THE PRICING OF LIABILITIES IN AN INCOMPLETE MARKET USING DYNAMIC MEAN-VARIANCE HEDGING WITH REFERENCE TO AN EQUILIBRIUM MARKET MODEL

BY

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ABSTRACT

In this article the method of pricing the liabilities of a financial institution by means of dynamic mean–variance hedging is applied to the situation of an incomplete market that is nevertheless in equilibrium. For a given stochastic asset–liability model that is consistent with the market, the article shows how to determine a unique price at which, subject to specified provisos, a prospective buyer or seller would be indifferent about concluding the transaction.

KEYWORDS

Market value of liabilities, dynamic mean–variance hedging, equilibrium market models, incomplete markets

1. INTRODUCTION

Because of moral hazard, legal constraints and the de facto incompleteness of markets, it is generally impossible to replicate the liabilities of a financial institution with traded assets. Under such conditions it is impossible to determine the price at which the liabilities would be traded by means of asset matching, risk-neutral pricing methods or deflators (Møller, 2002; Jarvis, Southall & Varnell, 2001). However, if a stochastic asset–liability model (ALM) is adopted, and the market, though incomplete, is in equilibrium, and the ALM is consistent with the market, then a unique price can, in principle, be obtained that is consistent both with the ALM and with the market (Thomson, 2002).

The notion of a price at which an asset or liability would trade if a complete market existed (e.g. Martin & Tsui, 1999: 357) is a fiction. It suggests that a complete market makes no difference to prices. That is clearly untrue: in an incomplete market, extra risks exist, which cannot be hedged. Those risks must affect prices. The notion of a price at which an asset or liability would trade if a liquid market existed in it (e.g. Cairns, 2001), while slightly weaker, is similarly a fiction. The price contemplated in this article is the price at which a prospective buyer or a seller who is willing but unpressured and fully informed would be indifferent about concluding the transaction, provided the effects of moral hazard and legal constraints would not be altered by the transaction. The price therefore allows for the fact that the non-systematic risks of the liabilities cannot be diversified away.

In practice, the determination of such a price is non-trivial. Hairs et al. (2002: 273–4) suggest 'the selection of a replicating portfolio by minimising the asset–liability cash-flow mismatches over time.' How they envisage that this would be achieved is not clear. Møller (*op. cit.*: 794–8) outlines solutions based on four different approaches in continuous time. Of those, one (the 'super-replication' approach) does not reflect the price contemplated above but a minimum price. The second (the 'utility' approach) depends on the utility function of the financial institution and is therefore not unique. Furthermore, it has been shown (Thomson, 1998) that, though expected-utility theory may be valid for normative purposes, it is not generally valid for definitive purposes. The third (the 'quadratic' approach) comprises two alternatives: 'risk minimisation' and 'mean–variance hedging'. Risk minimisation involves minimising a process reflecting the costs of financing a strategy that meets the cash flow exactly. Mean–variance hedging (Bouleau & Lamberton, 1989; Duffie & Richardson, 1991; Schweizer, 1992) involves approximating the cash flow as closely as possible to the terminal value of a self-financing strategy so as to minimise the variance of the difference. No theoretical justification is given for either of these alternatives, and no reference is made to an equilibrium market. The fourth ('quintile hedging and shortfall-risk minimisation') involves arbitrary parameters and therefore, like the utility approach, does not produce a unique result.

Bouleau & Lamberton (*op. cit.*), Duffie & Richardson (*op. cit.*) and Schweizer (*op. cit.*) all address the mean–variance hedging process in continuous time for claims contingent on share prices whose processes do not permit complete hedging due, for example, to jumps. The first, which is couched in language generally inaccessible to actuaries, assumes that the state space is a Markov process. The second and third assume that prices are geometric Brownian motions (the third more generally and rigorously than the second). None of them solves the pricing problem. Numerous subsequent papers have generalised their findings or applied them to particular cases.

Schäl (1994), Schweizer (1995) and Èerný (unpublished) derive the mean–variance hedging process in discrete time, but they also do not address the pricing problem.

Cairns (*op. cit.*) applies the utility approach to the pricing of liabilities. He assumes a single time period with normally distributed returns and investors with exponential utility and heterogeneous expectations (i.e. different ALMs) and a market in equilibrium. He determines the price of the liability by considering the introduction into the market of an asset defined in the same way as the liability, and finding the price at which equilibrium is restored. This is consistent with the (fictional) completion of the market.

In this article, mean–variance hedging is applied to the liabilities of a financial institution rather than to a contingent claim (though the latter is included as a special case). The state space is not necessarily assumed to be a Markov process. And a particular form for price processes is not assumed, except that they must be consistent with current market information and with the assumption that the market is in equilibrium with homogeneous expectations. The pricing problem is solved in discrete time.

From an actuarial point of view, a solution in discrete time is preferable. A solution in continuous time can be applied through numerical integration. But this tends to be a black-box approach, which does not necessarily permit direct interpretation of intermediate results. In the final analysis, all the processes are discrete. Cash-flow payments occur at discrete intervals. Market returns occur as and when quoted prices change. Returns on risk-free deposits occur overnight. Continuous-time models, though they may be more elegant, are not necessarily better representations of reality than discrete-time solutions.

The suggestion of Hairs et al. (*op. cit.*) would be realised if the liabilities were priced as the price of a hedge portfolio less the price of the remaining exposure to the undiversifiable risks. The latter price would be equal to the price of a portfolio with a probability distribution identical to that of a diversified market portfolio in an equilibrium market. While this does not presuppose the validity of expected-utility theory, it would be consistent with that theory: if two risky prospects have identical distributions, an agent will be indifferent between them. In this article, as a first-order approximation to such an approach, mean–variance hedging is used with reference to an equilibrium market model.

While it may at first appear counter-intuitive that the value of a liability should be lower for higher degrees of risk, the paradox is resolved by considering the liability as a short position in an asset; clearly, for an asset, the price is lower for higher degrees of risk. The fact that the institution has a short position in the asset does not affect its market price. The value determined does not constitute the capital required to cover the risk; that is a separate issue. In the context of pension funds, consideration is not often given to capital requirements. That is partly because the employer bears the balance of cost. Whether the capital required should be held in the fund or by the employer is a matter for discussion between the trustees and the employer. The determination of capital required is outside the scope of this article.

The assumption of the adoption of a particular ALM combined with the assumption of homogeneous expectations may at first appear problematic. However, provided the ALM is designed to reflect market expectations so far as they are evidenced by market conditions and historical returns, those assumptions are not inconsistent. Any pricing theory other than pure arbitrage pricing must inevitably assume some such model. And any assumption other than homogeneous expectations would give the modeller a deceptive advantage.

In sections 2 to 4 the problem is specified, formulated and solved. An algorithm for the implementation of the solution is set out in section 5. Section 6 concludes.

2. SPECIFICATION OF THE PROBLEM

Suppose that cash flows occur annually and that all time periods are measured in years. (The time interval is arbitrary; any other interval could be used.) The amounts of the cash flows are not deterministically specified; they depend on the state of the world from time to time, and in particular on the values at those times of certain variables. Suppose that we have a stochastic model of those variables, which enables us to generate a pseudo-random sample of the liability cash flows (including tax and other expenses) and the investment returns on all relevant categories of assets in the market portfolio. This means that, for given information at the start of the year, we shall be able to estimate the moments of the joint distribution of those cash flows and returns.

The price of the liabilities is then defined by the following requirements:

(A) At the start of each year:
- (1) the price of the liability for subsequent cash flows is equal to the sum of:
  - (a) the price of the hedge portfolio; and
  - (b) the (negative) price of the remaining exposure to undiversifiable risk;
- (2) the hedge portfolio is selected from the constituents of the market portfolio and the risk-free asset so that:
  - (a) the undiversifiable risk is minimised; and
  - (b) the expected return on the hedge portfolio during that year is equal to that on the liability; and
- (3) the price of the remaining exposure to undiversifiable risk is equal to that of a portfolio, comprising the market portfolio and the risk-free asset, whose risk is equal to that of the liabilities.

(B) When all the liability cash flows have been paid, the price of the liabilities is nil.

In this paper, 'risk' is defined as variance.

In the above requirements, the 'market portfolio' carries the same connotation as in the capital-asset pricing model (CAPM). It is the set of all capital assets in which market participants (including the institution) can invest. The state vector modelled must therefore contain sufficient variables to facilitate the calculation (by the ALM) of the return on the market portfolio.

The asset categories modelled by the ALM should be consistent with a market in equilibrium. Since market participants are assumed to operate in mean–variance space with homogeneous expectations, the CAPM should apply during each future year, conditional on the values of the state vector at the start of the year. That would also necessitate an assumption about the composition of the market portfolio by category in future years. Because of the capital irrelevance proposition (Modigliani & Miller, 1958), it may be assumed that the total return on the market portfolio will (apart from frictional effects) be unaffected by the composition of that portfolio. The stochastic modelling of the composition of the market portfolio by asset category would therefore be spurious. Taxation, which is a non-trivial frictional effect in the pricing of institutional liabilities, may detract from the validity of this assumption. But provided the rates of tax incurred on the investment returns earned on the various asset categories are not unduly asymmetric, the assumption is arguably not unreasonable. In fact, if allowance were made for optimisation of aggregate capital allocation, allowance would also have to be made for tax optimisation by government. It would be better to model the market on the assumption of constant composition by asset category.

Furthermore (on the expectations hypothesis) the ALM's expected future risk-free interest rates should be equal to the current forward-rate yield curve, in both nominal and real terms. Also, the difference between expected nominal returns and expected real returns should be consistent with expected inflation, allowing in addition for a reasonable inflation risk premium. The initial price volatilities of the asset categories modelled should conform to those implied by option prices.

In the selection of asset categories to be modelled, care should be taken to include those that would be relevant to the liabilities. The proliferation of asset categories should, however, be avoided, as this would tend to result in the inclusion of categories whose correlation with the liabilities is fortuitously significant. The resulting reduction in undiversifiable risk would be spurious. The selection must be sufficiently comprehensive to model the market portfolio. But excessive division into categories should be avoided. Further consideration is given in section 6 to the selection of the asset categories to be modelled.

3. FORMULATION OF THE PROBLEM

Let $\boldsymbol{X}_t$ be the p-component state vector of the stochastic model at time t. The model defines the conditional distribution:

$$F_{X_t}\left(\boldsymbol{x}\middle|\boldsymbol{X}_{t-1}\right).$$

An ALM defines the following variables as functions of $\boldsymbol{X}_t$:

C_t = the institution's net cash flow at time t;

$V_{\alpha t}$ = the market value at time t of an investment in asset category $\alpha = 1,\ldots, A$ per unit investment at time $t - 1$;

f_{t+1} = the amount of a risk-free deposit at time $t + 1$ per unit investment at time t.

Let

$$\boldsymbol{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_A \end{pmatrix}$$

represent the composition of the market portfolio, where:

m_α is the proportion in asset category α;

$m_a > 0$; and

$\sum_{a=1}^{A} m_a = 1$.

We denote by L_t the market value of the institution's liabilities at time t after the cash flow then payable.

Suppose that, in order to minimise the variance of the difference between $(C_t + L_t)$ and the value of its hedge portfolio at time t given $\boldsymbol{X}_{t-1} = \boldsymbol{x}$, the institution would invest an amount of $g_{a,t-1}$ in asset category α and h_{t-1} in the risk-free asset (together comprising the hedge portfolio). Let:

$$\boldsymbol{g}_t = \begin{pmatrix} g_{1,t} \\ \vdots \\ g_{A,t} \end{pmatrix} \text{ and } \boldsymbol{V}_t = \begin{pmatrix} V_{1t} \\ \vdots \\ V_{At} \end{pmatrix}.$$

Then:

$$C_t + L_t = \boldsymbol{g}'_{t-1}\boldsymbol{V}_t + h_{t-1} f_t + \boldsymbol{e}_t;$$

where $\boldsymbol{e}_t$, being the undiversifiable exposure, is independent of $\boldsymbol{V}_t$, $\mathrm{E}(\boldsymbol{e}_t) = 0$ (requirement (A)(2)(b)), and $\boldsymbol{g}_t$ is such that

$$\boldsymbol{s}_{et}^2 = \mathrm{Var}\left(C_t + L_t - \boldsymbol{g}'_{t-1}\boldsymbol{V}_t \middle| \boldsymbol{X}_{t-1} = \boldsymbol{x}\right) \qquad (1)$$

is minimised (requirement (A)(2)(a)).

Now to get the same expected return on the hedge portfolio as on the liability, we require (requirement (A)(2)(b)):

$$m_{et} = \mathrm{E}\left(C_t + L_t - \boldsymbol{g}'_{t-1}\boldsymbol{V}_t - h_{t-1} f_t\right) = 0 \ . \qquad (2)$$

Note that, as mentioned above, the ALM should conform to the CAPM. This means that:

$$m_{at} = f_t + b_{at}\left(m_{Mt} - f_t\right);$$

where:

$$\boldsymbol{b}_{at} = \frac{\boldsymbol{s}_{Mat}}{\boldsymbol{s}^2_{Mt}};$$

$$s_{Mat} = \mathrm{Cov}\left(M_t, V_{at}\right); \text{ and}$$

$$s^2_{Mt} = \mathrm{Var}\left(M_t\right).$$

If the circumstances warrant it, the CAPM formulae may be modified for non-standard effects as outlined by Elton & Gruber (1995: 311–31).

Besides the hedge portfolio, the institution has an exposure to $\boldsymbol{e}_t$. This exposure may be priced as an undiversifiable risk with reference to the risk-free deposit and the market portfolio. Suppose the price of the exposure is k_{t-1}, of which l_{t-1} is in the market portfolio and $(k_{t-1} - l_{t-1})$ is in the risk-free deposit. Then (requirements (A)(2)(b) and (A)(3)):

$$m_{et} = l_{t-1} m_{Mt} + \left(k_{t-1} - l_{t-1}\right) f_t = 0 \text{; and} \qquad (3)$$

$$s^2_{et} = l^2_{t-1} s^2_{Mt} \ . \qquad (4)$$

It may be noted that this specification corresponds to an extension of the capital market line to an expected return of zero.

The price of the liability comprises the price of the hedge portfolio plus the (negative) price of the exposure (requirement (A)(1)), i.e.:

$$L_{t-1} = \sum_{a=1}^{A} g_{a,t-1} + h_{t-1} + k_{t-1} \ . \qquad (5)$$

The problem is to find L_0 given that $L_N = 0$ (where N is the last possible cash-flow date—requirement (B)).

It should be noted that, whereas $\boldsymbol{g}_t$ and h_t represent constituents of the hedge portfolio, k_t does not; it is merely the price of a portfolio that satisfies equations (3) and (4). The unhedged exposure is not a position in the market portfolio, since that would generally result in non-zero covariance with the hedge portfolio, which is specifically excluded. The market portfolio is used merely to price undiversifiable risk in terms of its variance.

4. Solution of the Problem

In order to minimise σ_{et}^2 (equation (1)), we require:

$$\boldsymbol{\gamma}_{\boldsymbol{t-1}} = \boldsymbol{\Sigma}_{Vt}^{-1}\left(\boldsymbol{\sigma}_{CVt} + \boldsymbol{\sigma}_{LVt}\right);$$

where:

$$\boldsymbol{\Sigma}_{Vt} = \begin{pmatrix} \sigma_{11t} & \cdots & \sigma_{1At} \\ \vdots & & \vdots \\ \sigma_{A1t} & \cdots & \sigma_{AAt} \end{pmatrix},\ \boldsymbol{\sigma}_{CVt} = \begin{pmatrix} \sigma_{C1t} \\ \vdots \\ \sigma_{CAt} \end{pmatrix} \text{ and } \boldsymbol{\sigma}_{LVt} = \begin{pmatrix} \sigma_{L1t} \\ \vdots \\ \sigma_{LAt} \end{pmatrix};$$

$$\sigma_{ijt} = \mathrm{Cov}\left(V_{it}, V_{jt} \middle| \boldsymbol{X}_{\boldsymbol{t-1}} = \boldsymbol{x}\right);$$

$$\sigma_{Cit} = \mathrm{Cov}\left(C_t, V_{it} \middle| \boldsymbol{X}_{\boldsymbol{t-1}} = \boldsymbol{x}\right); \text{ and}$$

$$\sigma_{Lit} = \mathrm{Cov}\left(L_t, V_{it} \middle| \boldsymbol{X}_{\boldsymbol{t-1}} = \boldsymbol{x}\right).$$

The resulting value of σ_{et}^2 is:

$$\sigma_{et}^2 = \sigma_{Ct}^2 + 2\sigma_{CLt} + \sigma_{Lt}^2 - \boldsymbol{\gamma}'_{\boldsymbol{t-1}}\left(\boldsymbol{\sigma}_{CVt} + \boldsymbol{\sigma}_{LVt}\right).$$

In order to get $\mu_{et} = 0$ (equation (2)), we require:

$$\mu_{Ct} + \mu_{Lt} = \boldsymbol{\gamma}'_{\boldsymbol{t-1}}\boldsymbol{\mu}_{Vt} + \eta_{t-1}\phi_t;$$

where:

$$\mu_{Ct} = \mathrm{E}\left\{C_t \middle| \boldsymbol{X}_{\boldsymbol{t-1}} = \boldsymbol{x}\right\};$$

$$\mu_{Lt} = \mathrm{E}\left\{L_t \middle| \boldsymbol{X}_{\boldsymbol{t-1}} = \boldsymbol{x}\right\};$$

$$\boldsymbol{\mu}_{Vt} = \begin{pmatrix} \mu_{1t} \\ \vdots \\ \mu_{At} \end{pmatrix}; \text{ and}$$

$$\mu_{it} = \mathrm{E}\left\{V_{it} \middle| \boldsymbol{X}_{\boldsymbol{t-1}} = \boldsymbol{x}\right\};$$

i.e.:

$$\eta_{t-1} = \frac{1}{\phi_t}\left(\mu_{Ct} + \mu_{Lt} - \boldsymbol{\gamma}'_{\boldsymbol{t-1}}\boldsymbol{\mu}_{Vt}\right).$$

Also, from equations (3) and (4) we have:

$$\kappa_{t-1} = -\frac{\sigma_{et}}{\sigma_{Mt}}\left(\frac{\mu_{Mt}}{\phi_t} - 1\right).$$

Since the means, variances and covariances of C_t, L_t and $\boldsymbol{V}_t$ can be estimated for any value of $\boldsymbol{X}_{t-1}$ by means of the ALM, the corresponding value of L_{t-1} may be found from equation (5). Since $L_N = 0$, earlier values of L_t may be calculated recursively, as functions of $\boldsymbol{X}_t$. Since $\boldsymbol{X}_0$ is known, L_0 is unique.

5. IMPLEMENTATION ALGORITHM

The solution in section 4 may be implemented by means of the algorithm shown below. The algorithm involves successive calculations of $L_{t-1} \big| \boldsymbol{X_{t-1}} = \boldsymbol{x}$ for $t = N$ to 1; i.e. calculating backwards year by year from $t = N$, where $L_N = 0$, to $t = 1$, where $\boldsymbol{X}_0$ is known. For $t > 1$, I values of $\boldsymbol{X}_{t\text{-}1}$ are simulated. For each of these, J values of $\boldsymbol{X}_t$, conditional on the value of $\boldsymbol{X}_{t-1}$, are simulated. The latter pseudo-random sample is used to estimate L_{t-1} for that value of $\boldsymbol{X}_{t-1}$. This gives us values of L_{t-1} for I values of $\boldsymbol{X}_{t-1}$. For $t < N$, the value of L_t is required as a function of $\boldsymbol{X}_t$ in order to calculate L_{t-1} as a function of $\boldsymbol{X}_{t-1}$. This is found by interpolation from the values obtained from the calculation already made in respect of year $t + 1$.

The algorithm is as follows:

For $t = N$ to 1:

If $t = 1$, let $i = 1$; otherwise for $i = 1$ to I:

If t > 1 then, using the stochastic model, simulate $\boldsymbol{X}_{t-1} = \boldsymbol{x}_{t-1,i}$.

(Otherwise $\boldsymbol{X_{t-1}} = \boldsymbol{X}_0$.)

Using the ALM, calculate f_t.

For $j = 1$ to J:

Using the stochastic model, simulate:

$$\boldsymbol{X_t} \big| \left(\boldsymbol{X_{t-1}} = \boldsymbol{x_{t-1,i}} \right) = \boldsymbol{x_{tj}} \ .$$

Using the ALM calculate:

$C_{tj} \big| \boldsymbol{X}_t = \boldsymbol{x_{tj}}$; and

$\boldsymbol{V_{tj}} \big| \boldsymbol{X}_t = \boldsymbol{x_{tj}}$.

If $t = N$ then $L_{tj} \big| \boldsymbol{X_t} = 0$.

Otherwise interpolate $L_{tj} \big| \boldsymbol{X_t} = \boldsymbol{x_{tj}}$ from previous calculations (see below).

Calculate:

$$\hat{m}_{Ct} = \frac{1}{J} \sum_{j=1}^{J} C_{tj} \ ;$$

$$\hat{m}_{Lt} = \frac{1}{J} \sum_{j=1}^{J} L_{tj} \ ;$$

$$\hat{\boldsymbol{i}}_{Vt} = \frac{1}{J} \sum_{j=1}^{J} \boldsymbol{V}_{tj} \ ;$$

$$\hat{m}_{Mt} = \boldsymbol{m}' \hat{\boldsymbol{i}}_{Vt} \ ;$$

$$\hat{s}^2_{Ct} = \frac{1}{J-1} \sum_{j=1}^{J} \left(C_{tj} - \hat{m}_{Ct} \right)^2 \ ;$$

$$\hat{s}^2_{Lt} = \frac{1}{J-1} \sum_{j=1}^{J} \left(L_{tj} - \hat{m}_{Lt} \right)^2 \ ;$$

$$\hat{s}_{CLt} = \frac{1}{J-1}\sum_{j=1}^{J}\left(C_{tj} - \hat{m}_{Ct}\right)\left(L_{tj} - \hat{m}_{Lt}\right);$$

$$\hat{\boldsymbol{o}}_{CVt} = \frac{1}{J-1}\sum_{j=1}^{J}\left(C_{tj} - \hat{m}_{Ct}\right)\left(\boldsymbol{V}_{tj} - \hat{\boldsymbol{i}}_{Lt}\right);$$

$$\hat{\boldsymbol{o}}_{LVt} = \frac{1}{J-1}\sum_{j=1}^{J}\left(L_{tj} - \hat{m}_{Lt}\right)\left(\boldsymbol{V}_{tj} - \hat{\boldsymbol{i}}_{Lt}\right);$$

$$\hat{\boldsymbol{O}}_{Vt} = \frac{1}{J-1}\sum_{j=1}^{J}\left(\boldsymbol{V}_{tj} - \hat{\boldsymbol{i}}_{Vt}\right)\otimes\left(\boldsymbol{V}_{tj} - \hat{\boldsymbol{i}}_{Vt}\right)';$$

where $\otimes$ denotes the Kronecker product, i.e.:

$$[a_i]\otimes[b_i]' = [a_i b_j];$$

$$\hat{\boldsymbol{o}}_{Mt} = \boldsymbol{m}'\hat{\boldsymbol{O}}_{Vt}\boldsymbol{m}.$$

$$\boldsymbol{g}_{t-1} = \hat{\boldsymbol{O}}_{Vt}^{-1}\left(\hat{\boldsymbol{o}}_{CVt} + \hat{\boldsymbol{o}}_{LVt}\right);$$

$$h_{t-1} = \frac{1}{f_t}\left(\hat{m}_{Ct} + \hat{m}_{Lt} - \boldsymbol{g}'_{t-1}\hat{\boldsymbol{i}}_{Vt}\right);$$

$$\hat{s}_{et}^{2} = \hat{s}_{Ct}^{2} + 2\hat{s}_{CLt} + \hat{s}_{Lt}^{2} - \boldsymbol{g}'_{t-1}\left(\hat{\boldsymbol{o}}_{CVt} + \hat{\boldsymbol{o}}_{LVt}\right);$$

$$k_{t-1} = -\frac{\hat{s}_{et}}{\hat{s}_{Mt}}\left(\frac{\hat{m}_{Mt}}{f_t} - 1\right);\text{ and}$$

$$L_{t-1} = \sum_{a=1}^{A} g_{a,t-1} + h_{t-1} + k_{t-1}.$$

Record each value of $\boldsymbol{x}$ and the corresponding value of $L_{t-1}\middle|\boldsymbol{X}_{t-1} = \boldsymbol{x}$ for use in subsequent calculations (see above).

The final line gives L_0, the current price of the liabilities, when $t = 1$.

For each year's calculations, the algorithm requires I values of $\boldsymbol{x}$ as at the end of the previous year. It is not necessary for these values to be a pseudo-random sample. What is required is that they should cover the range of such a sample sufficiently well to facilitate interpolation (or extrapolation). I must be specified as a value such that the number of values of $L_{t-1}(\boldsymbol{x})$ calculated in the final step for $t = s$ is sufficient to interpolate $L_t(\boldsymbol{x})$ for $t = s - 1$ with adequate accuracy. Second- or third-order interpolation from the 3^p or 4^p closest values of $\boldsymbol{x}_s$ (where p is the number of components in $\boldsymbol{x}_t$ and the norm $\left\|\boldsymbol{x}_{s-1} - \boldsymbol{x}_s\right\|^2$ is used as the measure of closeness) would be preferable to linear interpolation.

For each of those values, the algorithm simulates J values of $\boldsymbol{x}_t$. For this purpose a pseudo-random sample is required. J should be sufficiently large to give adequate estimates of the parameters required to calculate L_{t-1}.

6. CONCLUSION

In order to ensure that the hedge portfolio is optimal, the asset categories should generally include a set of risk-free zero-coupon bonds (nominal or index-linked or both, depending on the nature of the liabilities) maturing at successive year ends. For liabilities corresponding to such bonds, the hedge portfolio will comprise those bonds and the price of the liabilities will be equal to the price of the bonds. The market portfolio will include not only those bonds but also new issues from year to year.

If the liability is a contingent claim on one of the asset categories modelled, the implementation algorithm is a numerical solution of the option-pricing problem. As the length of the time period reduces, the option price will tend to the continuous-time price for the ALM process used. This is because the undiversifiable risk tends to zero. By using a Wiener process, the program can be checked against the Black-Scholes formula.

The assumption that cash flows occur only at year-ends creates errors of approximation. Nevertheless, while the method is not completely accurate, it does allow for an arbitrarily small degree of inaccuracy, according to the length of the calculation intervals used. It could be modified to allow for the payment of 50% of each year's cash flow at the beginning of the year and 50% at the end so as to avoid bias. Greater accuracy might be spurious.

On the other hand, the computational demands increase as the calculation interval is reduced. The number of components in $\boldsymbol{X}_t$ required for the purposes of calculating $C_{tj} \middle| \boldsymbol{X}_t = \boldsymbol{x}_{tj}$, is strictly speaking equal to the number of data items that would be needed for a valuation of the liabilities of the institution. The number of possible combinations of values of those components required for the purposes of interpolation would soon become prohibitive. Cash flows that can be diversified by reinsurance or securitisation should be modelled accordingly so as to reduce the number of components in the state-space vector. For the modelling of the spot yield curve, it would not be necessary to model each year's maturity as a component of $\boldsymbol{X}_t$. Two or three principal components may be adequate for that purpose (Maitland, 2002).

Because of the effects of correlation between different liabilities, the price of a set of liabilities is not necessarily equal to the sum of the prices of the respective liabilities; i.e. the liability prices are not additive. This means that, if a subset of an institution's liabilities is transferred to another institution (for example on the transfer of employees from one pension fund to another), the transfer price at which the transferor would be indifferent would be equal to the difference between the price of the liabilities including those transferred and the price excluding them. The converse would apply to the transferee. But because the liabilities of the transferee before transfer would not generally be the same as those of the transferor after transfer, the transferor's price would not necessarily be equal to the transferee's. This would necessitate negotiation between the respective prices, resulting in either a win–win or a lose–lose outcome. The unique price applies only to the liabilities *in toto*. This contrasts with Cairns's (*op. cit.*) finding that liability prices are additive. The reason lies in the allowance made in this paper for the undiversifiability of unhedged liabilities, in contrast to Cairns's definition of fair value (section 1).

Nevertheless, for a particular institution with a given set of initial conditions and a particular ALM, the method outlined in this article does approximate, to an arbitrarily high level of accuracy, a price that is uniquely consistent with a market in equilibrium and decision-making in mean–variance space. This applies even if the market is incomplete. In particular, it is not assumed that there is an asset in the market that replicates the salary inflation of the fund concerned, nor even aggregate salary inflation.

It has been argued by actuaries in the employee benefits field that markets cannot be assumed to be in equilibrium. In particular, values of equities are reduced below market prices. (Kendal & Franklin, 1993) To some extent, this may arise from confusion between valuation and capital adequacy. But if actuaries are bold enough to write certain assets down on the grounds that the market is not in equilibrium, they should be recommending the sale of such assets.

The stochastic investment models currently available in the public domain (e.g. Wilkie, 1986, 1995; Carter, 1991; Thomson, 1996) do not purport to reflect equilibrium markets. In order to implement the mean–variance pricing of liabilities, a new generation of stochastic models is required.

In this article it has been assumed that agents operate in mean–variance space. In the absence of any assumption about the applicability or otherwise of expected-utility theory, this implies that the distribution functions of the annual growth rates of investments (and the values of liabilities) are elliptically symmetric. That is not necessarily true. Consideration therefore needs to be given to the pricing of higher-order moments than variances.

Wilkie (1997) shows that the CAPM fails in a multi-currency world. The implications of that finding for mean–variance hedging in incomplete equilibrium markets need to be explored.

In summary, further research is required in order to:

- reduce the computational demands of the implementation algorithm for a given level of accuracy;
- develop suitable stochastic models of state vectors that facilitate the modelling of unmarketable liabilities and of the market portfolio in equilibrium;
- establish a basis for the determination of the capital required by an institution with such liabilities;
- allow, in the case of a defined-benefits pension fund, for the risk of the insolvency of the employer;
- include higher-order moments than variances in the pricing of risk; and
- explore the adaptation of the proposed method to a multi-currency world.

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