Asset Allocation Via The Conditional First Exit Time

or

How To Avoid Outliving Your Money

Moshe Arye Milevsky Doctoral Candidate

Professor Chris Robinson

Faculty of Administrative Studies

Professor Kwok Ho

Atkinson College

York University, 4700 Keele St, North York

Ontario, Canada, M3J - 1P3

Telephone: (416) 736 - 5073

Fax: (416) 736 - 5687

Summary:

The risk of outliving your money with low risk, low return investments is very often more serious than the risk of losing money on high risk investments, until quite late in life. A stochastic process model incorporating mortality tables, rate of return and standard deviation of return on wealth, age, sex and desired level of consumption provides the analytical tool. A simulation using Canadian mortality tables and rates of return shows that the optimal allocation is related to age, sex and the ratio of initial wealth to desired consumption. The asset allocation literature in finance has ignored these factors, and recommends lower allocations to risky assets.

Répartition des avoirs par temps de première sortie conditionnelle

ou

Comment éviter de survivre à votre argent

Moshe Arye Milevsky, Candidat au doctorat

Professeur Chris Robinson
Collège des études de gestion

Professeur Kwok Ho
Collège d'Atkinson
Université de York, 4700 Keele St., North York
Ontario, Canada, M3J - 1P3
Téléphone : (416) 736 - 5073
Fax : (416) 736 - 5687

Résumé

Jusqu'à ce que vous parveniez à un âge très avancé, le risque de survivre à votre argent avec des investissements à risque faible et à faible rendement est très souvent plus sérieux que le risque le perdre de l'argent sur des investissements à grand risque. L'outil analytique utilisé ici est un modèle de procédé stochastique incorporant des tables de mortalité, un taux de rendement et un écart standard du rendement fondé sur la fortune, l'âge, le sexe et le niveau souhaité de consommation. Une simulation faisant intervenir des tables de mortalité et des taux de rendement canadiens montre que la répartition optimale est fonction de l'âge, du sexe et du rapport existant entre la fortune initiale et la consommation souhaitée. La littérature financière portant sur la répartition des avoirs a ignoré ces facteurs et recommande des affectations moins importantes sur des avoirs à risque.

Asset Allocation Via The Conditional First Exit Time
or
How to Avoid Outliving Your Money

How should an individual who is retired allocate his or her investments between high risk, high return assets and low risk, low return assets? The retiree faces two risks. If s/he invests in low return assets, s/he risks outliving the income stream such an investment generates. If s/he invests in high return assets, there is a chance that losses will diminish the asset base and also lead to starvation.

The conventional wisdom has always been that investors should pick equities or other high return investments earlier in their life cycle, and gradually switch to bonds and treasury bills later in the life cycle. By retirement age, they should be holding more than half their investments in bonds and near-cash securities. For example, [Malkiel 1990] recommends:

> As investors age they should start cutting back on the riskier investments and start increasing the proportion of the portfolio committed to bonds. By the age of fifty-five, investors should start thinking about the transition to retirement and moving the portfolio toward income production. ...In retirement, portfolio mainly in a variety of intermediate-term bonds (five to then years to maturity) and long-term bonds (over ten years to maturity) is recommended. The small proportion of stocks is included to give some income growth to cope with inflation. (pp. 356-7)

In the graphs that follow the chapter he recommends investors in the late sixties and beyond hold 60% bonds, 30% equity and 10% in a money market fund. Investors in their mid-fifties are recommended to have 50% in stocks, 45% in bonds.

[Ho, Milevsky, Robinson 1993] suggest that retiring individuals would be better served by undertaking a substantial amount of investment risk. The return patterns over time noted by [Butler and Domian 1993] seem to support this view.

The investment allocation in this paper incorporates the required rate of return to minimize the probability of failing to meet that rate of return on average over the weighted lifespan remaining to the person. We can speak in dramatic terms of the minimizing the probability of starvation, although we hope it isn't quite that serious. This implied utility function of minimizing shortfall is somewhat similar to the approach taken by [Leibowitz and Kogelman 1991]. They "measure risk by the "shortfall probability" relative to a minimum return threshold." A fund manager can choose any combination of minimum return and probability and then allocate the assets between a risky and a risk-free asset to attain a desirable position. Their procedure does not endogenize the time horizon of the investor, since fund managers do not necessarily have a specific time constraint. They do observe that for longer time horizons, the proportion invested in equity rises.

Many researchers have considered the general question of which investment horizon to use and what effect different horizons have on how we view risk and return. In general, they find that the risk of riskier assets declines if they are held without trading for long periods. Different assets do well in specific shorter periods of time; so the benefits of changing portfolio composition are considerable if the investor times successfully.[2] The conclusion for asset allocation is that you should use more equity for longer horizons. [Lloyd and Modani 1983] conclude:

> In general, the usefulness of time diversification is more evident for portfolios containing common stock. Further, the riskiness of any portfolio position is unclear unless the number of time periods the portfolio will

[2]See, for example, [Benari 1990]; [Butler and Domian 1991]; [Butler and Domian 1993]; [Grauer and Hakansson 1982]; [Lloyd and Modani 1983].

be held is also considered. (pp. 11)

Since we are solving the problem for an individual retiree, we incorporate this time dimension explicitly. In addition, we require annual consumption from the portfolio, which does not appear in other researchers' treatments of this problem. Substitution of standard Canadian mortality tables and reasonable estimates of return and variance for Canadian T-bills and equity provides surprising results. Women should invest in a higher proportion of equity than men in the otherwise identical situation, because they live longer. Individuals of retirement age should invest in more equity than the conventional wisdom recommends, unless they are very wealthy relative to the amount they intend to consume. In quite reasonable cases we would recommend 85% to 100% equity for someone aged 65 at date of retirement.

In the rest of the paper we derive the formal mathematical model for optimizing the retiree's asset allocation. Since the derivation does not produce a closed form solution, we developed a Monte Carlo simulation. We present three realistic numerical cases solved using this procedure and draw our conclusions from them.

1 Minimize The Probability of Outliving Wealth

Our model assumes that at retirement ($t = 0$) the retiring-investor deposits all of his/her current wealth (W_0) into an account that allows him/her to allocate funds to and from various asset categories, within the account, at fixed points in time. In addition, the retiring-investor consumes from this account fixed sums at fixed points in time, as long as there is enough wealth to cover the withdrawal. Hence, the account is instantaneously aware of its own current market value and does not allow a withdrawal that exceeds its own net worth. Furthermore, when the retiring-investor consumes, the account dispenses funds from each category in a way that is proportional to the market value of funds in each category.

First we analyze the probability of an individual outliving his/her wealth under one asset category with deterministic interest rates. We then generalize the model to a stochastic rate of return which depends on various asset categories and the respective asset allocation proportions in each.

We express everything in real dollars and real rates of return. The mathematics could be done in nominal rates and we would obtain the same results. The simplicity of real rates is the sole reason for doing it this way.

1.1 Deterministic Rates of Return:

For notational simplicity, time t is measured in units of years and each year is subdivided into k non-overlapping periods of equal length $\frac{1}{k}$ years. Therefore, n periods will be synonymous with $\frac{n}{k}$ years. Also, we use and refer to both an *effective annual* interest rate as well as to an *effective periodic* interest rate. From a purely mechanical point of view, we assume that the real annual consumption rate is C and hence at the end of each period the retiring-investor will consume $\frac{C}{k}$, furthermore we set the interest compounding periods to correspond with the withdrawal periods.

In the deterministic case, the effective annual interest rate is r. There are k compounding-withdrawal periods per year, therefore the effective the one period interest rate is $\sqrt[k]{1+r} - 1$.

Hence, at the end of the first period, ($\frac{1}{k}$ years after retirement and immediately after the first withdrawal), the investor's wealth W_1 can be represented as:

$$W_1 \equiv \max\left\{0,\left(W_0 \times \sqrt[k]{1+r} - \frac{C}{k}\right)\right\}$$

The reason for the max{} term is that wealth cannot become negative; in other words there is no line of credit. This means that the retiring- investor cannot withdraw $\frac{C}{k}$ dollars from an account whose market value is less than $\frac{C}{k}$ dollars.

Likewise, at the end of the second period, ($\frac{2}{k}$ years after retirement and immediately after the second withdrawal), the investor's wealth W_2 can be represented as:

$$W_2 \equiv \max\left\{0, \left(\left(W_0 \times \sqrt[k]{1+r} - \frac{C}{k}\right) \times \sqrt[k]{1+r} - \frac{C}{k}\right)\right\}$$

In general, at the end of the n'th period, ($\frac{n}{k}$ years after retirement and immediately after the n'th withdrawal), the investors wealth W_n can be represented as:

$$W_n \equiv \max\left\{0, \left(\left(\ldots\left(\left(W_0 \times \sqrt[k]{1+r} - \frac{C}{k}\right) \times \sqrt[k]{1+r} - \frac{C}{k}\right)\ldots\right) \times \sqrt[k]{1+r} - \frac{C}{k}\right)\right\} \quad (1)$$

which after some algebraic manipulation can be expressed as:

$$W_n \equiv \max\left\{0, \left(W_0 \times \left(\sqrt[k]{1+r}\right)^n - \frac{C}{k} \times \left(\sum_{i=0}^{n-1}(\sqrt[k]{1+r})^i\right)\right)\right\} \quad (2)$$

Moreover, by viewing the second term in equation [2] as the accumulated value of an annuity due, (or by adding the terms of a geometric series) one can re-write W_n as:

$$W_n \equiv \max\left\{0, W_0 \times (1+r)^{\frac{n}{k}} - \frac{C}{k} \times \left(\frac{(1+r)^{\frac{n}{k}} - 1}{\sqrt[k]{1+r} - 1}\right)\right\} \quad (3)$$

Now, as a result of the deterministic interest rate involved, one can calculate the exact period N^* when the investor's wealth W_{N^*} will equal zero for the first time.

$$W_0 \times (1+r)^{\frac{n}{k}} = \frac{C}{k} \times \left(\frac{(1+r)^{\frac{n}{k}} - 1}{\sqrt[k]{1+r} - 1}\right) \quad (4)$$

After some elementary algebra this translates [3] into:

$$N^* = \left\lceil \frac{\ln[C] - \ln[C - W_0 k((1+r)^{1/k} - 1)]}{\frac{1}{k}\ln[1+r]} \right\rceil \quad (5)$$

Where $\lceil a \rceil$ represents the smallest integer greater than or equal to a. Thus, when interest rates are known and constant, the investor will run out of money at (the end of) period N^*; of course this is provided s/he is alive at (the end of) N^*.

[3] Strictly speaking, the only time the investor can run out of money is during a withdrawal, *i.e.* when there is not enough money in the account to satisfy the consumption requirement. Therefore N^* is defined as the integer-valued withdrawal period when the investor no longer is able to satisfy the entire consumption requirement.

However, by incorporating mortality functions one can strengthen the above statement by observing that, under deterministic interest rates, the probability that a retiring-investor outlives his or her money is:

$$\mathbf{P}(starve) = P^{x}_{N^*/k} \tag{6}$$

P^x_t denotes the probability that an individual aged x (in years) will survive to age $x + t$ (in years), taken from any standard mortality table. In words, equation [6] represents the probability that an individual who is alive at age x will survive to age $x + N^*/k$, which happens to be the time (in years) when the individual runs out of money. Hence the probability of starvation is the probability that the retiring-individual *lives* to the point where s/he runs out of money.

In particular, (and extremely desirable) N^* may be infinite. This occurs when the (deterministic) interest rate r is large enough so as to establish an eternal perpetuity. Mathematically this occurs when the argument in the logarithmic expression of equation [5] becomes negative,[4] which is when:

$$r \geq \left(1 + \frac{C/W}{k}\right)^k - 1 \tag{7}$$

In which case $N^* = \infty$ and then:

$$\mathbf{P}(starve) = P^{x}_{\infty} = 0 \tag{8}$$

Under such circumstances, one will never starve to death because there is enough initial wealth to create a consumption perpetuity that will last for ever.

One final point about deterministic interest rates is to investigate when the frequency k of compounding-withdrawal approaches infinity, in which case, from equation [5], the appropriate *time* of starvation $T^* = N^*/k$, will be:

$$T^* = \lim_{k\to\infty} \frac{\ln[C] - \ln[C - W_0 k((1+r)^{1/k} - 1)]}{\ln[1+r]} = \frac{\ln[C] - \ln[C - W_0 \ln[1+r]]}{\ln[1+r]}$$

[4]One can derive the appropriate interest rate that will ensure a perpetuity, without having to resort to equation [5].

The proof of which can be obtained by noticing that for $a > 0$:

$$\lim_{k \to \infty} k(a^{1/k} - 1) = \ln[a]$$

Likewise, from equation [7], $T^* = \infty$ for continuous compounding-consumption, when:

$$r \geq \lim_{n \to \infty} \left(1 + \frac{C/W}{k}\right)^k - 1 = e^{C/W} - 1$$

Which gives us a practical *rule of thumb* formula for analyzing the investment-consumption problem (under deterministic interest rates) without resorting to a specific withdrawal frequency.

We are now ready to generalize the above discussion to stochastic rates of return.

1.2 Stochastic Rates of Return:

As we are now dealing with a collection of m asset categories, each with its own stochastic behaviour, we let $\vec{\alpha}$ denote the m dimensional *vector of asset allocation proportions*. At retirement the retiring-investor specifies a particular $\vec{\alpha}$ which he/she would like to maintain. We assume that the retiring-investor chooses an $\vec{\alpha}$ and adheres to it throughout his/her remaining lifetime. This may sound like a very strict requirement, and in practice we paper would recommend that the investor *update* his/her $\vec{\alpha}$ after every withdrawal. However, for the purpose of the model, the static assumption is necessary for obtaining an estimate of the probability of outliving wealth. This information should act as a guide for the asset allocation decision now, even if the investor will almost surely change the proportions at the next withdrawal period.[5]

[5]In reality, a *dynamic* policy would be optimal and may indeed reduce the probability of outliving wealth; however this is beyond the scope of this paper and perhaps may be the subject of future research. A stochastic control theory approach that would incorporate the Hamilton- Jacoby-Bellman equations would be the obvious technique to use.

The stochastic scenario analogue of the deterministic one period interest rate $\sqrt[k]{1+r} - 1$ will be a random variable $\mathbf{R^k}(\vec{\alpha})$ which is an explicit function of the vector $\vec{\alpha}$ as well as the number k of compounding-withdrawal periods per year. In addition it is an implicit function of the underlying asset return specification which is a multivariate distribution denoted by $\mathbf{\Lambda}$. In this paper we assumed that all financial assets can be modeled as a Geometric Brownian Motion.[6]

Therefore, one year returns are Multivariate Lognormally distributed with parameters (μ, Σ) where μ is the one year mean vector of logarithmic returns and Σ is the one year variance-covariance matrix of logarithmic returns. Furthermore, using the properties of the Lognormal distribution [Crow, Shimizu 1985], one *period* returns are Multivariate Lognormally distributed with parameters $(\frac{1}{k}\mu, \frac{1}{k}\Sigma)$ Furthermore, if $\vec{\alpha}$ is the vector of asset allocation proportions whose elements are α_i and if we let I_i denote an m dimensional row vector with a 1 in position i and 0 in all other positions, then the desired random variable $\mathbf{R^k}(\vec{\alpha})$ is:

$$\mathbf{R^k}(\vec{\alpha}) = \sum_{i=1}^{m} \alpha_i \cdot \mathbf{\Lambda}(I_i \cdot \frac{1}{k}\mu, I_i \cdot \frac{1}{k}\Sigma \cdot I_i') \tag{9}$$

The notation, $\mathbf{R^k}(\vec{\alpha})_i$ would represent a realization of the one period return random variable, in other words the *actual* return in the i'th period.

Continuing as in the deterministic case, at the end of the first period the investors

[6]The authors would like to avoid a lengthy discussion about the appropriate model for financial asset returns. Suffice it to say that the Geometric Brownian Motion assumption, which translates into the Lognormal distribution assumption, is still used extensively throughout the continuous time finance literature. In addition, the methodology developed in this paper to analyze the probability of outliving wealth can be applied to any stochastic specification of returns by employing a Monte Carlo analysis as we do. Thus, for example, if one is convinced that the appropriate model for investment returns is a *Contaminated GBM*, *Jump Poisson Process* or a *GARCH process*, then $\mathbf{R^k}(\vec{\alpha})_i$ would represent a generalized one period return. However, regardless of the exact specification of $\mathbf{R^k}(\vec{\alpha})_i$, it still is an implicit function of the asset allocation proportions $\vec{\alpha}$, and hence can be simulated as such.

wealth W_1 is a random variable that can be represented as:

$$W_1 \equiv \max\left\{0, \left(W_0 \times \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_1 - \frac{C}{k}\right)\right\}$$

Likewise, at the end of the second period the investor's wealth W_2 is a random variable that can be represented as:

$$W_2 \equiv \max\left\{0, \left(\left(W_0 \times \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_1 - \frac{C}{k}\right) \times \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_2 - \frac{C}{k}\right)\right\}$$

In general, at the end of the n'th period, the investors wealth W_n is a *random variable* that can be represented as:

$$W_n \equiv \max\left\{0, \left(\left(\ldots\left(\left(W_0 \times \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_1 - \frac{C}{k}\right) \times \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_2 - \frac{C}{k}\right) \times \ldots\right) \times \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_n - \frac{C}{k}\right)\right\} \tag{10}$$

which, after some algebraic manipulation, can be expressed as:

$$W_n \equiv \max\left\{0, \left(W_0 \times \prod_{i=1}^{n} \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_i - \frac{C}{k} \times \left(\sum_{i=1}^{n-1} \prod_{j=1}^{i} \mathbf{R}^{\mathbf{k}}(\vec{\alpha})_j\right) - \frac{C}{k}\right)\right\} \tag{11}$$

As before, we are interested in the first time W_n reaches zero, denoted by N^*. However, since returns are stochastic, N^* is a random variable (otherwise known as a stopping time). Specifically:

$$N^* = \inf\{n \geq 0 ; W_n = 0\} \tag{12}$$

N^* is the First Exit Time of the stochastic process W_n from the set of non zero numbers. Technically speaking, $P[N^* = i]$ is an implicit function of $W_0, C, k, \vec{\alpha}, \mathbf{\Lambda}$ which for obvious reasons satisfies:

$$\sum_{i=1}^{\infty} P[N^* = i] = 1$$

A non-zero $P[N^* = \infty]$ denotes the probability that W_n never equals zero *i.e.* that the investor is set for an eternal life.

As in the deterministic case, we would like to calculate the probability of living to the time (period) when the money runs out. However, since we do not know the exact N^* when the wealth will first be zero, the best we can do is compute the probability that $N^* = i$ for all possible i, then compute the probability of surviving to age $x + i/k$, after which we multiply those two numbers and then finally add them up over all i. Mathematically, we obtain that the probability of outliving wealth[7] (under stochastic rates of return) is:

$$\mathbf{P}(starve) = \sum_{i=1}^{\infty} P^x_{\frac{i}{k}} \cdot P[N^* = i] \tag{13}$$

One should view $\mathbf{P}(starve)$ as the conditional (on being alive) First Exit Time of the stochastic process W_n from the set of positive real numbers.

The general objective of this paper is to:

- Under a Lognormal distributional assumption for Λ together with a given k, W_0, C; compute the numerical value of $\mathbf{P}(starve)$ in equation [13] for various $\vec{\alpha}$ values.
- For the above conditions, find a suitable $\vec{\alpha^*}$ that will minimize $\mathbf{P}(starve)$ in equation [13] and hence minimize the probability of outliving wealth.

[7]We further must assume independence between P^x_t and $P[N^* = i]$ for the expression in equation [13] to be valid. Qualitatively this means that mortality must be independent of wealth at all points in time. Thus, a stock market crash will not be allowed to cause heart failures, likewise a sustained bull market does not improve one's health nor does it prolong one's life.

2 Numerical Examples:

The authors developed a computer program, in Turbo C+, that *estimates* the magnitude of **P**(*starve*) in equation [13] for $k = 12$ (which is monthly withdrawals) for various $\vec{\alpha}$ values of dimension $m = 2$ (which is two asset categories). This was achieved via a Monte Carlo Simulation of $P[N^* = i]$ for a Lognormal distribution, in conjunction with monthly Male/Female mortality rates estimated from a yearly actuarial mortality tables. The two asset categories were Canadian equity and Canadian treasury bills, representing high risk, high return investments and low risk, low return investments, respectively.

The mortality data was provided by Statistics Canada Health Reports Supplement No. 13 1990 Vol 2 No. 4. The statistical values for the real rates of return were provided by [Hatch and White 1988]. The average real return on Canadian equity was estimated to be 7.5% per year, with a standard deviation of 17.5%. The average real return on Canadian treasury bills was estimated to be 1.5% percent with a standard deviation of 3.5%.

We illustrate the results from this model with three cases encompassing reasonable situations. As a general benchmark, we note that the average family income in the province of Ontario is $57,727. Our three cases each consider consumption well below that level. We show both male and female weightings for equity. There is a problem with this split, because the majority of plans encompass a couple. If the reader will suspend this question until after the examples, we will explain why the results would apply even more strongly to couples. All the monetary amounts are in constant dollars at the date of retirement.

One particular commonality in the patterns is that all but one have interior optima. We can see that the probability of failing to earn enough to meet desired consumption drops as the equity allocation increases, then rises again after a minimum point. There are some ambiguities, because these results are simulated. Thus, while the pattern is clear, there are some slightly anomalous points.

Case 1

A person earns $80,000 per year for 30 years, saving $15,000 per year, and retires at age 55. The savings accumulate at a real rate of return of 5% to reach about $1 million. The person desires to consume $40,000 per year in real dollars, which yields a $\frac{C}{W}$ ratio of .04. This is a very low level of consumption of wealth, compared with what most persons or families can expect.

We then simulate the pattern of consumption and earnings to estimate the probability of failing to be able to consume at least the desired amount for increasing proportions of equity investment. The results are shown in Table 1. The optimal allocation for the male is 55% equity. This seems somewhat in line with Malkiel's advice, but the optimal equity investment for the female in the same situation is 75%. If the woman stays invested wholly in Treasury Bills, she has a 39% chance of not being able to consume at the desired level. Even at her optimum of 75% equity she faces a 7% chance of failing to be able to consume at the desired level.

Case 2

A person earns $40,000 per year for 40 years, saving $5,000 per year, and retires at age 65. The savings accumulate at a real rate of return of 5% to reach about $600,000. The person desires to consume $25,000 per year in real dollars, which yields a $\frac{C}{W}$ ratio of .0417. As in Case 1, this is a very low level of consumption of wealth, compared with what most persons or families can expect.

The simulation results in Table 1 show much lower required equity positions for both male and female. We would expect this, since they have fewer years to live, and hence need less income on the invested amount. The male should be 40% in equity; the female 40-50%. These values are somewhat higher than most financial planners tend to recommend at age 65, and we have someone consuming quite a low proportion of invested wealth per year.

Case 3

A person earns \$35,000 per year for 30 years, saving \$5,000 per year, and retires at age 65. The savings accumulate at a real rate of return of 5% to reach about \$330,000. The person desires to consume \$25,000 per year in real dollars, which yields a $\frac{C}{W}$ ratio of .076. This consumption ratio is higher than in the first two cases, but it is still quite modest.

Table 2 displays the simulation results. The male seems to have an optimum somewhere in the 85-100% equity allocation range, although it is possible that 100% equity is not enough and borrowing would be required. He will have to face a substantial risk of not earning enough for his goal – 22-23%. The female seems to require 100% equity as a minimum, and probably needs to borrow a lot, in spite of the risk, in order to reach a reasonable risk of 'starvation.' Even at 100% equity, she has a one- third probability of not meeting her desired consumption level.

Extending the Model to Families

Without doing further analytical work, we can realize that the main conclusion – put more of your retirement investment in equity than most advisers suggest – follows even more strongly if we make each of the three examples into a couple with the joint wealth and consumption values in each of the three cases. The joint probability that at least one of the couple will survive and require support at a given age is higher than the probability for a single. Thus, even higher returns would be required, leading to even greater equity allocations.

3 Conclusion and Extensions:

We have developed a rigorous model to answer the question: how should a retiree allocate investment assets between low risk, low return assets and high risk, high return assets. This model incorporates the mortality tables, sex of the retiree, desired consumption to invested wealth ratio and rates of return and standard deviation.

Three realistic cases with historic mean and standard deviation of Canadian equity and Treasury Bills convey three general messages:

- Retirees should consider their desired consumption, existing wealth, age and sex, before deciding how to allocate their investment assets.
- Retirees in most cases should invest a higher proportion in high risk high return assets than most planners have traditionally recommended; and,
- Women need to invest even higher proportions of their wealth in riskier assets, because they live longer, on average, and need to earn more from their retirement funds than men do.

We think the results are quite robust, but the model could be improved in two ways, if they prove feasible:

1. Incorporate the joint probabilities of death for a couple, with declining consumption for the survivor compared with the couple. The model as it stands would continue to be valid for single persons.
2. Solve the dynamic problem of reallocation of assets over time.

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TABLE 1

Case 1

Age: 55 years
Wealth: $1,000,000
Desired consumption: $40,000 per annum in real dollars

The best probability is shown in **bold**.

Optimal Allocation

% in Stock	Probability of Not Reaching Goal	
	Male	Female
0	.2078	.3859
5	.1673	.3263
10	.1271	.2602
20	.0788	.1659
30	.0523	.1084
40	.0436	.0855
50	.0433	.0786
55	**.0394**	.0714
60	.0427	.0744
65	.0422	.0729
70	.0429	.0776
75	.0447	**.0681**
80	.0515	.0818
85	.0533	.0836
90	.0528	.0824

Case 2

Age: 65 years
Wealth: $600,000
The best probability is shown in **bold**.

Optimal Allocation

% in Stock	Probability of Not Reaching Goal	
	Male	Female
0	.0506	.1297
10	.0256	.0708
20	.0148	.0410
30	.0112	.0299
40	**.0110**	**.0276**
45	.0113	**.0276**
50	.0118	**.0276**
55	.0130	.0298
60		.0307

TABLE 2

Case 3

Age: 65 years
Wealth: $330,000
Desired consumption: $25,000 per annum in real dollars

The best probability is shown in **bold**.

Optimal Allocation

% in Stock	Probability of Not Reaching Goal: Male	Probability of Not Reaching Goal: Female
0	.4939	.6855
10	.4632	.6595
20	.4278	.6265
30	.3893	.5846
40	.3497	.5351
50	.3084	.4778
60	.2745	.4244
70	.2591	.3956
80	.2415	.3642
85	**.2327**	.3505
90	**.2350**	.3487
95	**.2321**	.3424
100	**.2222**	**.3262**