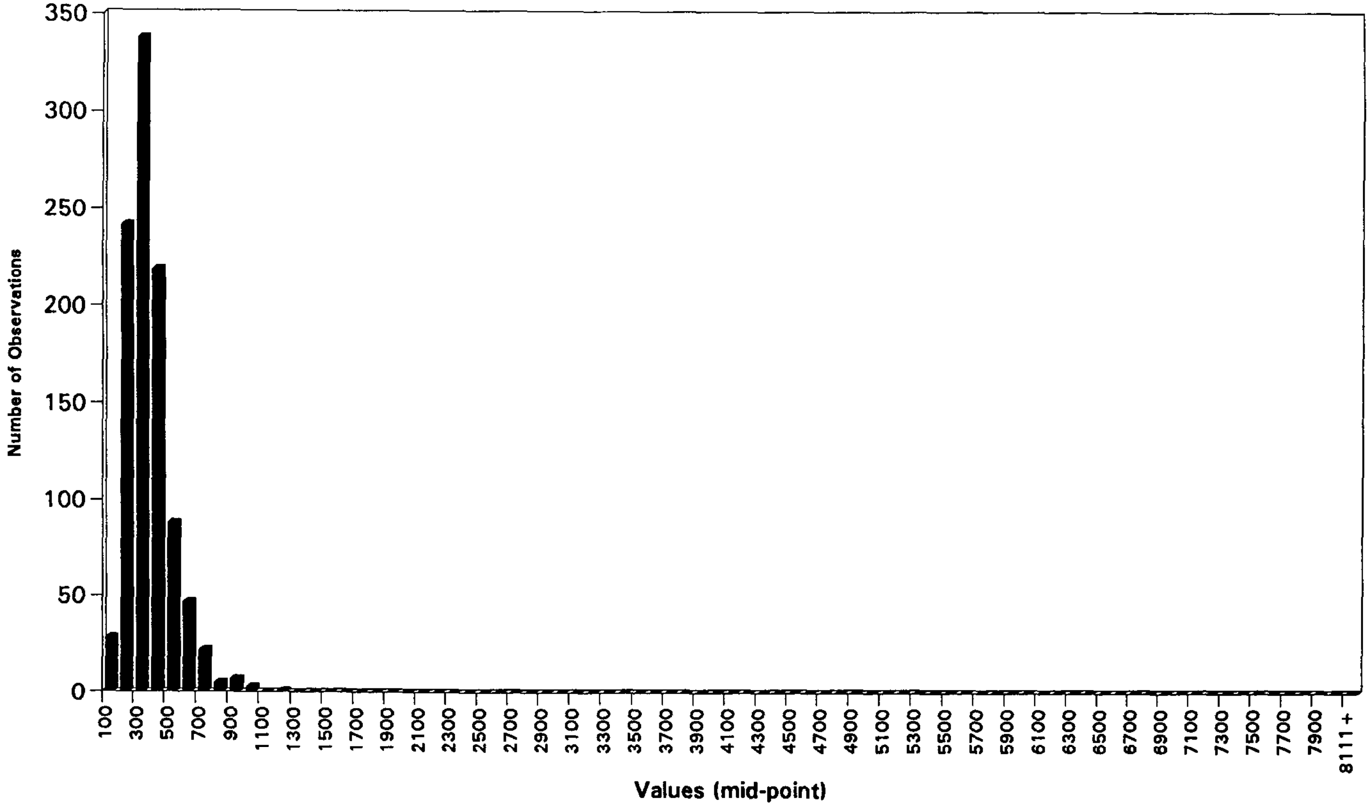
A number of comments can be made from TABLE IV. Firstly, the average monthly return is 0.3335% per month and 1.2949% per month for the 70s and the 80s respectively. Secondly, the estimate of α is not very sensitive to the time period of the sample. The long run unconditional mean of the standard deviation of returns appears to be constant.

However, both β and a_1 appear sensitive to the time period of the sample. It is clear that a better model would allow for a time varying mean in equity return. This has been left for further research.

Overview of The Model

The model, with the estimates from the full period, was used to run one thousand simulations for a period of twenty-five years. The first return is set to the average ie $X_0 = \mu$. The results for periods consistent with TABLE I are summarised in TABLE V below and in graphs VII to XI. TABLE V should be compared with TABLE I which is based on Carter's Model. From the two tables, it is clear that the standard deviation of returns from the proposed model increases at a much slower rate as time lapses while the standard deviation of returns of Carter's model increases linearly.

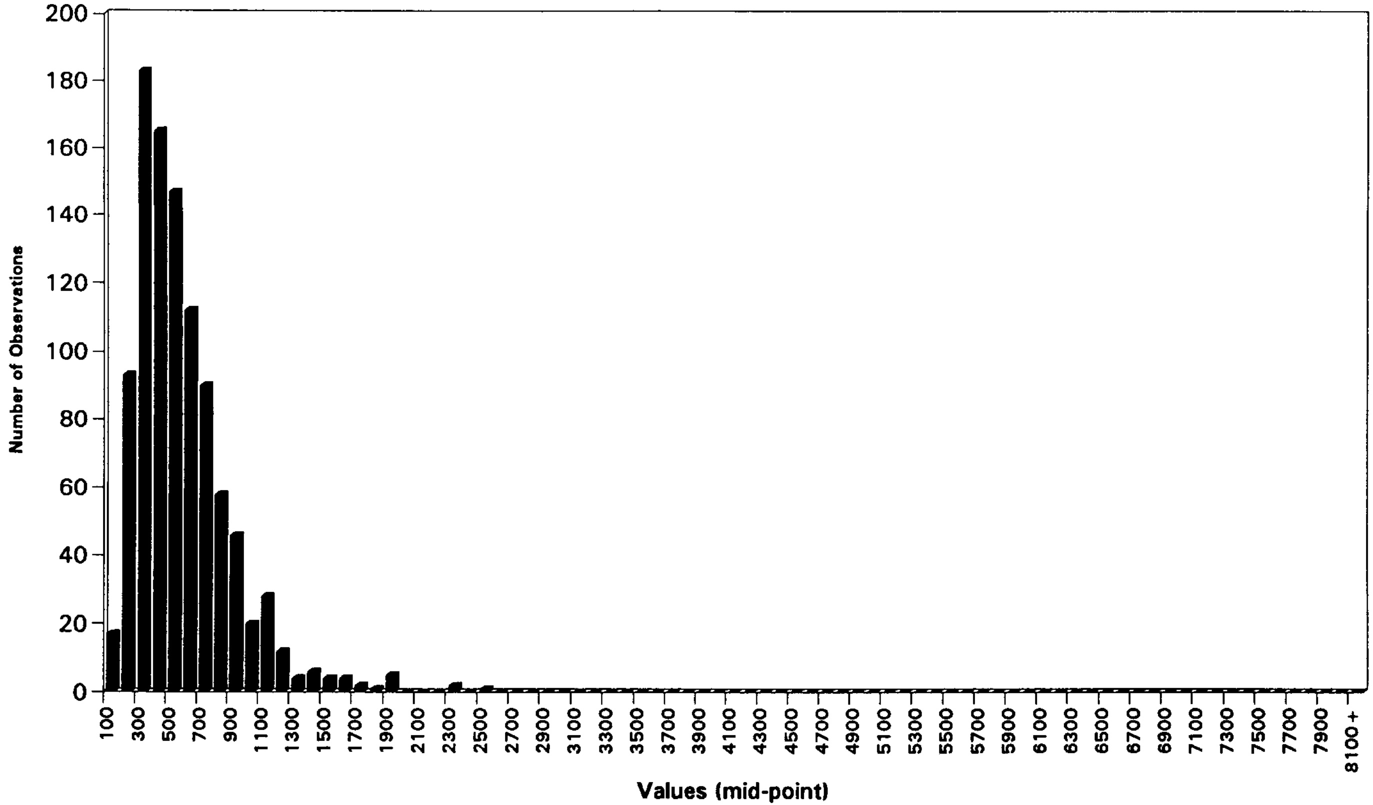
Distribution of 1000 Simulation of The All Ordinaries Index - Five Years Ahead



GRAPH VII

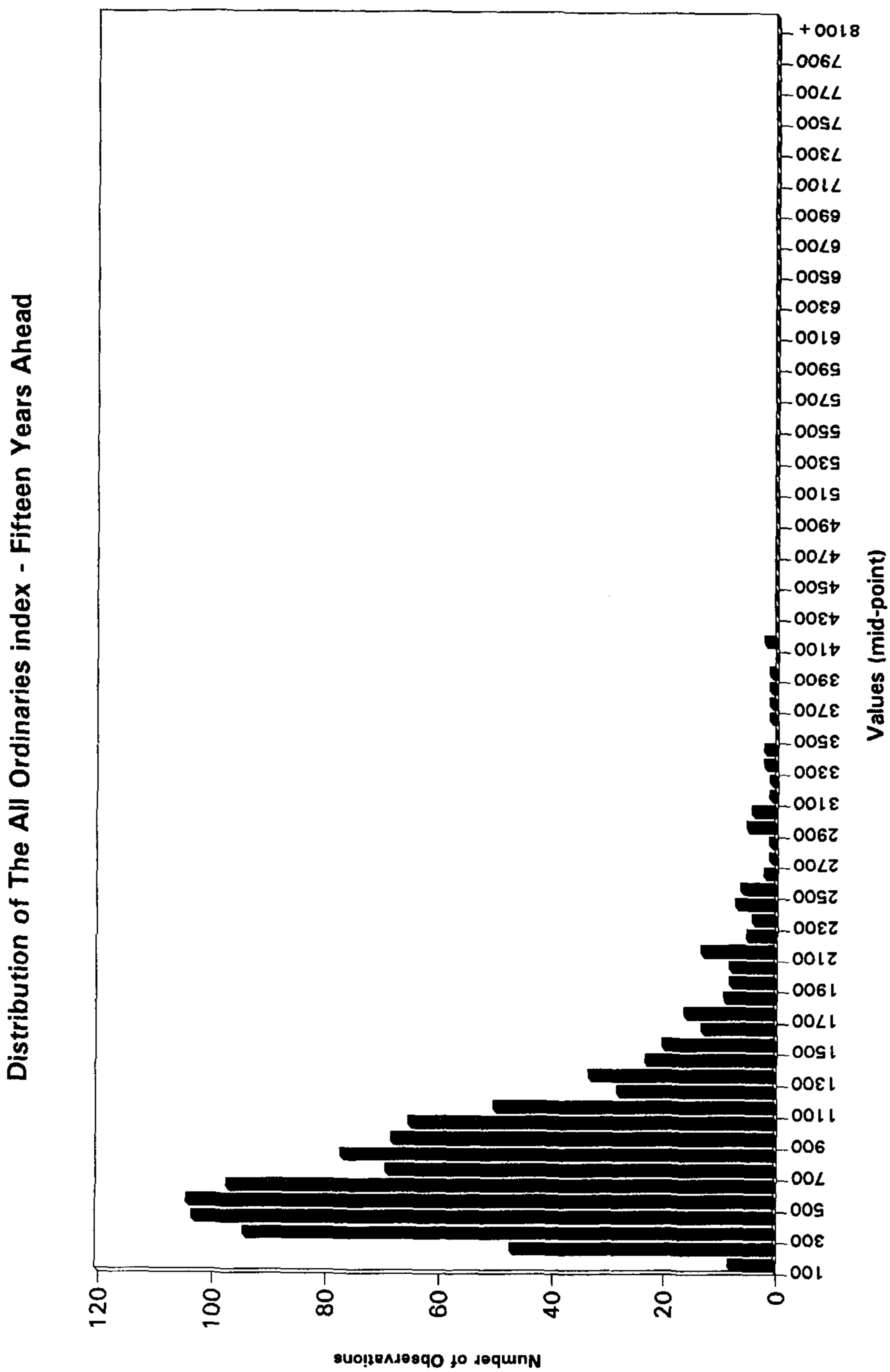
Mean = 342.17 Standard Deviation = 142.08

Distribution of The All Ordinaries index - Ten Years Ahead



GRAPH VIII

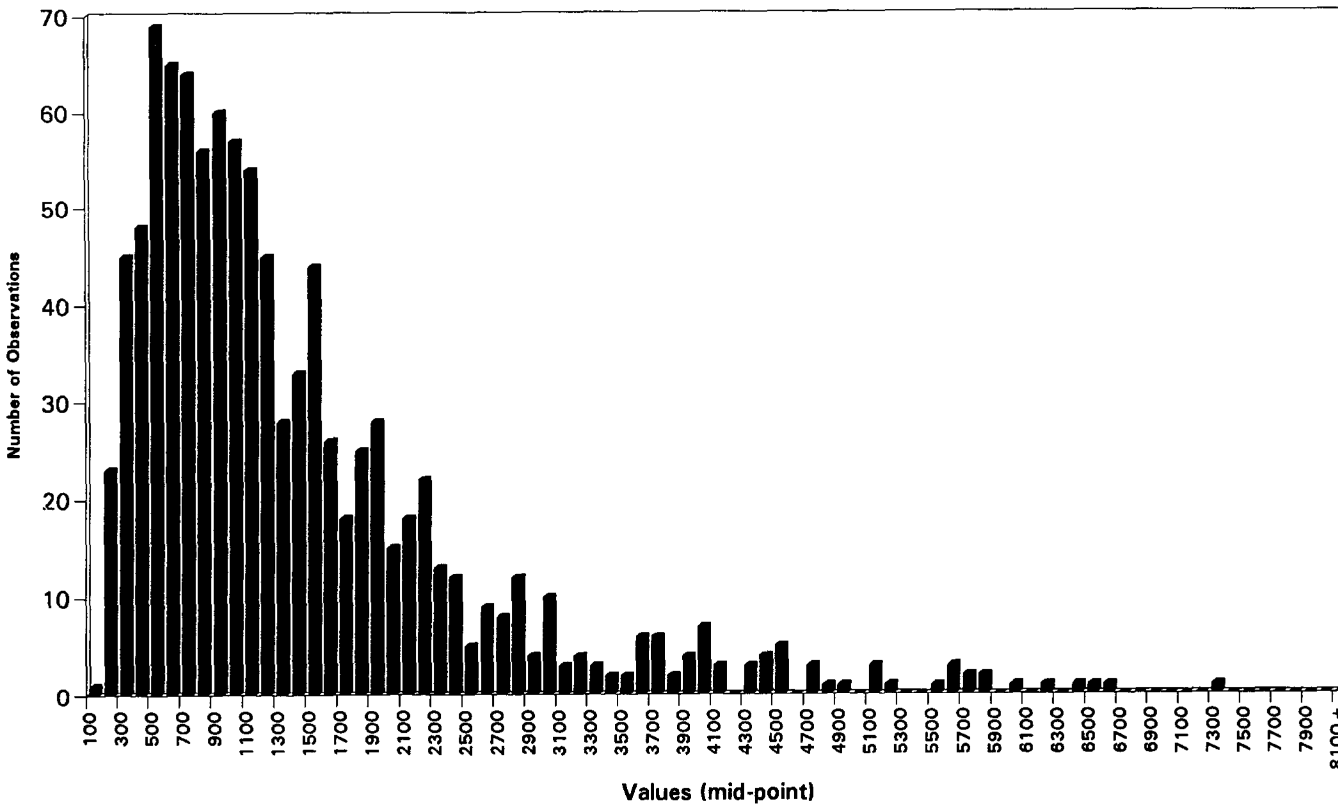
Mean = 546.62 Standard Deviation = 312.05



Mean = 874.00 Standard Deviation = 618.90

GRAPH IX

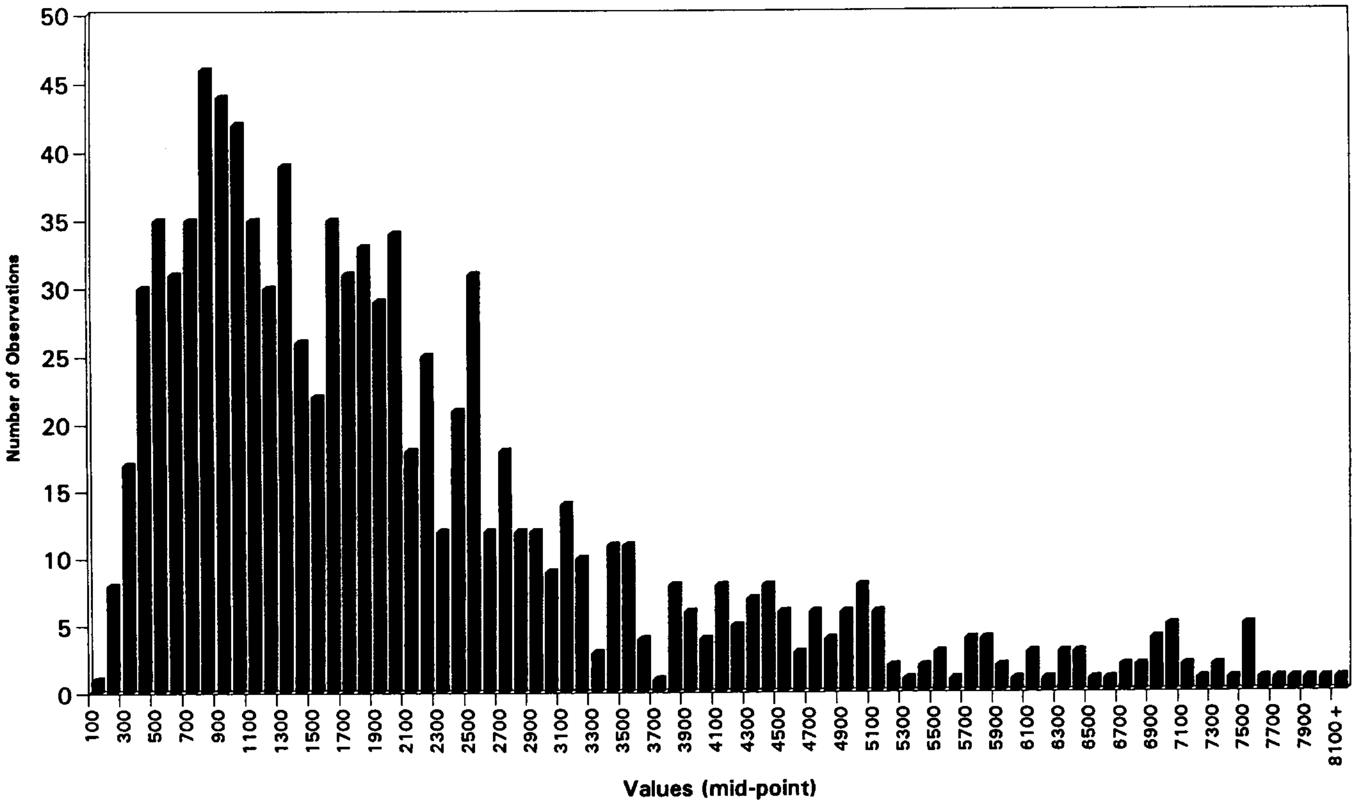
Distribution of The All Ordinaries index - Twenty Years Ahead



Mean = 1440.80 Standard Deviation = 1262.50

GRAPH X

Distribution of The All Ordinaries index - Twenty -Five years Ahead



GRAPH XI

Mean = 2355.80 Standard Deviation = 2199.70

TABLE V : Mean & Standard Deviation of The All Ordinaries Index. 1000 Simulations

	Period Of Simulation				
	5 Years	10 Years	15 Years	20 Year	25 Years
Mean	342.17	546.62	874.00	1440.80	2355.80
Standard Deviation	142.08	312.05	618.90	1262.50	2199.70

Index at time zero = 100

Further Research

This model is only a share price yields model. Investors receive dividends as well as capital gains and the dividend return needs to be modelled for a useable model. Further, as previously mentioned, a time varying mean should be included in the model.

CONCLUSION

This paper fits a simple ARCH model to Australian equity returns using several different period of data. It demonstrates that models for equity returns should allow for both a time varying conditional mean and a time varying conditional variance. The paper confines its scope to that of time varying conditional variance.

Hopefully the use of these models in actuarial work will increase as the benefits of this approach are better understood by actuaries.

APPENDIX I

Equations Used

The following assumptions are made :

1. $R_t = \mu + \sigma_t \epsilon_t$
2. $\log(\sigma_t) \sim N(\alpha, \beta^2)$ and $\beta > 0$
3. $\log(\sigma_t) = \alpha + a_1(\log(\sigma_{t-1} - \alpha) + Z_t$
4. ϵ_t and Z_t are independent N(0,1) variable and they are uncorrelated.

There are four parameters to be estimated. μ is estimated by the sample mean. The method used to estimate the other three are outlined below.

The method of moment is used to estimate α and β. As $E[\sigma_t] = \exp\{\alpha+0.5\beta^2\}$ and $E[\sigma^2_t] = \exp\{2\alpha+2\beta^2\}$, therefore,

$$\beta^2 = \log\{\frac{E[\delta_t^2]}{E[\delta]^2}\} = \log\{\frac{\mu_\delta^2 + \delta_\delta^2}{\mu_\delta^2}\}$$

and

$$\alpha = \log\{\frac{E[\delta_t]^2}{\sqrt{E[\delta_t^2]}}\} = \log\{\frac{\mu_\delta^2}{\sqrt{\mu_\delta^2 + \delta_\delta^2}}\}$$

μ_σ and σ_σ and be estimated by :

$$\delta_\delta^2 = \frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})^2 - \{\frac{\frac{1}{n}\sum_{t=1}^{n}|x_t - \bar{x}|}{\sigma}\}^2$$

and

$$\hat{\mu}_\delta = \frac{\frac{1}{n}\sum_{t=1}^{n}|x_t - \bar{x}|}{\sigma}$$

If $Z_t \sim N(0,1)$, σ is approximately 0.798.

The estimate of β outlined above is not strictly correct since β is dependant on the value of a_1. However, the above estimate can be used as an initial value to be used in the equation so that both a_1 and β can be estimated iteratively. One such method is given in the next paragraph.

It can be shown that (See Taylor(1989)) the first order autoregressive process given above has an autocorrelation at lag k equal to a^k_1. We now give an outline of the method that can be used to estimate a_1.

Define :

1. $S_t = (X_t - \mu)^2$

2. $M_t = |X_t - \mu|$

It can be shown that :

$$\rho_{k,S} = \rho_{k,\delta^2}\left\{\frac{var(\delta_t^2)}{var(S_t)}\right\} \qquad --- \qquad (A1)$$

where k is lag and ρ is the autocorrelation.

$$\rho_{k,M} = \sigma^2 \rho_{k,\delta}\left\{\frac{var(\delta_t)}{var(M_t)}\right\} \qquad --- \qquad (A.2)$$

$$\mathrm{Var}(\sigma_t) = \exp\{2\alpha+\beta^2\}(\exp\{\beta^2\} - 1) \qquad --- (A3)$$

$$\mathrm{Var}(M_t) = \exp\{2\alpha+\beta^2\}(\exp\{\beta^2\} - \sigma^2) \qquad --- (A4)$$

$$\mathrm{Var}(S_t) = \exp\{4\alpha+4\beta^2\}(3\exp\{4\beta^2\} - 1) \qquad --- (A5)$$

If $Z_t \sim N(0,1)$ then $\sigma^2 = 2/\pi$ and the following equations can be obtained :

$$\frac{\rho_{k,S}}{\rho_{k,\delta^2}} = \frac{\{e^{4\beta^2} - 1\}}{\{3e^{4\beta^2} - 1\}} = A(\beta) \qquad --- \qquad (A6)$$

$$\frac{\rho_{k,M}}{\rho_{k,\delta}} = \frac{2\{e^{4\beta^2} - 1\}}{\{\pi e^{\beta^2} - 2\}} = B(\beta) \qquad --- \quad (A7)$$

A set of sample autocorrelations $r_{k,s}$, $r_{k,m}$ for several k, say lags 1 to K, provides information useful for estimating a_1. Assuming that $\{Z_t\}$ is strict white noise when estimating a_1, then

$$\rho_{k,S} = A(\beta)\,\rho_{k,\delta^2} \quad , \quad \rho_{k,M} = B(\beta)\,\rho^{k,\delta}$$

All the autocorrelations above are functions of β and a_1.

Sample autocorrelations are calculated by the formula :

$$r_{k,s} = \{\frac{n}{n-k}\}\frac{\sum_{t=1}^{n-k}(s_t - \bar{s})(s_{t+k} - \bar{s})}{\sum_{t=1}^{n}(s_t - \bar{s})^2} \qquad --- \quad (A8)$$

The factor n/(n-k) is included to reduce bias in the estimates.

The discrepancies between sample and theoretical autocorrelations will be measured by statistics based on :

$$n\sum_{k=1}^{K}[r_{k,s} - \rho_{k,S}]^2 \quad , \quad n\sum_{k=1}^{K}[r_{k,m} - \rho_{k,M}]^2$$

β and a_1 are estimated by minimising :

$$F_1(\beta, a_1) = n\sum[r - A(\beta)\,\rho_{k,\delta^2}]^2$$

or

$$F_2(\beta, a_1) = n\sum[r_{k,m} - B(\beta)\,\rho_{k,\delta}]^2$$

These optimisations are performed using an iterative technique.

BIBLIOGRAPHY

Abrahams and Ledolter, (1983), "Statistical Methods for Forecasting", Wiley Interscience.

Box and Jenkins, (1970), "Time Series Analysis : Forecasting and Control", Holden Day, San Francisco.

Carter, Jon, (1991), "The Derivation and Application of An Australian Stochastic Investment Model", TIAA.

Clark, P. K. (1973), "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices", Econometrica, p 135-155.

Cryer, J.D. (1990), "Time Series Analysis", PWS Publishers.

Geoghegan, T.J., et al., (1992), "Report on The Wilkie Stochastic Investment Model", JIA.

Perry, P. R. (1982), "The Time-Variance Relationship of Security Returns : Implications for The Return Generating Stochastic Process", Journal of Finance, p.857-870.

"Reserve Bank Bulletin", (1970-1992), Reserve Bank of Australia, Sydney.

Tauchen G. E. and Pitts M. (1983), "The Price Variability - Volume Relationship on Speculative Markets", Econometrica, p. 485-505.

Taylor, Stephen, (1989), "Modelling Financial Time Series", John Wiley & Sons.

Wilkie, A.D., (1984), "A Stochastic Investment Model for Actuarial Use", JIA.

Wilkie, A.D., (1985), "Some Applications of Stochastic Investment Models", JASSA.