

Alternative Investments and Risk Measurement

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Abstract

In this paper we will present a model to generate simultaneous random returns on traditional and alternative investments. We will use this method to estimate the risks associated with the inclusion of alternative investments in a traditional investment portfolio, where the investment horizon is one month. As we will show, a rather sophisticated model is needed to get an adequate impression of the inherent risk of investing in alternative investments. The traditional portfolio consists of bonds and equity and the returns of this portfolio have been modeled with the normal distribution. Alternative investments include, among others, hedge funds, high yield bonds, commodities, convertibles, real estate and emerging markets debt. The returns of the alternative investments have been modeled with the normal inverse gaussian distribution, as this distribution allows for skewness and heavy tails. The dependence between the returns of the traditional portfolio and the returns of the portfolio consisting of the alternative investments has been modeled with the student copula.

The risks associated with the portfolio of traditional and alternative investments are measured with Value at Risk and Expected Shortfall. These risk measures are computed with Monte Carlo simulation. With the use of the normal inverse gaussian distribution instead of the normal distribution, the Value at Risk and Expected Shortfall are much larger.

Keywords

Market risk; Alternative investments; Normal Inverse Gaussian Distribution; Copula; Value at Risk; Expected Shortfall; Generating random variates

Introduction

Investing in alternative investments has proved to be a good way to diversify a traditional portfolio consisting of equity and investment grade bonds. However, the inclusion of alternative investments in a traditional portfolio creates problems in the assessment of the risks of the new portfolio, since this introduces skewness and kurtosis (fat tails) to the probability distribution of the returns of a combined portfolio. The returns of a traditional portfolio, consisting of bonds and equity offer no real problems in risk management because the distribution of these returns can be well modeled with the normal distribution. The historical returns of many alternative investments however show characteristics that are highly incompatible with the normal distribution.

We have tested the assumption of a normal distribution for the monthly returns of the traditional and alternative investments with the Bera-Jarque statistic [Bera87], using monthly dollar denominated indices. We also have included three portfolios of equity and bonds: “trad 25-75”, “trad 50-50” and “trad 75-25”. The first number is the percentage of equity in the portfolio and the second number is the percentage of bonds. We further included two portfolios of alternatives: “alt 1” and “alt 2”. The first consists of equal parts hedge funds, commodities, high yield bonds, convertibles, real estate and emerging markets debt. The second alternative portfolio consists of large portions of convertibles and commodities (both 30%) and equal parts hedge funds, high yield bonds, real estate and emerging markets debt.

The Bera-Jarque test uses the skewness and the (excess) kurtosis of the returns to test the assumption of normality. If $\hat{\gamma}_3$ represents the skewness and $\hat{\gamma}_4$ represents the kurtosis of the returns, than the Bera-Jarque statistic is defined as:

$$\hat{T} = n(\hat{\gamma}_3^2 / 6 + \hat{\gamma}_4^2 / 24)$$

with n the number of returns in the observed period. If the returns follow the normal distribution, the Bera-Jarque statistic has a asymptotic χ_2^2 distribution. The confidence level of the test is 95%, the critical value by this confidence is 5.99.

We have used the following data:

equity	MSCI World total return
bonds	Salomon World index total return
hedge funds	Hedge Fund Research fund weighted composite index
commodities	Goldman Sachs Commodity Index
high yield	Merrill Lynch high yield 175 total return index
convertibles	IA Monthly Convertible total return index (January 1994 – January 1998) Goldman Sachs global convertible total return index (February 1998 – March. 2002)
real estate	Salomon Smith Barney total return index
em. markets	J.P. Morgan EM. Markets Bond Index + Composite – Return Ind. (OFCL)

The longest period for which we were able to obtain data of all indices was January 1994 to March 2002.

Bera-Jarque test

	skewness	kurtosis	\hat{t}	reject normality
equity	-0.62	0.67	8.19	yes
bonds	0.41	0.67	4.61	no
trad 25-75	0.08	0.41	0.81	no
trad 50-50	-0.27	-0.12	1.30	no
trad 75-25	-0.50	0.26	4.39	no
hedge funds	-0.55	2.81	37.54	yes
commodities	0.38	0.44	3.15	no
high yield	-0.76	3.35	55.88	yes
convertibles	0.05	0.05	0.05	no
real estate	-0.50	0.70	6.11	yes
em. markets	-2.04	9.27	422.85	yes
alt 1	-0.89	2.89	47.63	yes
alt 2	-0.16	0.89	3.68	no

Period: January 1994 - March 2002

Although normality is rejected for equity returns, normality cannot be rejected for a combination of equity and bonds. The skewness and kurtosis of the traditional portfolio are even closer to zero than the skewness and kurtosis of the returns on just bonds alone.

For hedge funds, high yield bonds, emerging markets debt and real estate, the situation is different. These assets have highly nonnormal distributed returns. These returns exhibit negative skewness, so negative returns are on average larger than the positive returns. Furthermore, they have very fat tails, which means that there are more and greater outliers. Even if we combine them, the distribution of the resulting returns is still skewed and has fat tails. Only if they are combined with large fractions of convertibles and commodities, the distribution of the returns will be close to the normal distribution.

The aim of this paper is to develop a method which is capable to assess the risks of a portfolio of traditional and alternative investments. The risks we want to quantify are the risks of very unlikely and unfavorable outcomes of the investments: excessive negative monthly returns. The appropriate risk measures are Value at Risk and Expected Shortfall. Value at Risk gives the maximum possible loss over a specified time horizon of a portfolio with a certain specified confidence. The Expected Shortfall gives the average return on a portfolio over a specified time horizon, given that the return will be beneath a certain lower bound. With these two risk measures, one has information about the worst outcomes of an investment.

Why should a new model be developed? Value at Risk and Expected Shortfall can be directly estimated from the history of appropriate indices. The only problem is the required length of the historic period. If one is interested in the risks associated with large negative returns that will occur less than once in a hundred times, more than hundred months of history are required. For a reliable estimation several thousands of returns are required. Since indices for most alternative investments go back no longer than the early nineties, a simulation approach is needed.

In order to simulate, we will construct a model that provides an adequate reflection of the relevant characteristics of the returns of traditional and alternative investments.

The model

We will model two portfolios: the traditional portfolio and the alternative portfolio. For these two portfolios the distribution of the returns and the dependence structure have to be specified.

As we have seen in the introduction, the assumption of normally distributed monthly returns cannot be rejected for a portfolio consisting of investment grade bonds and equity. The skewness and kurtosis were in fact very close to zero, so the normal distribution is a good model for the returns of the traditional portfolio. Furthermore, we assume that the returns are identically and independently distributed, which is a valid assumption for monthly returns. The indices we used showed no significant autocorrelation for either equity or bonds.

We will let the proportion of bonds or equity in the portfolio unspecified. We only need the mean μ_{trad} and standard deviation σ_{trad} of the traditional portfolio.

For the alternative portfolio, the normal distribution is not the appropriate distribution to model the returns with, due to negative skewness and fat tails, which increase the likelihood of large negative returns. Therefore we need a distribution with adjustable skewness and kurtosis, which gives a higher probability to outliers than the normal distribution. We choose the normal inverse gaussian distribution to model the distribution of the returns of the alternative portfolio with. This distribution is used in [Prau99] and [Bølv00] to model equity returns. The normal inverse gaussian distribution is included in the class of generalized hyperbolic distributions, introduced by [Barn77].

The normal inverse gaussian distribution has four parameters: a location parameter μ , a scale parameter δ , and two shape parameters α and β . We denote the normal inverse gaussian distribution by $\text{NIG}(\mu, \delta, \alpha, \beta)$. If we set $\mu = 0$ and $\delta = 1$, we have the standard normal inverse gaussian distribution, which we denote by $\text{SNIG}(\alpha, \beta)$.

The probability density function of the $\text{NIG}(\mu, \delta, \alpha, \beta)$ distribution is defined as:

$$f_{\text{NIG}}(x; \mu, \delta, \alpha, \beta) = \frac{\alpha}{\pi\delta} \exp\left(\sqrt{\alpha^2 - \beta^2} + \beta \frac{x - \mu}{\delta}\right) \frac{K_1\left(\alpha \sqrt{1 + \left(\frac{x - \mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x - \mu}{\delta}\right)^2}}$$

with the parameter restrictions:

$$\mu \in (-\infty, \infty)$$

$$\delta > 0$$

$$\alpha > 0$$

$$|\beta| \leq \alpha$$

The function K_ν is the modified Bessel function of the third kind with index ν :

$$K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp\left(-\frac{1}{2}z(y + y^{-1})\right) dy$$

This probability distribution has considerably fatter tails than the normal distribution as can be seen by the fact that $f_{\text{SNIG}}(x) \sim \text{const.} |x|^{-\frac{3}{2}} \exp(-\alpha|x| + \beta x)$ if $x \rightarrow \pm\infty$ as opposed to the tail behavior of the standard normal density: $\varphi(x) \sim \text{const.} \exp(-\frac{1}{2}x^2)$.

There is a simple relation between the normal inverse gaussian distribution and the standard normal inverse gaussian distribution:

$$X \sim \text{SNIG}(\alpha, \beta) \Leftrightarrow \mu + \delta X \sim \text{NIG}(\mu, \delta, \alpha, \beta)$$

In appendix 1 we derive formulas for the mean, variance, skewness and kurtosis of the normal inverse gaussian distribution. It follows that $\text{Kurt}[X] > 0$, so this distribution has more kurtosis than the normal distribution. In fact, the kurtosis can assume every desired positive value. Skewness is limited by the kurtosis, since $|\text{Skew}[X]| \leq \sqrt{\frac{3}{5} \text{Kurt}[X]}$.

With explicit formulas for the mean, variance, skewness and kurtosis, we have a simple way to estimate the parameters μ , δ , α and β by solving the equations of the mean, variance, skewness and kurtosis (see appendix 1). For the remainder of this paper we consider the parameters μ_{alt} , δ_{alt} , α_{alt} and β_{alt} of the distribution of the portfolio of alternative investments as given.

We now focus on the modeling of the dependence between the traditional and the alternative portfolio. It is a well known fact that simultaneous large negative returns on different asset classes tend to occur far more often than models based on the multivariate distribution predict. For example, most asset classes showed large negative returns in August 1998 and September 2001. This phenomenon is called tail dependence. Mathematically, two stochastic variables R_1 and R_2 with distribution functions F_1 and F_2 have (lower) tail dependence if:

$$\lambda_L = \lim_{\alpha \downarrow 0} \text{P}[R_2 \leq F_2(\alpha) \mid R_1 \leq F_1(\alpha)] > 0$$

Tail dependence is a property of the dependence structure, for which the correct name is copula, and is independent from the marginal distributions F_1 and F_2 . The copula associated with two stochastic variables R_1 and R_2 with distribution functions F_1 and F_2 is the bivariate distribution function of the two dimensional uniform stochastic variable $(F_1(R_1), F_2(R_2))$. With copulas, the dependence structure can be modeled independent from the marginal distribution functions.ⁱ

In order not to underestimate the risks associated with the occurrence of simultaneous outliers of both the returns of the traditional and the alternative portfolio, we will model the dependence between the two portfolio with a copula that has tail dependence.

Since the bivariate normal distribution does not possess this quality, we cannot copy the copula of this distribution. Therefore we use the copula of the bivariate student distribution, the so-called student copula, which does possess tail dependence and has some nice computational properties.

The copula of the bivariate student distribution with ν degrees of freedom and correlation ρ is defined as:

$$C^\nu(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \left(1 + \frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{\nu(1-\rho^2)} \right)^{-\frac{\nu+2}{2}} ds_1 ds_2$$

with t_ν^{-1} the inverse student distribution function with ν degrees of freedom.

The tail dependence that goes with the student copula is:ⁱⁱ

$$\lambda_L = 2t_{v+1} \left(-\sqrt{(v+1) \left(\frac{1-\rho}{1+\rho} \right)} \right) > 0$$

From this formula, we see that tail dependence decreases with a increase in the degrees of freedom. So smaller degrees of freedom give a higher probability of simultaneous extreme negative returns. For $v \rightarrow \infty$ this copula converges to the gaussian copula from the bivariate normal distribution with correlation ρ and tail dependence $\lambda_L = 0$.

We now have all required ingredients for a realistic model of the monthly returns on a traditional and an alternative portfolio. We have specified the marginal distributions of the returns and the dependence between the returns on the different portfolios. In this paper, we will not give attention to the estimation of the parameters of the marginal distributions and the student copula. Estimation is a complex topic, which deserves its own discussion. ⁱⁱⁱ

Computation of the risk measures

Our aim is the evaluation of the risk measures Value at Risk and Expected Shortfall for a portfolio consisting of $100p\%$ traditional investments and $100(1-p)\%$ alternative investments, where our investment horizon is month ahead. Let $R_{\text{trad}} \sim N(\mu_{\text{trad}}, \sigma_{\text{trad}}^2)$ be the stochastic monthly return on the traditional portfolio and $R_{\text{alt}} \sim \text{NIG}(\mu_{\text{alt}}, \delta_{\text{alt}}, \alpha_{\text{alt}}, \beta_{\text{alt}})$ be the stochastic monthly return on the alternative portfolio. The stochastic monthly return on the total portfolio is therefore $R_{\text{tot}} = pR_{\text{trad}} + (1-p)R_{\text{alt}}$. The distribution function of these returns is F_{tot} . Since this distribution function is continuous, the Value at Risk with $1-\alpha$ confidence is defined according to the following simple formula: $\text{VaR}_\alpha = F_{\text{tot}}^{-1}(\alpha)$. So the probability of a return on the portfolio below VaR_α is α . The Expected Shortfall with $1-\alpha$ confidence is the expected return on the portfolio, given that this return is below VaR_α , more formally: $\text{ES}_\alpha = E[R_{\text{tot}} | R_{\text{tot}} \leq \text{VaR}_\alpha]$.

With our choices for the marginal distributions and the copula, these risk measures cannot be computed analytically, so we will use Monte Carlo. We shall develop methods to generate N random variates $(r_{\text{trad}}^1, r_{\text{alt}}^1), \dots, (r_{\text{trad}}^N, r_{\text{alt}}^N)$ from the correct bivariate distribution and compute the random portfolio returns $r_{\text{tot}}^i = pr_{\text{trad}}^i + (1-p)r_{\text{alt}}^i$. Next, we order these returns from the lowest to the highest return to end with the numbers $r_{\text{tot}}^{(1)}, \dots, r_{\text{tot}}^{(N)}$. We are now able to compute VaR_α and ES_α . Define $n_\alpha = \lfloor \alpha N \rfloor$, with $\lfloor \cdot \rfloor$ the function that rounds down towards the nearest integer, we assume $n_\alpha > 0$. We then have:

$$\text{VaR}_\alpha = r_{\text{tot}}^{(n_\alpha)}$$

and

$$\text{ES}_\alpha = n_\alpha^{-1} \sum_{i=1}^{n_\alpha} r_{\text{tot}}^{(i)}$$

These quantities become of course more accurate with increasing N .

The only problem left is the generation of the numbers $(r_{\text{trad}}^i, r_{\text{alt}}^i)$. The most direct way to do this is first to generate numbers $(u_{\text{trad}}^i, u_{\text{alt}}^i)$ with the student copula, which is in fact a bivariate uniform probability distribution and then transform these numbers with the inverse marginal distribution functions:

$$(r_{\text{trad}}^i, r_{\text{alt}}^i) = (F_{\text{trad}}^{-1}(u_{\text{trad}}^i), F_{\text{alt}}^{-1}(u_{\text{alt}}^i))$$

While it is certainly possible to generate random variates with this approach, we choose a slightly different approach in order to circumvent the computation of F_{alt}^{-1} , which is quite hard to evaluate. We first generate the random variates r_{alt}^i , appendix 2 describes how, scale them to ensure they have the right mean and variance and transform according to the formula:

$$t_{\text{alt}}^i = t_v^{-1} \left(F_{\text{SNIG}} \left(\frac{r_{\text{alt}}^i - \mu_{\text{alt}}}{\delta_{\text{alt}}} \right) \right)$$

with t_v^{-1} the inverse student distribution function with ν degrees of freedom and F_{SNIG} the distribution function of the normal inverse gaussian distribution with parameters α_{alt} and β_{alt} . In appendix 2 we describe a simple algorithm to compute F_{SNIG} . The numbers t_{alt}^i are random variates from the student distribution.

Now we use the following fact about the bivariate t_v distribution. Let (X, Y) have the bivariate student distribution with ν degrees of freedom, then the following relation holds:

$$\sqrt{\frac{\nu+1}{\nu+Y^2}} \frac{X - \rho Y}{\sqrt{1-\rho^2}} \mid Y = y \sim t_{\nu+1}$$

We use this fact to generate the t_v -distributed random variates

$$t_{\text{trad}}^i = \rho t_{\text{alt}}^i + t_v \sqrt{\frac{(1-\rho^2)(\nu + (t_{\text{alt}}^i)^2)}{\nu+1}}$$

with $t \sim t_{\nu+1}$ and subsequently transform these numbers to the normal distribution to arrive at:

$$r_{\text{trad}}^i = \mu_{\text{trad}} + \sigma_{\text{trad}} \Phi^{-1}(t_v(t_{\text{trad}}^i))$$

with Φ^{-1} the inverse normal distribution function. Due to the finite number of simulations, the sample mean and variance of the constructed numbers r_{trad}^i shall in general deviate from their exact values μ_{trad} and σ_{trad}^2 , so they should be scaled to correct this.

Application

In this section, we will demonstrate our model. We start with a traditional portfolio consisting of 50% equity and 50% bonds. To this portfolio we will add hedge funds, keeping the ratio equity to bonds fixed. We will compute the Value at Risk and Expected Shortfall for $\alpha = 5\%$ and $\alpha = 0.5\%$ for each fraction of hedge funds in the portfolio, both with the model developed in this paper (model 1) and under the assumption that the returns of all investments are normally distributed (model 2).

The parameters of our model will be estimated using actual data. However, caution is called for. Several studies indicate that because of survivorship bias, backfilling bias and stale price bias estimates of the mean, variance, skewness and kurtosis of hedge fund returns will give a flattering picture of the situation. The real values of the mean and skewness would be lower and the real values of the variance and kurtosis would be higher.^{iv} To deal with this situation, we will not directly estimate the expected returns but base them on subjective views.

asset class	expected return
equity	8.0%
bonds	4.5%
hedge funds	7.0%

With these values, monthly expected returns on the traditional portfolio and the alternative portfolio are 0.51% and 0.57% respectively. The volatility, skewness and kurtosis of the traditional and the alternative portfolio are not based on subjective views, but are estimated using the same indices as we used in the introduction. For bonds, equity and hedge funds, we were able to obtain data over the longer period January 1990 to March 2002. For illustrative purpose, the skewness and kurtosis of the traditional portfolio are showed too, but they have no further use in this application.

	volatility	skewness	kurtosis
traditional portfolio	2.52%	-0.10	-0.11
alternative portfolio	2.10%	-0.71	2.90

Period: January 1990 - March 2002

The parameters of the normal inverse gaussian distribution of the monthly returns of the alternative portfolio can now be determined using the results of appendix 1:

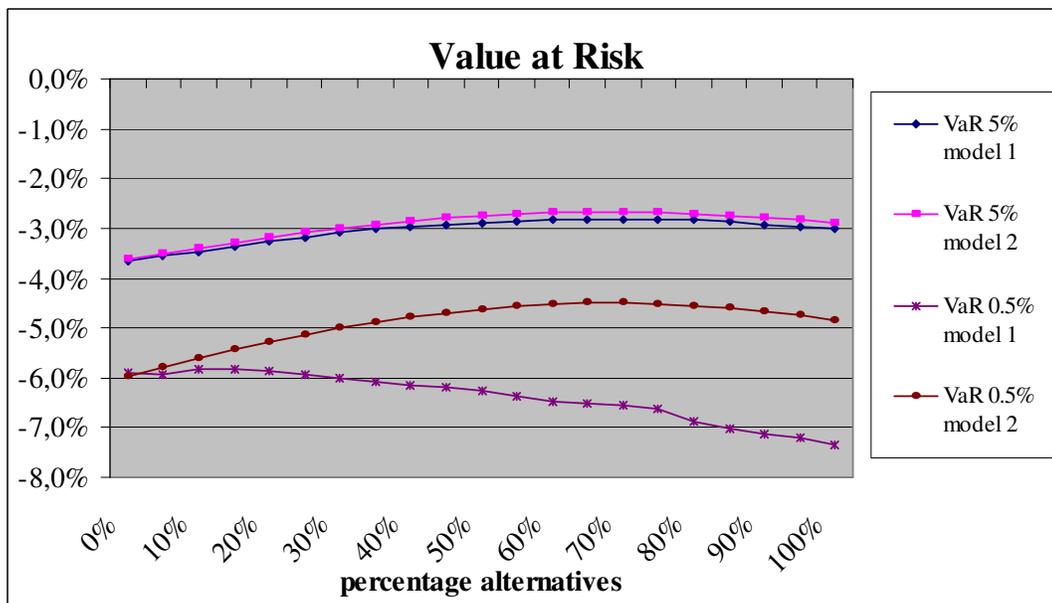
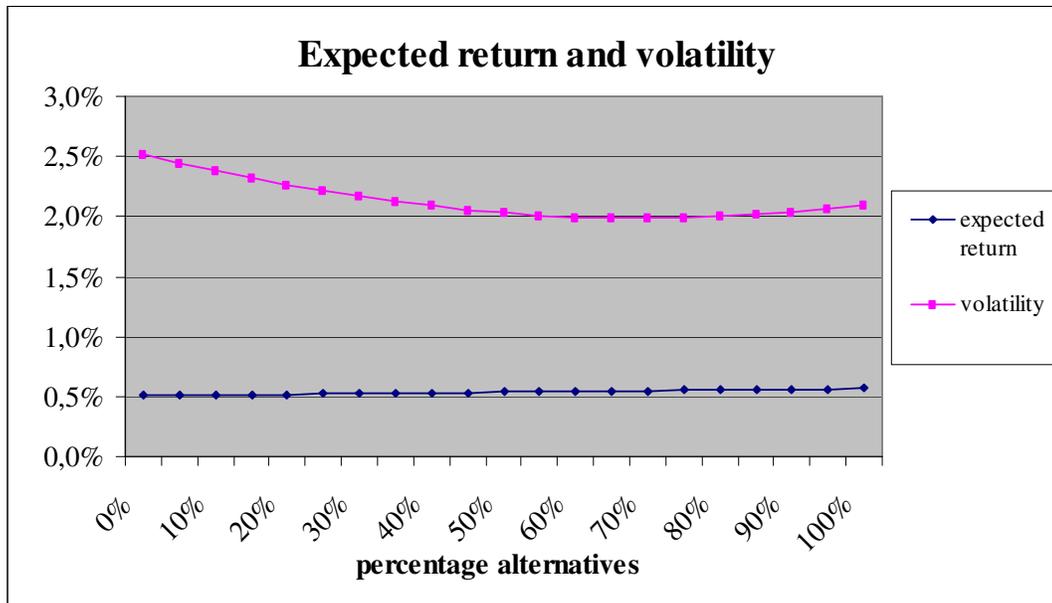
μ_{alt}	1.23%
δ_{alt}	2.34%
α_{alt}	1.40
β_{alt}	-0.38

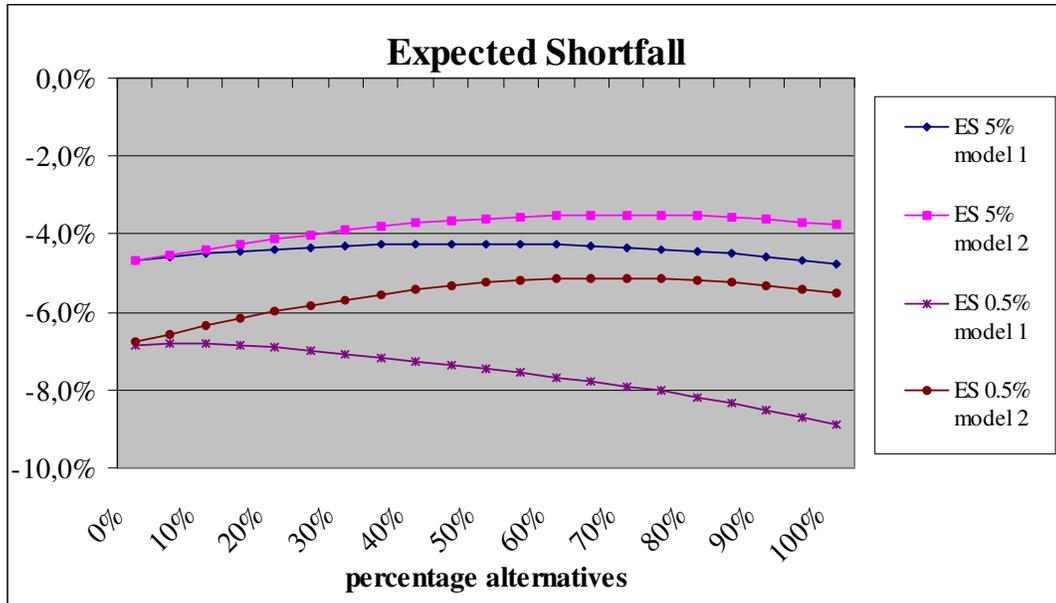
We also have to determine the parameters of the copula describing the dependence between the monthly returns of the traditional and the alternative portfolio. The estimated correlation, using the same indices and the same period is 0.54. With trial and error we will determine the value for the copula parameter ρ that gives us the correct correlation between the simulated returns. Before we can do this, we first have to specify ν , the degrees of freedom of the copula. With $\nu = 2$, we have a reasonably large tail dependence of 0.39. With all other parameters specified, we can determine ρ . With $\rho = 0.57$ the correlation between the simulated returns equals the actual estimated correlation.

We have simulated 10.000 portfolio returns. The results are shown in the graphs below for different percentages of alternatives in the portfolio. The first graph shows the expected return and volatility. The second and third graph show the Value at Risk and the Expected Shortfall. The Value at Risk and the Expected Shortfall are computed for both model 1 and model 2.

For $\alpha = 5\%$, it makes not much difference whether the Value at Risk is computed with model 1 or 2. The Expected Shortfall for $\alpha = 5\%$ is somewhat larger with model 1. The difference between the two models becomes really large for $\alpha = 0.5\%$. Both the Value at Risk and the Expected Shortfall are much larger with model 1, the relative difference can amount to 150% for the Value at Risk and to 160% for the Expected Return.

We can examine which portfolio minimizes the risk. With model 2, the portfolio consisting of 65% to 70% hedge funds is the portfolio with minimal risk, irrespective of whether risk measure is chosen. With model 1 the situation is different. For fixed α , the portfolio minimizing the Value at Risk consists of a larger fraction of hedge funds than the portfolio minimizing the Expected Shortfall. Also, with lower values for α , the risk minimizing portfolios consists of a smaller fraction of hedge funds. With model 1 and $ES_{0.5\%}$ as risk measure, the portfolio with minimal risk consists of at most 15% hedge funds, so skewness and fat tails have a large impact on portfolio selection.





Conclusion

Returns on alternative investments have properties, such as negative skewness and fat tails, which cannot be captured if they are modeled with the (log-)normal distribution. In this paper we constructed a model, using the normal inverse gaussian distribution for the returns on alternative investments and the normal distribution for returns on traditional investments, which explicitly allows for these properties. We developed algorithms with which we could generate random returns for a portfolio consisting of both traditional and alternative investments. As these random returns have been drawn from a skewed and fat tailed distribution, the resulting Value at Risk and Expected Shortfall provide a better impression of the inherent risks. We applied this model to a traditional portfolio consisting of 50% equity and 50% bonds to which we added hedge funds, keeping the ratio equity to bonds fixed. For large fractions of hedge funds, the Value at Risk (0.5%) and the Expected Shortfall (0.5%) became about one and a half times larger if we allowed for skewness and fat tails, while for small amounts of hedge funds the Value at Risk and the Expected Shortfall were almost equal to the values we obtained using normal distributed hedge fund returns.

The results of the simulations were that without the explicit consideration of skewness and fat tails, risks associated with catastrophic events that will happen with a very small probability like 1% or less, will be greatly underestimated.

Appendix 1: Mean, variance, skewness and kurtosis of the normal inverse gaussian distribution

In this appendix, we will derive formulas for the mean, variance, skewness and kurtosis of the normal inverse gaussian (NIG).

The probability density of the NIG distribution with $\mu = 0$ and $\delta = 1$ was:

$$f_{\text{SNIG}}(x; \alpha, \beta) = c(\alpha, \beta) \frac{K_1(\alpha\sqrt{1+x^2})}{\sqrt{1+x^2}} e^{\beta x}$$

$$c(\alpha, \beta) = \frac{\alpha}{\pi} \exp(\sqrt{\alpha^2 - \beta^2})$$

and we also have:

$$c(\alpha, \beta) = \left(\int_{-\infty}^{\infty} \frac{K_1(\alpha\sqrt{1+x^2})}{\sqrt{1+x^2}} e^{\beta x} dx \right)^{-1}$$

because $\int_{-\infty}^{\infty} f_{\text{SNIG}}(x; \alpha, \beta) dx = 1$.

Now we derive the moment generating function of $X \sim \text{SNIG}(\alpha, \beta)$:

$$\begin{aligned} M(t; \alpha, \beta) &= E e^{tX} \\ &= \int_{-\infty}^{\infty} e^{tx} f_{\text{SNIG}}(x; \alpha, \beta) dx \\ &= c(\alpha, \beta) \int_{-\infty}^{\infty} \frac{K_1(\alpha\sqrt{1+x^2})}{\sqrt{1+x^2}} e^{(\beta+t)x} dx \\ &= \frac{c(\alpha, \beta)}{c(\alpha, \beta+t)} \\ &= \exp(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+t)^2}) \end{aligned}$$

With the cumulant generating function:

$$K(t; \alpha, \beta) = \ln(M(t; \alpha, \beta)) = \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+t)^2}$$

the first four cumulants can be computed. We use the variable $\rho = \beta / \alpha$.

$$\begin{aligned} \kappa_1 &= \left. \frac{dK(t; \alpha, \beta)}{dt} \right|_{t=0} = E[X] = \frac{\rho}{\sqrt{1-\rho^2}} \\ \kappa_2 &= \left. \frac{d^2K(t; \alpha, \beta)}{dt^2} \right|_{t=0} = \text{Var}[X] = \frac{1}{\alpha(1-\rho^2)^{\frac{3}{2}}} \\ \kappa_3 &= \left. \frac{d^3K(t; \alpha, \beta)}{dt^3} \right|_{t=0} = \frac{3\rho}{\alpha^2(1-\rho^2)^{5/2}} \\ \kappa_4 &= \left. \frac{d^4K(t; \alpha, \beta)}{dt^4} \right|_{t=0} = \frac{3(1+4\rho^2)}{\alpha^3(1-\rho^2)^{7/2}} \end{aligned}$$

Let $Y = \mu + \delta X$, so $Y \sim \text{NIG}(\mu, \delta, \alpha, \beta)$. We can easily compute $E[Y]$ and $\text{Var}[Y]$:

$$E[Y] = \mu + \frac{\delta\rho}{\sqrt{1-\rho^2}}$$

$$\text{Var}[Y] = \frac{\delta^2}{\alpha(1-\rho^2)^{\frac{3}{2}}}$$

The skewness and kurtosis are also readily obtained:

$$\text{Skew}[Y] = \text{Skew}[X] = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{3\rho}{\sqrt{\alpha^4}\sqrt{1-\rho^2}}$$

$$\text{Kurt}[Y] = \text{Kurt}[X] = \frac{\kappa_4}{\kappa_2^2} = \frac{3(1+4\rho^2)}{\alpha\sqrt{1-\rho^2}}$$

If we set $\beta = 0$, which gives us the symmetric NIG distribution, we get $\text{Kurt}[Y] = 3/\alpha$, which shows that the NIG distribution can have any desired level of kurtosis. From these formulas we also see:

$$\frac{(\text{Skew}[Y])^2}{\text{Kurt}[Y]} = \frac{3\rho^2}{1+4\rho^2}$$

which gives us $\text{Kurt}[Y] > 0$ and $-\sqrt{\frac{3}{5}}\text{Kurt}[Y] \leq \text{Skew}[Y] \leq \sqrt{\frac{3}{5}}\text{Kurt}[Y]$, since $\rho^2 < 1$. With explicit formulas for the cumulants, we have an easy method to estimate μ , δ , α and β . The first step is to estimate the cumulants. This can be done with the unbiased estimators:

$$\hat{\kappa}_1 = \frac{1}{n} \sum_{i=1}^n Y_i$$

$$\hat{\kappa}_2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \hat{\mu})^2$$

$$\hat{\kappa}_3 = \frac{n}{(n-1)(n-2)} \sum_{i=1}^n (Y_i - \hat{\mu})^3$$

$$\hat{\kappa}_4 = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \sum_{i=1}^n (Y_i - \hat{\mu})^4 - 3 \frac{1}{(n-2)(n-3)} \left(\sum_{i=1}^n (Y_i - \hat{\mu})^2 \right)^2$$

next we solve the formulas for the cumulants and obtain:

$$\hat{\rho} = \frac{\hat{\kappa}_3}{\sqrt{3\hat{\kappa}_2\hat{\kappa}_4 - 4\hat{\kappa}_3^2}}$$

$$\hat{\alpha} = \frac{1+4\hat{\rho}^2}{\sqrt{1-\hat{\rho}^2}} \frac{3\hat{\kappa}_2^2}{\hat{\kappa}_4}$$

$$\hat{\beta} = \hat{\rho}\hat{\alpha}$$

$$\hat{\delta} = \sqrt{\hat{\alpha}\hat{\kappa}_2(1-\hat{\rho}^2)^{3/2}}$$

$$\hat{\mu} = \hat{\kappa}_1 - \hat{\delta} \frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}}$$

Appendix 2: Computation of the distribution function of the normal inverse gaussian distribution

The generalized hyperbolic distribution can be represented¹ as a mixture of normal densities with different means and variances:

$$f_{\text{GH}}(x; \lambda, \alpha, \beta, \delta, \mu) = \int_0^{\infty} \varphi(x; \mu + \beta w, w) \text{gig}(w; \lambda, \delta^2, \alpha^2 - \beta^2) dw$$

with $\varphi(x; \mu, \sigma^2)$ the probability density of the normal distribution with mean μ and variance σ^2 and $\text{gig}(x; \lambda, \chi, \psi)$ the probability density of the generalized inverse gaussian distribution:

$$\text{gig}(w; \lambda, \chi, \psi) = \frac{(\psi / \chi)^{\lambda/2}}{2 K_{\lambda}(\sqrt{\psi \chi})} w^{\lambda-1} \exp\left(-\frac{1}{2}(\chi w^{-1} + \psi w)\right)$$

With $\lambda = -\frac{1}{2}$, $\mu = 0$ and $\delta = 1$ we have the standard normal inverse gaussian distribution:

$$f_{\text{SNIG}}(x; \alpha, \beta) = \int_0^{\infty} \varphi(x; \beta w, w) \frac{1}{\sqrt{2\pi}} e^{\sqrt{\alpha^2 - \beta^2}} w^{-3/2} \exp\left(-\frac{1}{2}(w^{-1} + (\alpha^2 - \beta^2)w)\right) dw$$

The weights assigned to the normal densities are given by the probability density of the inverse gaussian distribution, for which the usual parameterization is:

$$\text{ig}(x; a, b) = \frac{a}{\sqrt{2\pi b}} x^{-3/2} \exp\left(-\frac{1}{2} \frac{(a - bx)^2}{bx}\right), \quad x > 0$$

from which we see $a = \sqrt{\alpha^2 - \beta^2}$ and $b = \alpha^2 - \beta^2$.

The distribution function of the standard normal inverse gaussian distribution can also be represented as a normal mean-variance mixture:

$$F_{\text{SNIG}}(x; \alpha, \beta) = \int_0^{\infty} \Phi(x; \beta w, w) \text{ig}\left(w; \sqrt{\alpha^2 - \beta^2}, \alpha^2 - \beta^2\right) dw$$

with $\Phi(x; \mu, \sigma^2)$ the probability distribution of the normal distribution with mean μ and variance σ^2 . We will use this formula for the computation of F_{SNIG} . We also use the identity:

$$\int_0^{\infty} f(w) dw = \int_0^1 t^{-2} f(1/t - 1) dt$$

For large N we therefore have:

$$F_{\text{SNIG}}(x; \alpha, \beta) \approx N \sum_{k=1}^{N-1} k^{-2} \Phi(x; \beta(N/k - 1), N/k - 1) \text{ig}\left(N/k - 1; \sqrt{\alpha^2 - \beta^2}, \alpha^2 - \beta^2\right)$$

We have omitted the cases $k=0$ and $k=N$ because in our case we have $\lim_{t \downarrow 0} t^{-2} f(1/t - 1) = \lim_{t \uparrow 1} t^{-2} f(1/t - 1) = 0$.

For the usual values for α and β , $N=1000$ will give us F_{SNIG} with enough precision.

¹ See [Prau99].

Appendix 3: Generating random variates from the normal inverse gaussian distribution

The normal inverse gaussian distribution can be described, as we have seen in appendix 2, as a mixture of normal densities, where the mean, variance and weight of each normal density is given by the density function of the inverse gaussian distribution:

$$f_{\text{SNIG}}(x; \alpha, \beta) = \int_0^{\infty} \varphi(x; \beta w, w) \text{ig}(w; a, b) dw$$

with $\varphi(x; \mu, \sigma^2)$ the probability density of the normal distribution with mean μ and variance σ^2 and ig the probability density of the inverse gaussian distribution:

$$\text{ig}(w; a, b) = \frac{a}{\sqrt{2\pi b}} w^{-3/2} \exp\left(-\frac{1}{2} \frac{(a-bw)^2}{bw}\right), \quad w > 0$$

The parameters a and b are functions of α and β : $a = \sqrt{\alpha^2 - \beta^2}$, $b = \alpha^2 - \beta^2$.

Let $Z \sim \text{IG}(\sqrt{\alpha^2 - \beta^2}, \alpha^2 - \beta^2)$, $Y \sim \text{N}(0,1)$ and Z and Y independent. The fact that f_{SNIG} is a inverse gaussian mixture of normal densities gives us $\beta Z + Y\sqrt{Z} \sim \text{SNIG}(\alpha, \beta)$.

With this result, we can easily generate a random variate x from the $\text{NIG}(\mu, \delta, \alpha, \beta)$ distribution. We first generate a random variate z from the inverse gaussian distribution with parameters $a = \sqrt{\alpha^2 - \beta^2}$ and $b = \alpha^2 - \beta^2$, then generate a random variate y from the normal distribution with mean βz and variance z and finally transform y with the transformation $x = \mu + \delta y$ to end with the desired random variate from the normal inverse gaussian distribution.

Now we only have to find a method to generate random variates from the inverse gaussian distribution. We use the algorithm devised in [Mich76]. This algorithm uses the following property of the inverse gaussian distribution:

$$X \sim \text{IG}(a, b)$$

$$\frac{(a - bX)^2}{bX} \sim \chi_1^2$$

A random variate from the χ_1^2 distribution is obtained by squaring a random variate y from the standard normal distribution. A random variate z from the inverse gaussian distribution is therefore obtained by solving $(a - bz)^2 - by^2z = 0$. However, this equation has two roots:

$$z_1 = \frac{a}{b} + \frac{y^2}{2b} - \frac{|y|}{2b} \sqrt{4a + y^2}$$

$$z_2 = \frac{a}{b} + \frac{y^2}{2b} + \frac{|y|}{2b} \sqrt{4a + y^2}$$

In [Mich76] it is proved that a random variate from the $\text{IG}(a, b)$ distribution is

obtained if the first root (z_1) is chosen with probability $p = \frac{a}{a + bz_1}$ and the second

root (z_2) with probability $1 - p$.

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ⁱ For more theory on copulas and their use in finance, see [Bouy00].

ⁱⁱ For a derivation of this result, see [Embr99].

ⁱⁱⁱ Theory and further references can be found in the above mentioned literature, which, at the time of writing this paper, could all be downloaded for free from the internet.

^{iv} See for example [Fung02] and [Kat01b].