

Shortfall Returns and Shortfall Risk

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Summary

Shortfall risk denotes the risk that a specified minimum return may not be earned by a financial investment. Measures of shortfall risk have recently attracted considerable interest in investment theory and investment management.

The present paper gives a general analysis of the probability law of shortfall returns and introduces a general conception of shortfall risk. Some analytical results are given and the multi-period case is considered.

In addition excess returns (returns exceeding the target return) are considered and a general conception of excess value is introduced. It is claimed that excess value is superior to the expected value as a measure of value of a financial investment.

Profits insuffisants et risque de découvert

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Résumé

Le risque de découvert est le risque que le montant fixé d'un revenu minimum ne soit pas atteint par un investissement financier. La mesure du risque de découvert a récemment fait l'objet d'un intérêt considérable dans le domaine de la théorie de l'investissement et dans celui la gestion des investissements.

Le présent article offre une analyse générale de la loi des probabilités de profits insuffisants et introduit le concept générale du risque de découvert. Il présente quelques résultats analytiques et une étude de cas s'étendant sur plusieurs périodes.

De plus, l'article traite des rendements excédentaires (revenus dépassant l'objectif fixé) et introduit un concept généralisé de la valeur excédentaire. Certains experts affirment que cette valeur excédentaire est supérieure à la valeur prévue comme mesure de la valeur d'un investissement financier.

I. INTRODUCTION

Shortfall-risk (downside-risk) denotes the risk that a specified minimum return (target return, threshold return, minimum acceptable return) may not be earned by a financial investment. Measures of shortfall risk have recently attracted considerable interest in investment theory and investment management. The main theoretical applications are portfolio optimization and capital asset pricing, the main practical applications are concerned with asset allocation.

Shortfall probabilities (mainly used as shortfall constraints within a portfolio optimization framework) have been considered by *Albrecht* (1993) and *Leibowitz et al.* (1989, 1990, 1991a, 1991b, 1992a, 1992b). *Harlow* (1991) treats shortfall probabilities, shortfall expectation and shortfall semivariance. *Bawa/Lindenberg* (1977), *Lee/Rao* (1988) and *Harlow/Rao* (1989) are concerned with capital asset pricing within a mean-shortfall semivariance framework. *Fishburn* (1977) gives a general analysis of mean-lower partial moments models. *Cariño/Fan* (1993a,b) introduce the concept of penalized shortfall and use piecewise linear penalization functions. Finally *Fishburn* (1984) investigates the utility theoretical basis of a conception of risk connected with below target returns.

The present paper gives a general analysis of the probability law of shortfall returns and introduces a general conception of shortfall risk. Some analytical results are given and the multi-period case is considered.

In addition excess returns (returns exceeding the target return) are considered and a general conception of excess value is introduced. It is claimed that excess value is superior to the expected value as a measure of value of a financial investment.

2. ONE-PERIOD SHORTFALL RETURNS

In the following let denote R the one-period return of a single financial investment or a portfolio of financial investments. We assume that R possesses a density function $f(r)$ and denote the distribution function by $F(r)$.

In addition let denote m the (one-period) minimum acceptable return for the investment as specified by the investor. Then R can be decomposed additively relative to the

minimum return as follows:

$$R = m + R_+(m) - R_-(m) \quad . \quad (1)$$

where

$$\begin{aligned} R_+(m) &:= (R - m)^+ := \max(R - m, 0) \\ &= \begin{cases} 0 & r \leq m \\ r - m & r > m \end{cases} \end{aligned} \quad (2a)$$

and

$$\begin{aligned} R_-(m) &:= (m - R)^+ := \max(m - R, 0) \\ &= \begin{cases} m - r & r \leq m \\ 0 & r > m \end{cases} . \end{aligned} \quad (2b)$$

Obviously $R_-(m)$ characterizes the shortfall magnitude of the realizations of R being below the target return, whereas $R_+(m)$ characterizes the excess magnitude of the realizations being above the target return. We will call $R_-(m)$ the $(m-)$ shortfall return, usually suppressing m in order not to overload the notation. $R_-(m)$ characterizes the extent of the danger of not achieving the desired minimum return.

For a later application we note that

$$\max(m - R, 0) = \max[(1 + m) - (1 + R), 0] \quad . \quad (3)$$

The shortfall return of R with respect to m is identical to the shortfall return of $1 + R$ with respect to $1 + m$.

In the following we will analyze some aspects of the probability law of shortfall returns.

We have $P(R_- = 0) = P(R > m) = 1 - F(m)$ and for $r > 0$ $P(R_- \leq r) = P(m - R \leq r) = P(R \geq m - r) = 1 - F(m - r)$. Let $F_-(r)$ denote the distribution function of R_- , then we have

$$F_-(r) = \begin{cases} 0 & r < 0 \\ 1 - F(m - r) & r \geq 0 \end{cases} \quad (4)$$

Let $f_-(r)$ denote the density function of R_- , then we have

$$f_-(r) = \begin{cases} 0 & r < 0 \\ 1 - F(m) & r = 0 \\ f(m - r) & r > 0 \end{cases} \quad (5)$$

The density function $f_-(r)$ has a point of discontinuity in $r = 0$, and we have

$$\lim_{x \downarrow 0} f_-(r) = f(m) \quad .$$

The distribution of R_- is concentrated at the left, the probability mass $P(R > m)$ is concentrated in $r = 0$. As a consequence we have

$$\int_0^{\infty} f_-(r) \, dr = F(m) \quad .$$

3. A GENERAL CONCEPTION OF SHORTFALL RISK

The shortfall return $R_{-}(m)$ contains all relevant information with respect to the potential of shortfall risk. However, in order to obtain a (one dimensional) risk measure we have to perform a valuation of $R_{-}(m)$. In analogy to the approach of statistical decision theory, where risk is measured as expected loss, we introduce a loss function (cost function) L in order to be able to evaluate the consequences of different magnitudes of shortfall. On this basis it is possible to develop a general measure of *shortfall risk* SR_m with respect to the minimum return m , as follows:

$$SR_m(R) = E[L(R_{-})] \quad . \quad (6)$$

We note, that the concept of a penalized shortfall, as introduced in *Cariño/Fan* (1993a) is identical to this conception. Using expression (5) we obtain

$$\begin{aligned} SR_m(R) &= \int_{-\infty}^{+\infty} L(r) f_{-}(r) dr \\ &= \int_0^{\infty} L(r) f(m - r) dr = \int_{-\infty}^m L(m - r) f(r) dr \quad . \end{aligned} \quad (7)$$

Expression (6) gives a general measure of shortfall risk and clarifies the structural approach. Specific risk measures can be derived on the basis of a specification of the loss function, which expresses the valuation of the decision maker of shortfalls of different magnitudes.

Choosing a power function, i.e. $L(x) = x^n$, as loss function, we obtain the moments of

R_- as specific measures of shortfall risk. We have:

$$E(R_-^n) = E[\max(m - R, 0)^n] = \int_{-\infty}^m (m - r)^n f(r) dr = LPM_n(m) \quad (8)$$

The n -th order moments of the shortfall return R_- are identical to the n -th order lower partial moments of the return R with respect to the minimum return m .

In the following chapter we look at three basic cases ($n = 0, 1, 2$) and give analytical results for normally and lognormally distributed returns.

4. SPECIFIC RISK MEASURES

We start from expression (8) and consider the cases $n = 0, 1, 2$. In case of $n = 0$ we obtain a measure of shortfall risk known as *shortfall probability*

$$\begin{aligned} SR_m &:= E(R_-^0) = \int_{-\infty}^m f(r) dr \\ &= P(R \leq m) = F(m) \end{aligned} \quad (9)$$

The risk measure shortfall probability is based on the loss function $L(x) \equiv 1$, which means that every shortfall is valued identically without consideration of its magnitude.

Example 1: Normally distributed returns

We consider a normally distributed return, i.e. $R \sim N(\mu, \sigma^2)$. We then have

$$SR_m = \Phi \left[\frac{m - \mu}{\sigma} \right] = \Phi(m_N), \quad (10a)$$

where $m_N := (m - \mu) / \sigma$.

Example 2: Lognormally distributed returns

We assume $\ln(1 + R) \sim N(\mu, \sigma^2)$. Considering $1 + R$ instead of R takes account of return realizations $r \geq -1$. With $q = 1 + m$ we have

$$\begin{aligned} SP_m &= P(R \leq m) = P(1 + R \leq q) \\ &= \Phi \left[\frac{\ln q - \mu}{\sigma} \right] = \Phi [q_{LN}] , \end{aligned} \quad (10b)$$

where $q_{LN} := (\ln q - \mu) / \sigma$.

In case of $n = 1$ we obtain

$$\begin{aligned} SE_m(R) &:= \int_{-\infty}^m (m - r) f(r) dr \\ &= m \int_{-\infty}^m f(r) dr - \int_{-\infty}^m r f(r) dr \\ &= m F(m) - E^m(R) , \end{aligned} \quad (11)$$

where (cf. the appendix) $E^m(R)$ denotes the first partial moment of R .

The risk measure given by expression (11) is the shortfall expectation (shortfall magnitude, target shortfall), a measure of the mean shortfall with respect to m . The loss function behind the shortfall expectation is given by $L(x) = x$, which means that the potential shortfalls are valued proportionally to their magnitude.

Example 1: (continued)

$$SE_m = (m - \mu) \Phi(m_N) + \sigma \varphi(m_N) . \quad (12a)$$

$E^m(R)$ for a normally distributed random variable R is obtained from expression (A2) of the appendix.

Example 2: (continued)

$$\begin{aligned}
 SE_m &= E[R_-(m)] = E[(1 + R)_-(q)] \\
 &= q \Phi [q_{LN}] - \exp(\mu + \frac{\sigma^2}{2}) \Phi [q_{LN} - \sigma] .
 \end{aligned}
 \tag{12b}$$

$E^m(R)$ for a lognormally distributed random variable R is obtained from expression (A4) of the appendix.

In case of $n = 2$ we obtain

$$\begin{aligned}
 SSV_m &:= \int_{-\infty}^m (m - r)^2 f(r) dr \\
 &= m^2 F(m) - 2 m \int_{-\infty}^m r f(r) dr + \int_{-\infty}^m r^2 f(r) dr \\
 &= m^2 F(m) - 2 m E^m(R) + E^m(R^2) ,
 \end{aligned}
 \tag{13}$$

where (cf. the appendix) $E^m(R^2)$ denotes the second partial moment of R .

The risk measure given by expression (13) is the *shortfall semivariance* (target semivariance), a measure for the mean quadratic variation of the possible shortfalls. The loss function behind (A3) is the quadratic loss function $L(x) = x^2$.

On the basis of the shortfall semivariance we can define the shortfall semi-standard-

deviation SST_m as

$$SST_m := + \sqrt{SSV_m} \quad (14)$$

Example 1: (continued)

Using expressions (A2) and (A3) from the appendix we obtain

$$SSV_m = [(m - \mu)^2 + \sigma^2] \Phi(m_N) + \sigma(m - \mu) \varphi(m_N) . \quad (15)$$

Example 2: (continued)

With $q := 1 + m$, and using expression (3) as well as expressions (A4) and (A5) from the appendix we obtain

$$\begin{aligned} SSV_m &= E[R_-^2(m)] = E[(1 + R_-)^2(q)] \\ &= q^2 \Phi[q_{LN}] \\ &\quad - 2 q \exp(\mu + \frac{\sigma^2}{2}) \Phi[q_{LN} - \sigma] \\ &\quad + \exp[2(\mu + \sigma^2)] \Phi[q_{LN} - 2\sigma] . \end{aligned} \quad (15b)$$

5. MULTI-PERIOD SHORTFALL RISK

The point of departure of our following considerations are successive one-period returns R_1, \dots, R_T . These can be transformed to an equivalent one period return on the basis of the *arithmetically annualized* return RA given by

$$RA = \frac{1}{T} (R_1 + \dots + R_T) , \quad (16)$$

or alternatively on the basis of the geometrically annualized return RG , given by

$$RG = \sqrt[T]{\prod_{t=1}^T (1 + R_t)} - 1 . \quad (17)$$

This implies that an analysis of multi-period shortfall can be performed on the basis of the shortfall returns (relative to a specified minimum return m)

$$RA_{-}(m) := \max(m - RA, 0) \quad (18a)$$

resp.

$$RG_{-}(m) := \max(m - RG, 0) . \quad (18b)$$

In addition we consider the *total return* RT at the end of the investment horizon given by

$$RT = \prod_{t=1}^T (1 + R_t) \quad (18 c)$$

and relating to a permanent accumulation of capital situation. In case that a minimum

total return MT is aimed at over the total investment horizon, *the total shortfall return* RT_- as given by

$$RT_- := \max(MT - RT, 0) \quad (19)$$

will be the adequate basis of the analysis.

Obviously relations (4), (5), and (8) remain valid, if F resp. f denote the distribution function resp. density function of the random variables RA resp. RG resp. RT . General definitions of multi-period shortfall risk are given by $E[L(RA_-)]$ resp. $E[L(RG_-)]$ resp. $E[L(RT_-)]$ on the basis of the introduction of a loss function L .

In the following, we give some analytical results for $L(x) \equiv 1$, $L(x) = x$, $L(x) = x^2$ assuming *stochastically independent* normally resp. lognormally distributed successive returns R_t , beginning with an analysis of the arithmetically annualized return RA according to (16).

Let $R_t \sim N(\mu_t, \sigma_t)$ and $\bar{\mu}_T = (\sum_{t=1}^T \mu_t) / T$ resp. $\bar{\sigma}_T^2 = (\sum_{t=1}^T \sigma_t^2) / T^2$. Then we have

$R_A \sim N(\bar{\mu}_T, \bar{\sigma}_T)$ resp. $R_A \sim N(\mu, \sigma/\sqrt{T})$ in case of identically distributed random variables. We obtain the following expressions for the shortfall probability SP_A , the shortfall expectation SE_A and the shortfall semivariance SV_A for the Return RA with respect to the minimum return m ($\bar{m}_N := (m - \bar{\mu}_T) / \bar{\sigma}_T$):

$$\begin{aligned}
SP_A &= \Phi(\bar{m}_N) \\
SE_A &= (m - \bar{\mu}_T) \Phi(\bar{m}_N) + \bar{\sigma}_T \varphi(\bar{m}_N) \\
SSV_A &= \left[(m - \bar{\mu}_T)^2 + \bar{\sigma}_T^2 \right] \Phi(\bar{m}_N) \\
&\quad + \bar{\sigma}_T (m - \bar{\mu}_T) \varphi(\bar{m}_N) .
\end{aligned} \tag{20}$$

In case of the geometrically annualized return RG resp. the total return RT we assume lognormally distributed one-period returns, i.e. $1 + R_A \sim LN(\mu, \sigma)$. We then have

$\ln(1 + RG) \sim N(\bar{\mu}_T, \bar{\sigma}_T^2)$ resp. $\ln(1 + RG) \sim N(\mu, \sigma^2 / T)$ in the i.i.d. case. With $\mu_T = \sum \mu_i$ resp. $\sigma_T^2 = \sum \sigma_i^2$ we have $\ln(1 + RT) \sim N(\mu_T, \sigma_T^2)$ in the i.i.d. case.

We obtain the following expressions for the shortfall probability SP_G , the shortfall expectation SE_G and the shortfall semivariance SV_G for the return RG with respect to

a minimum return m ($q := 1 + m$, $\bar{q}_{LN} := (\ln q - \bar{\mu}_T) / \bar{\sigma}_T$):

$$\begin{aligned}
SP_G &= \Phi(\bar{q}_{LN}) \\
SE_G &= q \Phi(\bar{q}_{LN}) \\
&\quad - \exp\left(\bar{\mu}_T + \frac{1}{2}\bar{\sigma}_T^2\right) \Phi(\bar{q}_{LN} - \bar{\sigma}_T) \\
SSV_G &= q^2 \Phi(\bar{q}_{LN}) \\
&\quad - 2q \exp\left(\bar{\mu}_T + \frac{1}{2}\bar{\sigma}_T^2\right) \Phi(\bar{q}_{LN} - \bar{\sigma}_T) \\
&\quad + \exp\left[2(\bar{\mu}_T + \bar{\sigma}_T^2)\right] \Phi(\bar{q}_{LN} - 2\bar{\sigma}_T) .
\end{aligned} \tag{21}$$

Finally we obtain the following expressions for the shortfall probability SP_T , the

shortfall expectation SE_T and the shortfall semivariance SSV_T for the return RT with respect to the minimum total return MT ($QT := 1 + MT$, $QT_{LN} := (\ln QT - \mu_T) / \sigma_T$):

$$\begin{aligned}
 SP_T &= \Phi(QT_{LN}) \\
 SE_T &= QT \Phi(QT_{LN}) \\
 &\quad - \exp(\mu_T + \frac{1}{2} \sigma_T^2) \Phi(QT_{LN} - \sigma_T) \\
 SSV_T &= QT^2 \Phi(QT_{LN}) \\
 &\quad - 2 QT \exp(\mu_T + \frac{1}{2} \sigma_T^2) \Phi(QT_{LN} - \sigma_T) \\
 &\quad + \exp[2(\mu_T + \sigma_T^2)] \Phi(QT_{LN} - 2 \sigma_T) .
 \end{aligned} \tag{22}$$

6. SHORTFALL CONTROL

On the basis of the general definition of shortfall risk according to (6) we propose as an adequate *control criterion* for the control of the shortfall risk the following restriction:

$$E[L(R)] \leq C . \tag{23}$$

The specification of the tolerated risk magnitude C depends on the selection of the loss function. In the case $L(x) \equiv 1$, where the shortfall risk is measured by the shortfall probability $SP_m(R)$ the following traditional control criterium is obtained

$$P(R \leq m) \leq \varepsilon \quad , \quad (24)$$

the shortfall probability is restricted by a small probability ε . In Albrecht (1993) a number of results are obtained on the basis of this conception of shortfall control.

In case of $L(x) = x$, where the shortfall risk is measured by the shortfall expectation $SE_m(R)$ the following criterion seems to be appropriate ($0 < c < 1$):

$$SE_m(R) \leq c E(R) \quad . \quad (25)$$

The shortfall expectation should not be exceed a certain percentage of the entire expected return.

7. EXCESS RETURNS AND EXCESS VALUE

The preceding analysis was concentrated on the development of a general conception of risk, where risk was associated with the danger of not achieving a minimal acceptable return. While in decision theoretic and financial literature the question of the adequate measurement of risk is heavily discussed, there seems to be an unanimous agreement that the expected value is a good measure of value (return). However, within a target return framework this seems not to be a consistent approach for within the measure for value possible shortfalls are considered as well. Therefore we propose to proceed in complete analogy to the case of shortfall risk, namely to consider the excess returns $R_+(m)$ according to expression (2a) and perform a valuation of the excess returns on the basis of a value function G . By doing so we obtain the following general measure of excess value XV with respect to the target return m

$$XV_m(R) = E[G(R_+)] \quad . \quad (26)$$

In analogy to (5) the density $f_+(r)$ of $R_+(m)$ is given by

$$f_+(r) = \begin{cases} 0 & x < 0 \\ F(r) & x = 0 \\ f(r - m) & x > 0 \end{cases} , \quad (27)$$

which gives

$$\begin{aligned} XV_m(R) &= \int_{-\infty}^{+\infty} G(r) f_+(r) dr \\ &= \int_0^{\infty} G(r) f(r - m) dr = \int_m^{\infty} G(r - m) f(r) dr . \end{aligned} \quad (28)$$

As a possible alternative to the expected value we propose the case $G(x) = x$, which leads to the *excess expectation* XE_m as a measure of value:

$$XE_m(R) = E[R_+] = \int_m^{\infty} (r - m) f(r) dr . \quad (29)$$

Clearly the expected value possesses significant advantages with respect to calculation, e.g. the property of linearity. However, this can not be the dominant criterion for the question of the development of an adequate conception of value.

8. CONSEQUENCES FOR MODELS OF CHOICE

On the basis of our results in chapters 3 and 7 we propose to analyze preference functionals of the following type

$$\Phi(X) = H[E[L(X_-)], E[G(X_+)]] \quad (30)$$

resp. alternatively of the type

$$\Phi(X) = H[E[L(X_-)], E(X)] \quad (31)$$

The preference functionals (30) resp. (31) are examples of *risk-value models*, for an overview cf. *Sarin/Weber* (1993), where the valuation of a random variable depends on a measure R of risk and a measure V of value, i.e.

$$\Phi(X) = H[R(X), V(X)] \quad (32)$$

The portfolio selection theory of Markowitz for example is based on a risk value approach, where risk is measured by the variance and value by the expected value, i.e. on a mean-variance approach. The function H quantifies the trade-off between risk and value.

Let us note that for models of type (31) with $L(x) = x^n$, *Fishburn* (1977) discusses the congruence with the expected utility approach of *von Neumann and Morgenstern*, but questions of congruence are not the main concern of the present paper.

In case the trade-off function $H(x, y)$ is completely specified, then the problem of choice, the determination of the optimal decision is solved by the following optimization problem

$$\Phi(X) \rightarrow \max ! \quad (33)$$

An interesting specification of H leading to a model with a simple structure would be $H(x, y) = y - x$ giving

$$\Phi(X) = E[G(X_+) - L(X_-)] \rightarrow \max ! \quad (34)$$

resp.

$$\Phi(X) = E[X - L(X_-)] \rightarrow \max ! \quad (35)$$

The valuation in this case depends on the (Euklidian) distance between value and risk. The larger this distance, the higher the value of R.

An alternative model of choice explicitly paying attention to a necessity of shortfall control would be

$$\begin{aligned} \Phi(X) &= H[V(X), R(X)] \rightarrow \max ! \\ \text{subject to} & \\ E[L(X_-)] &\leq C \quad . \end{aligned} \quad (36)$$

9. CONSEQUENCES FOR PORTFOLIO SELECTION

The portfolio selection problem is a special problem of choice. The random variable X of chapter 8, which has to be evaluated, here is identical to the *portfolio return* RP

$$RP = \sum_{i=1}^n x_i R_i \quad , \quad (37)$$

where R_1, \dots, R_n are the one-period returns of n securities $i = 1, \dots, n$ and x_i ($0 \leq x_i \leq 1$)

denotes the percentage of investment into security i . The traditional approach of solving the portfolio selection problem is solving the optimization problem (33) with $R(X) = \text{Var}(X)$ and $V(X) = E(X)$ (mean-variance model).

In contrast to $E(X)$ and $\text{Var}(X)$ the measures of shortfall risk as well as excess value presented in this paper are not linear functionals, so we have to work directly with RP. For example the *portfolio shortfall return* $RP_{(m)}$ can be defined as follows

$$RP_{(m)} = \max(m - RP, 0) \quad . \quad (38)$$

Let us assume $R_i \sim N(\mu_i, \sigma_i^2)$ and $\rho(R_i, R_j) = \rho_{ij}$.

With $\mu_p = \sum_{i=1}^n x_i \mu_i$ and $\sigma_p^2 = \sum_{i=1}^n x_i^2 \sigma_i^2 + 2 \sum_{i < j} x_i x_j \rho_{ij} \sigma_i \sigma_j$, we then

have

$$RP \sim N(\mu_p, \sigma_p) \quad . \quad (39)$$

As a consequence we obtain for the shortfall probability SP_p , the shortfall expectation SE_p and the shortfall semivariance SSV_p of the portfolio return R_p with respect to the minimum acceptable return m the following results ($m_p := (m - \mu_p) / \sigma_p$):

$$\begin{aligned} SP_p &= \Phi(m_p) \\ SE_p &= (m - \mu_p) \Phi(m_p) + \sigma_p \varphi(m_p) \\ SSV_p &= [(m - \mu_p)^2 + \sigma_p^2] \Phi(m_p) \\ &\quad + \sigma_p (m - \mu_p) \varphi(m_p) \quad . \end{aligned} \quad (40)$$

Similar expressions can be obtained for value measures of the portfolio excess return $RP_{+}(m)$, so a new approach to the portfolio selection problem could e.g. be based on expression (34), i.e.

$$E[G[RP_+(m)] - L[RP_-(m)]] \rightarrow \max ! \quad (41)$$

We want to note that *Cariño/Fan* (1993 a) propose a related approach to the portfolio selection problem based on (35), namely (in our notation)

$$E[RP - L[RP_-(m)]] \quad (42)$$

using a piecewise linear loss function.

In a series of papers *Leibowitz et al.* (1989, 1990, 1991 a, 1991 b, 1992 a, 1992 b) basically use an approach to the portfolio selection problem based on a special case of (36), namely

$$\Phi(X) = H[E(X), Var(X)] \rightarrow \max !$$

subject to (43)

$$P(RP \leq m) \leq \varepsilon .$$

In a model of this kind the classical mean-variance approach is combined with the criterion of shortfall control (more then one constraint is possible).

Finally other authors, e.g. *Hogan/Warren* (1972), *Lee/Rao* (1988), *Cariño/Fan* (1993 a, b), are concerned with the computation of the efficient boundary in special models of

the type (31) which leads to the following *parametric* optimization problem.

$$\begin{aligned}
 & E[L(RP)] \rightarrow \min ! \\
 & \text{subject to} \\
 & E(RP) \geq r .
 \end{aligned}
 \tag{44}$$

10. APPENDIX: PARTIAL MOMENTS

Let X denote a random variable. We assume that X possesses a density function $f(x)$. The *partial moment of n -th order from a to b of X* ($-\infty \leq a < b \leq \infty$) then is given by

$$E_a^b(X^n) := \int_a^b x^n f(x) dx .
 \tag{A1}$$

In the present paper we only are concerned with partial moments over the area $(-\infty, b]$, for this case we use the notation $E^b(X^n)$.

In the following we give expressions for the first and second partial moments of normally as well as lognormally distributed random variables. For the determination of partial moments in general cf. WINKLER et al. (1972).

Let the random variable X follow a normal distribution, $X \sim N(\mu, \sigma)$. Let $\Phi(x)$ resp. $\varphi(x)$ denote the distribution function resp. the density function of a random variable following a *standard normal* distribution and let X_N denote the following standardized quantity $x_N = (x - \mu) / \sigma$.

For the first partial moment $E^b(X)$ we then have

$$E^b(X) = \mu \Phi(b_N) - \sigma \varphi(b_N) .
 \tag{A2}$$

For the second partial moment we obtain

$$E^b(X^2) = (\mu^2 + \sigma^2) \Phi(b_N) - \sigma(\mu + b) \varphi(b_N) \quad . \quad (\text{A3})$$

Now we come to the case of a lognormally distributed random variable X , i.e. $X \sim \text{LN}(\mu, \sigma^2)$ resp. $\ln X \sim N(\mu, \sigma^2)$. Let x_{LN} denote the following quantity $x_{LN} = (\ln x - \mu)/\sigma$. For the first partial moment $E^b(X)$ we then have

$$E^b(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \Phi(b_{LN} - \sigma) \quad . \quad (\text{A4})$$

For the second partial moment we obtain

$$E^b(X^2) = \exp(2(\mu + \sigma^2)) \Phi(b_{LN} - 2\sigma) \quad . \quad (\text{A5})$$

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