Portfolio Management and Transaction Costs

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Summary

Transaction costs are considered for Markowitz diversification in discrete time, for option pricing in continuous time, as well as for general hedging problems in discrete as well as continuous time. For Markowitz diversification in discrete time, incorporation of transaction costs leads to optimal trading strategies with less active trading and with almost the same performance. For option pricing in continuous time, an adjustment of volatility (Leland 1985) is the optimal solution. In general hedging problems and discrete time, major numerical problems arise, while in continuous time, an adjustment of volatility is the solution. In this context it is of importance whether hedging strategies are monotone or not. Solutions are given only for cases admitting monotone hedging strategies which are optimal without transaction costs.

Gestion de portefeuilles et coûts de transaction

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Résumé

Les coûts de transaction sont considérés ici pour une diversification de Markowitz en temps discret, pour l'établissement du prix de levée en temps continu et pour des problèmes généraux de couverture en temps discret et en temps continu. Pour une diversification de Markowitz en temps discret, l'incorporation des coûts de transaction mène à de meilleures stratégies opérationnelles offrant un nombre moins important d'opérations en bourse et pratiquement la même performance. Pour l'établissement du prix de levée en temps continu, un ajustement de la volatilité (Leland 1985) constitue la meilleure solution. Dans les problèmes généraux de couverture en temps discret, de nombreux problèmes numériques sont soulevés, alors qu'en temps continu un ajustement de la volatilité constitue la solution. Dans ce contexte, il est important de déterminer si les stratégies de couverture sont univoques ou non. Des solutions sont proposées pour des cas admettant uniquement des stratégies de couverture univoques optimales sans coûts de transaction.

Portfolio Management and Transaction Costs

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Abstract

Transaction costs are considered for Markowitz diversification in discrete time, for option pricing in continuous time, as well as for general hedging problems in discrete as well as continuous time. For option pricing in continuous time, an adjustment of volatility is the optimal solution. In general hedging problems and discrete time, major numerical problems arise, while in continuous time, an adjustment of volatility is the solution.

1 Introduction and Summary

1.1 Discrete Time

Assume that trading is possible in k different stocks and one riskless asset (rfa). If all stock prices are discounted, we may assume that there is no return for riskless investment. In addition we shall assume that there are no transaction costs for riskless investments. Let $t_0, \ldots, t_N$ be a finite number of trading times, $S_0, \ldots, S_N$ the corresponding stock price vectors with S_0 deterministic and $S_1, \ldots, S_N$ random. An investment strategy is a set of vectors $\vartheta_0, \ldots, \vartheta_{N-1}$ such that for $i = 0, \ldots, N-1$ the vector ϑ_i depends on $S_0, \ldots, S_i$ only. The number $\vartheta_i^{(j)}$ is the number of shares of stock j held at time t_i, $j = 1, \ldots k$, $i = 0, \ldots, N-1$. (We shall throughout assume that appropriate moment conditions hold in order to ensure the existence of all expected values that occur.) The corresponding portfolio value at time t_i, $i = 1, \ldots, N-1$, - neglecting transaction costs - is

$$P_i^0 = \sum_{n=1}^{i} \sum_{m=1}^{k} \vartheta_{n-1}^{(m)} (S_n^{(m)} - S_{n-1}^{(m)}).$$

Here we adopt the view that the portfolio has been built up without initial capital, and at each trading time it is possible - without any restriction or expenses- to borrow or lend money.
If proportional transaction costs are taken into account with cost factor γ, then the portfolio value is given by

$$P_i = P_i^0 - \gamma \sum_{n=1}^{i-1} \sum_{m=1}^{k} |\vartheta_n^{(m)} - \vartheta_{n-1}^{(m)}| S_n^{(m)}.$$

Classical problems in portfolio management are

1. Markowitz diversification
2. pricing of contingent claims, and
3. asset liability management.

Problem 1 is usually described in the case $N = 1$. In order to incorporate transaction costs we shall consider the case $N = 2$. Assume that ϑ_0, the initial portfolio mix, as well as the stock prices S_0, S_1 are given. An optimal portfolio mix for the second period, ϑ_1, is the one which maximizes EP_2 under the constraint $VarP_2 \leq \sigma^2$. Since costs are nonrandom, $VarP_2 = VarP_2^0$. Hence the constraint is convex:

$$VarP_2^0 = \sum_{m_1, m_2} \vartheta_1^{(m_1)} \vartheta_1^{(m_2)} Cov(S_2^{(m_1)}, S_2^{(m_2)}) \leq \sigma^2. \qquad (1)$$

The objective function is, however, no longer linear but only piecewise linear: maximizing EP_2 is equivalent to maximizing

$$V(\vartheta) = \sum_{m=1}^{k} \vartheta^{(m)} E(S_2^{(m)} - S_1^{(m)}) - \gamma |\vartheta^{(m)} - \vartheta_0^{(m)}| S_1^{(m)}. \qquad (2)$$

This optimization problem is not easy, "standard" nonlinear optimization techniques fail (we tried the polyhedra method of Nelder and Mead (see [17].) For small to moderate k the following method can be used. Let ϑ^* be the unrestricted maximum of EP_2 (if it exists). If ϑ^* satisfies the constraint, then ϑ^* is the solution. If not, then the following procedure yields the solution ϑ_1 : For arbitrary $k-$vector ε with all entries equal to one of the values $-1, 0, 1$, let $\vartheta(\varepsilon)$ be the solution to the optimization problem "maximize $V_\varepsilon(\vartheta)$ where

$$V_\varepsilon(\vartheta) = \sum_{m=1}^{k} \vartheta^{(m)} ES_2^{(m)} - \gamma\varepsilon^{(m)}(\vartheta^{(m)} - \vartheta_0^{(m)})S_1^{(m)} \tag{3}$$

under the constraints (1), and

$$\vartheta^{(m)} = \vartheta_0^{(m)} \text{ if } \varepsilon^{(m)} = 0."$$

Then ϑ_1 is the vector in the set of all $\vartheta(\varepsilon)$ which maximizes $V(\vartheta)$. For a proof see section 2.1.

This method involves 3^k solutions of a system of linear equations. For k moderate or large and under additional constraints for ϑ such as $\vartheta^{(m)} \geq 0$ (no shortselling) or $\vartheta^{(m)}$ integral, a random search algorithm and a high speed computer might help.

Notice that for Markowitz diversification, incorporation of transaction costs makes sense not only because they must be paid but also because the resulting optimal trading strategies are more stable than those computed without transaction costs (see the numerical results in section 3, and the recent article of Pelsser and Vorst [18]).

Problem 2 in discrete time and discrete space is solved for the case of European options in Boyle and Vorst [2]. Their method also works for American options (see Weber [24]).

In the discrete setup of Boyle and Vorst [2] the duplication strategy for options is unique and optimal. For other securities duplication must not be optimal (see [1] and [6]). In other situations even duplication is not unique, and then a utility based approach is justified such as in [3] and [4].

Problem 3 can be formulated as follows: given a liability L, what is the optimal trading strategy $\Theta = (\vartheta_0, \ldots, \vartheta_{N-1})$ for which

$$E(L + \Gamma - P_N)^2 \tag{4}$$

is minimized? Here, Γ is the total cost for strategy Θ :

$$\Gamma = \gamma \sum_{n=1}^{N-1} \sum_{m=1}^{k} |\vartheta_n^{(m)} - \vartheta_{n-1}^{(m)}| S_n^{(m)}.$$

The solution to this problem without transaction costs is the usual backward calculation method. Explicite simple solutions are possible in few cases only (see Schäl [19]), explicite solutions in the general case are given by Schweizer [22] and numerical solutions are given in [8]. With transaction costs, the path dependence of total transaction costs yields major numerical problems: even for moderate N, numerical results can be obtained only with the help of high speed computers. Explicite formulas for the case $N = 2$ can be found in section 2.1.

1.2 Continuous Time, Optimal Replication

Continuous time models are frequently considered as approximations of large scale discrete models, also with transaction costs. The limiting transactions costs generally depend on the distribution of increments in the discrete model, even if the first two moments agree: Boyle and Vorst [2] use Bernoulli distributed increments and obtain option prices which are different from Leland's [12] results for the case of normal increments. We shall use a cost term called *natural costs* which correspond to normal increments. Assume that the vector of stock prices satisfies the following stochastic differential equation:

$$dS(t) = a(S(t), t)dt + b(S(t), t)dW(t), \; S(0) = s_0. \tag{5}$$

Here, s_0 is fixed, $W(t)$ is standard $k-$variate Brownian motion, a is a smooth $k-$vector function and b a smooth $(k, k)-$matrix function. If $\vartheta(t)$ is a smooth function of t and $S(t)$, then the natural costs produced by the trading strategy $\vartheta(t)$ in the time interval $[0, T]$ is

$$\Gamma(\vartheta) = \gamma\sqrt{\frac{2}{\pi}}\int_0^T |\vartheta'(t)^T b(S(t), t)S(t)|dt. \tag{6}$$

Here, $\vartheta'(t)$ is the vector with components

$$\frac{\partial}{\partial x_i}\vartheta(\mathbf{x}; t)|_{\mathbf{x}=S(t)},$$

where $\vartheta(t) = \vartheta(\mathbf{x}; t)$ and $\mathbf{x} = (x_1, \ldots, x_k)$, and T denotes transposition. The term (6) is the limit if $[0, T]$ is divided into N equal intervals with endpoints as trading times, and proportional transaction costs with cost factor $\gamma/\sqrt{NT}$ are paid, and finally, $N \to \infty$:

$$\Gamma(\vartheta) = lim_N \frac{\gamma}{\sqrt{NT}} \sum_{n=1}^{N-1} |\vartheta(\frac{n}{N}T) - \vartheta(\frac{n-1}{N}T)| S(\frac{n}{N}T).$$

If cost factors go to zero at the rate $1/\sqrt{N}$, then total costs remain bounded. Recall that for constant cost factors, total costs will explode due to the fact that in the model, stock prices have paths with unbounded variation. Our term (6) is the one for normally distributed increments. For other distributions, the factor $\sqrt{2/\pi}$ has to be changed appropriately: for Bernoulli increments the factor is 1, and in general it is

$$E|X|/\sqrt{Var(X)},$$

where X is a generic increment.

In this paper, the admissible trading strategies $\vartheta(t)$ are assumed to depend on the past through $S(t)$ only. This Markovian dependence is possibly too restrictive, there might be trading strategies with a better performance. However, for more general trading strategies, transactions costs have to be defined differently. In order to avoid technical problems we do not consider these more general strategies here.

Using Itô calculus it can be shown (see section 2.3) that for the Black-Scholes model $k = 1$ and

$$dS(t) = aS(t)dt + \sigma S(t)dW(t) \tag{7}$$

there exist trading strategies $\phi_1(S(t), t)$ and $\phi_2(S(t), t)$ and real numbers C_1, C_2 such that

$$(S_T - K)^+ = C_1 + \int_0^T \phi_1(S(t), t)dS(t) - \gamma\sqrt{\frac{2}{\pi}} \int_0^T |\phi_1'(t)\sigma S^2(t)|dt \tag{8}$$

and

$$(S_T - K)^+ = C_2 + \int_0^T \phi_2(S(t), t)dS(t) + \gamma\sqrt{\frac{2}{\pi}} \int_0^T |\phi_2'(t)\sigma S^2(t)|dt \tag{9}$$

provided γ is not too large,

$$\sqrt{\frac{8}{\pi}}\gamma < \sigma.$$

C_1 is the usual Black-Scholes price for the European call with exercise price K, but with adjusted volatility

$$\sigma_1^2 = \sigma^2(1 + \sqrt{\frac{8}{\pi}}\gamma/\sigma),$$

and C_2 the corresponding price with volatility

$$\sigma_2^2 = \sigma^2(1 - \sqrt{\frac{8}{\pi}}\gamma/\sigma).$$

The numbers C_1 andC_2 are bounds for bid and ask prices of the call, respectively. They are the ones given by Leland [12]. Relations (8) and (9) hold in more general models (see section 2.3).

If C_0 is the Black-Scholes price for the call without transaction costs, then $C_1 \geq C_0 \geq C_2$.

An interesting improvement upon Leland's trading strategy has been suggested by Lott [13]. His strategy is an asymptotic boundary strategy which has the same transaction costs as Leland's, but a smaller matching risk. The asymptotic matching risk being zero for both strategies, the comparison is based on higher order terms.

In the discrete model of Boyle and Vorst [2] option replication is unique. The Black and Scholes continuous time model is the limit of these discrete models. It is, however, not obvious that also in continuous time the replication strategies ϕ_1 and ϕ_2 are unique. It can be shown that Leland's bounds are optimal in the following sense:

If ϕ_1^* is a replicating trading strategy for the seller with fair price C_1^*, then $C_1^* \geq C_1$.

If ϕ_2^* is a replicating trading strategy for the buyer with fair price C_2^*, then $C_2^* \geq C_2$.

The simple proof for this optimality is given in section 2.3.

1.3 Continuous Time, Hedging

Assume that $S(t)$ is a solution to (7), and let L be a stochastic liability. We want to find the optimal trading strategy $\vartheta(t)$ for which in the time interval $[0, T]$ the accumulated gains

$$G(\vartheta) = \int_0^T \vartheta(t) dS(t)$$

minus transactions costs, is as close to L as possible, in the sense that

$$E(L + \Gamma(\vartheta) - G(\vartheta))^2 \text{ is minimized.}$$

In the case of no transactions costs, a solution to this problem is given by Schweizer[21]. First, the liability L is represented as

$$L = \int_0^T g(t)dS(t) + M$$

with a random variable M orthogonal to the semimartingale $S(t)$:

$$EM \int_0^T \rho(t)dS(t) = 0$$

for all predictable processes $\rho(t)$ for which the expectation is defined. Then the optimal trading strategy is given by

$$\vartheta(t) = (L_t - P_t)a/(bS(t)),$$

where $P_t = \int_0^t \vartheta(s)dS(s)$ is the present value of the portfolio, and L_t is the conditional expectation of L at time t under the minimal martingale measure for $S(t), 0 \leq t \leq T$.

In order to incorporate transactions costs, we shall restrict ourselves to the case of constant L which implies $g(t) = 0$ and $L_t = L$. We assume that there exists an optimal hedging strategy $\vartheta_0(t)$ for which

$$L - \int_0^T \vartheta_0(t)dS(t)$$

is always nonnegative. Let $\vartheta_*(t)$ be the optimal hedging strategy without transactions costs for the model with volatility

$$\sigma_2^2 = \sigma^2(1 + \sqrt{\frac{8}{\pi}}\gamma/\sigma)$$

instead of σ^2. Then $\vartheta_*(t)$ is a smooth function of $S(t)$, i.e. $\Gamma(\vartheta_*)$ is defined, and $\vartheta_*(t)$ is an optimal hedging strategy for our model with transactions costs. A proof can be found in section 2.3. Basic for this result is the fact that $\vartheta_*(t)$, as a function of $S(t)$, is decreasing.

2 Technicalities

2.1 Computational Problems in Discrete Time

We first justify the algorithm for the computation of portfolio optimization with transactions costs. If $E(S_2^{(m)} - S_1^{(m)}) \geq \gamma S_1^{(m)}$, $m = 1, 2$, then the optimum is attained at the boundary of the constraint, and the assertion is easy. If not, then there are two possibilities: a) the unrestricted optimum satisfies the constraint: then this is our solution; and b) the unrestricted optimum does not exist or does not satisfy the constraint: then the optimum lies again on the boundary of the constraint, and the assertion is easy.
Our hedging problem in discrete time is computationally hard even for moderate numbers of periods N. We shall give formulas for the case $N = 2$ only. Let S_0, S_1, S_2 be the stock prices at the three time points, and for arbitrary but fixed numbers P and ϑ_0 define

$$U = L - P + \vartheta_0(S_2 - S_1),\ V^+ = (S_2 - S_1) - \gamma S_1,\ V^- = (S_2 - S_1) + \gamma S_1.$$

With these quantities we define the optimal number of shares ϑ_1 at time 1, if P is the present value of the portfolio at time 1, and ϑ_0 is the number of shares held at the beginning of the first period. So, if ϑ_0 is fixed, we have

$$P = \vartheta_0(S_1 - S_0).$$

We write E_1 for the conditional expectation at time 1.

1. If $E_1UV^+ > 0$ and $E_1UV^- \geq 0$ then $\vartheta_1 = \vartheta_0 + E_1UV^+/E(V^+)^2$.

2. If $E_1UV^- < 0$ and $E_1UV^+ \leq 0$ then $\vartheta_1 = \vartheta_0 + E_1UV^-/E(V^-)^2$.

3. If $E_1UV^+ \leq 0$ and $E_1UV^- \geq 0$ then $\vartheta_1 = \vartheta_0$.

4. If $EUV^+ > 0$ and $EUV^- < 0$ then ϑ_1 is equal to the expression in 1) if $E_1^2UV^+/E_1(V^+)^2 > E_1^2UV^-/E_1(V^-)^2$, and equal to the one in 2) elsewhere.

2.2 Going to the Limit

Leland's [12] proof for the replication strategy is based on a discrete time portfolio adjustment and a limit argument. His basic relation (2) on p. 1285

$$\frac{\Delta S}{S} = \mu\Delta t + \sigma z\sqrt{\Delta t} + O(\Delta t^{3/2})$$

is wrong: a term

$$\frac{1}{2}\sigma^2\Delta t(z^2 - 1)$$

is missing. See the stochastic Taylor formula in Kloeden and Platen [11], p. 351, (4.1). However, omission of this term does not affect the result of the trading strategy nor its transactions costs. This remark indicates that stochastic differential equations and the corresponding infinitesimal calculus (Itô calculus) need extra care. The monograph [11] is an excellent reference for discretization problems.

2.3 Itô Calculus

Basic relations for Itô calculus are Itô's Lemma and Itô-representation. Let $S(t)$ be given by the following $k-$variate stochastic differential equation

$$dS(t) = a(S(t), t)dt + b(S(t), t)dW(t), \;\; S(0) = s_0.$$

Itô's Lemma:
If $C(x, t)$ is smooth $k-$variate, then $C(S(t), t)$ satisfies the equation

$$\begin{aligned} dC(S(t), t) &= C_x(S(t), t)dS(t) + C_t(S(t), t)dt \\ &+\frac{1}{2}b^T(S(t), t)C_{xx}(S(t), t)b(S(t), t)dt. \end{aligned}$$

Here C_t is the $k-$vector of parial derivatives with respect to t, and C_x is the $(k, k)-$matrix of partial derivatives with respect to $\mathbf{x}$. The hypermatrix C_{xx} is (k, k, k) and consists of all second order partial derivatives of C_x.

Itô-representation:
If H is a real valued function depending on $S(t)$: $0 \leq t \leq T$, if a/b is bounded uniformly in t and $\mathbf{x}$, and if

$$E|H|^{2+\varepsilon} < \infty,$$

then there exists a real number h_0 and a progressivly measurable $k-$variate function h for which

$$H = h_0 + \int_0^T h(t)^T dS(t) \text{ almost surely.}$$

This relation yield the modern approach to the valuation of contingent claims (see, e.g. [16]): If H is the contingent claim, then h is the corresponding duplication strategy, and therefore h_0 is the arbitrage price for H in the absence of transaction costs. Both relations toghether also yield a special solution to arbitrage pricing with transaction costs (our equations (8) and (9)):
Let $k = 1$ and H be a convex function of $S(T)$. For $i = 0, 1, 2$ let $C^{(i)}(S(t), t)$ be the arbitrage price for H at time t for the model

$$dS(t) = a(S(t), t)dt + b_i(S(t), t)dW(t), \; S(0) = s_0.$$

Then $C^{(i)}(x, t)$ is smooth and satisfies (according to Itô's lemma)

$$\begin{aligned} dC^{(i)}(S(t), t) = {} & C_x^{(i)}(S(t), t)dS(t) + C_t^{(i)}(S(t), t)dt \\ & + \frac{1}{2} C_{xx}^{(i)}(S(t), t) b_i^2(S(t), t)dt. \end{aligned}$$

Under the martingale measure for $S(t)$ in model (i), $C^{(i)}(S(t), t)$ is a martingale. Therefore,

$$C_t^{(i)}(x, t) = -\frac{1}{2} C_{xx}^{(i)}(x, t) b_i^2(x, t).$$

Let $S(t)$ be driven by the stochastic differential equation with $i = 0$, and

$$\phi_i(x, t) = C_x^{(i)}(x, t), \; i = 1, 2.$$

Then

$$\phi_i'(x, t) = C_{xx}^{(i)}(x, t) \geq 0 \tag{10}$$

and - with $C_i = C^{(i)}(S(0), 0)$ and Itô's lemma

$$H = C_i + \int_0^T \phi_i(S(t), t)dS(t) + \int_0^T C_t^{(i)}(S(t), t)dS(t)$$

$$+\frac{1}{2}\int_0^T \phi_i'(S(t),t)b_0^2(S(t),t)dt$$
$$= C_i + \int_0^T \phi_i(S(t),t)dS(t)$$
$$+\frac{1}{2}\int_0^T \phi_i'(S(t),t)(b_0^2(S(t),t) - b_i^2(S(t),t))dt.$$

If $b_1^2(x,t)$ can be chosen such that

$$\frac{1}{2}(b_0^2(x,t) - b_1^2(x,t)) = -\gamma\sqrt{\frac{2}{\pi}}xb_0(x,t), \tag{11}$$

then ϕ_1 is a replicating trading strategy for the seller of H including transaction costs, and C_1 is an upper bound for the bid price of H. Similarly, if $b_2(x,t)$ can be chosen such that

$$\frac{1}{2}(b_0^2(x,t) - b_2^2(x,t)) = \gamma\sqrt{\frac{2}{\pi}}xb_0(x,t), \tag{12}$$

then ϕ_2 is a replicating trading strategy for the buyer of H including transaction costs, and C_2 is a lower bound for the ask price of H. The only restriction for a possible solution to (11) and (12) is the condition

$$b_0^2(x,t) - \gamma\sqrt{\frac{8}{\pi}}xb_0(x,t) > 0 \tag{13}$$

which holds in the Black and Scholes model if

$$\sqrt{\frac{8}{\pi}}\gamma < \sigma,$$

and in this case,

$$b_1^2(x,t) = x^2\sigma^2(1 + \gamma\sqrt{\frac{8}{\pi}}/\sigma),$$
$$b_2^2(x,t) = x^2\sigma^2(1 - \gamma\sqrt{\frac{8}{\pi}}/\sigma).$$

For this case, the solution is also given in Lott [13]. Other cases are possible, e.g.

$$b_0(x,t) = \sigma(t)xf(x)$$

with $\sqrt{\frac{8}{\pi}}\gamma < \sigma(t)$ and $f(x) \geq 1$. A proof for (10) can be found in Lott [13], Bemerkung 3.7.

Proof of optimality for Leland's bounds

Assume that there exists a replicating trading strategy ψ_1 and a real number C_1^* such that

$$H = C_1^* + \int_0^T \psi_1^*(S(t),t)dS(t) - \gamma\sqrt{\frac{2}{\pi}}\int_0^T |\psi_1'(S(t),t)b(S(t),t)|dt.$$

Then

$$H \leq C_1^* + \int_0^T \psi_1(S(t),t)dS(t) - \gamma\sqrt{\frac{2}{\pi}}\int_0^T \psi_1'(S(t),t)b(S(t),t)dt.$$

C_1^* is the arbitrage price of a contingent claim $\geq H$ under the model with diffusion b_1 without transaction costs. Since C_1 is the arbitrage price for H under this model, we have

$$C_1 \leq C_1^*.$$

Assume now that there exists a replicating trading strategy ψ_2 and a real number C_2^* such that

$$H = C_2^* + \int_0^T \psi_2(S(t),t)dS(t) + \gamma\sqrt{\frac{2}{\pi}}\int_0^T |\psi_2'(S(t),t)b(S(t),t)|dt.$$

Then

$$H \geq C_2^* + \int_0^T \psi_2(S(t),t)dS(t) + \gamma\sqrt{\frac{2}{\pi}}\int_0^T \psi_2'(S(t),t)b(S(t),t)dt.$$

C_2^* is the arbitrage price of a contingent claim $\leq H$ under the model with diffusion b_2 without transaction costs. Since C_2 is the arbitrage price for H under this model, we have

$$C_2^* \leq C_2.$$

Notice that only monotone trading strategies will lead to optimal bounds, and ϕ_i are unique in the class of monotone strategies (since Itô representations are unique).

2.4 Optimality of a Hedging Strategy

Let $S(t)$ be a solution of the stochastic differential equation

$$dS(t) = a(S(t), t)dt + b(S(t), t)dW(t),$$

and let $S^*(t)$ be a solution of the stochastic differential equation

$$dS(t) = a(S(t), t)dt + b_*(S(t), t)dW(t),$$

where

$$b_*^2(S(t), t) = b^2(S(t), t) + \gamma\sqrt{\frac{8}{\pi}}S(t)b(S(t), t),$$

without transactions costs. We are looking for a smooth hedging strategy $\vartheta(t)$ for which

$$E(L - \int_0^T \vartheta(t)dS(t) + \Gamma(\vartheta))^2$$

is minimized. Let $\vartheta_*(t)$ be the optimal hedging strategy for the model with volatility $b_*(S(t), t)$ without transactions costs. As a working hypothesis we assumed that an optimal hedging strategy $\vartheta_0(t)$ for our initial problem exists, and that for this the accumulated gain $\int_0^T \vartheta_0(t)dS(t)$ never exceeds L. Then

$$E(L - \int_0^T \vartheta_0(t)dS(t) + \Gamma(\vartheta_0))^2 \geq$$

$$E(L - \int_0^T \vartheta_0(t)dS(t) + \gamma\sqrt{\frac{2}{\pi}}\int_0^T \vartheta_0'(t)S(t)dt)^2 =$$

$$E(L - \int_0^T \vartheta_0(t)dS^*(t))^2 \geq$$

$$E(L - \int_0^T \vartheta_*(t)dS_*(t))^2 =$$

$$E(L - \int_0^T \vartheta_*(t)dS(t) + \gamma\sqrt{\frac{2}{\pi}}\int_0^T \vartheta_*'(t)S(t)dt)^2 =$$

$$E(L - \int_0^T \vartheta_*(t)dS(t) + \gamma(\vartheta_*))^2.$$

For the last equation we need the realtion $\vartheta'(t) < 0$ which can bee seen as follows:

$$\frac{d}{dS}((L - P_t)a/(bS)) = -\frac{d}{dS}P_t a/(bS) - (L - P_t)a/(bS^2).$$

Here we omitted the argument t for notational convenience. The first term is negative since P_t is an increasing function of $S(t)$, and the second is negative since $L - P_t$ is positive for all t : If

$$Z = exp(-\int_0^T a/(b^2 S(t))dW(t) - \frac{a^2}{2b^2}T)$$

and if Z_t is the conditional expectation of Z at time t under the minimal martingale measure, then

$$L - P_t = \alpha Z_t$$

for some constant $\alpha > 0$. This completes the proof.

3 Numerical Results

Numerical calculations show that optimal trading strategies with transaction costs behave "better" than those without transaction costs. The computation are based on two German stocks, Daimler (automobile) and RWE (electricity), in the time January 1, 1987, to February 2, 1988. We used portfolio optimization according to maximizing expected return under the constraint that the volatility is below 0.00005 (per day), with investment in Daimler and RWE as well as in a riskfree asset with return rate 8% per year. This is done with proportional transaction costs of 0.5 % , and without transaction costs. Expected returns and covariances were estimated each day from the last 105 days. Figure 1 gives the performance during the time of portfolio optimization (June 1, 1987, to February 2, 1988) with and without transactions costs, and figures 2 and 3 the trading activity of portfolio optimization with and without transaction costs ("without" means also that transaction costs are **not** paid). The naive portfolio is unchanged during the whole period, having 50% Daimler and 50% RWE of the initial whealth of 1,000 at the beginning.

We have very active trading in the case without, and less active trading in the case with transaction costs. Surprisingly, the performance is better with transactions costs of 2% compared to the performance with 1%. The performance of the strategies is rather good, but only by chance. Other periods would give other performance results. The effect of stable trading when transaction costs are included is, however, quite general.

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Fig.1: Performance of portfolio optimization with transaction costs

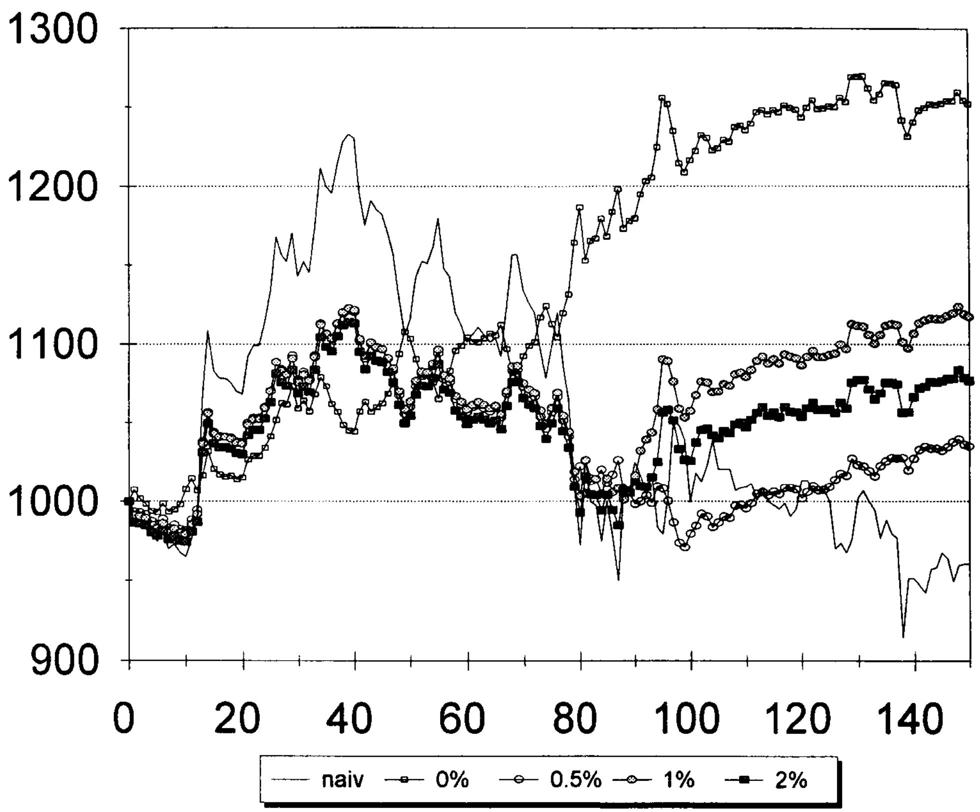

Fig.2: Trading activity in RWE

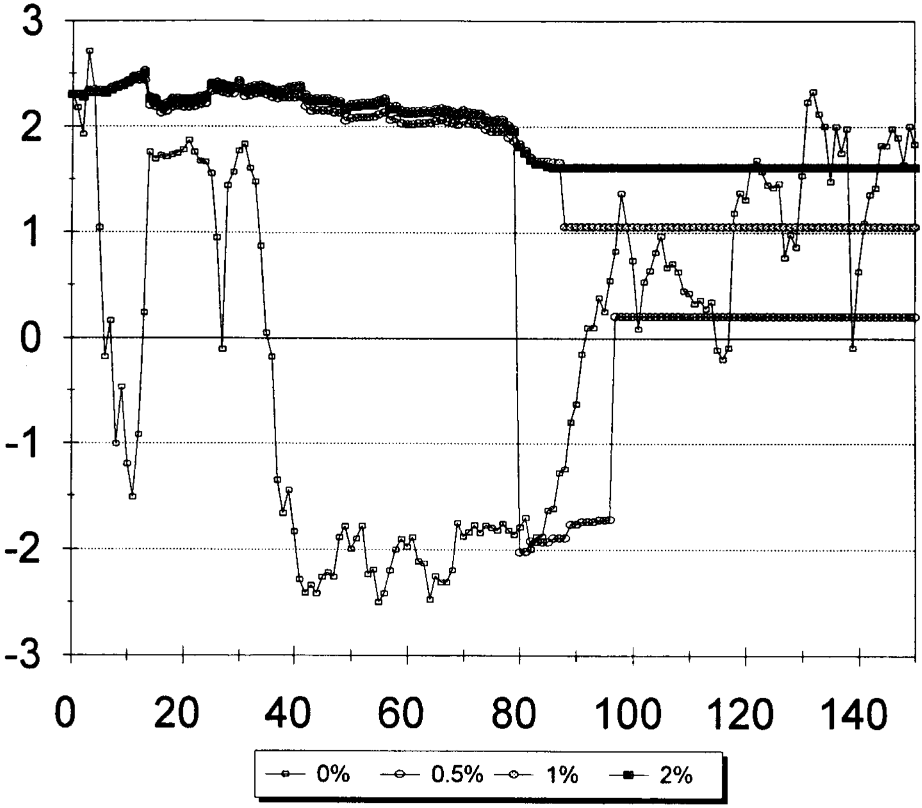

Fig.3: Trading activity in Daimler