

An Integrative Risk Evaluation Model for Market Risk and Credit Risk

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Abstract

Not only financial institutions but all firms have their own portfolio consisting of various financial assets. These portfolios are exposed to many kinds of risks, such as market risk, credit risk, liquidity risk, operational risk, and so on. Basel Accord in 1998 prompted us to recognize the importance of quantitatively evaluating these financial risks, and risk valuation models have been developed, first for market risk and then credit risk. However, in these models, each risk is evaluated separately, not integratively. Recently, Kijima and Muromachi [17] proposed a general framework for the integrative evaluation of market risk (interest rate risk) and credit risk. In this paper, we extend Kijima and Muromachi [17], and propose a framework to evaluate in a integrative way market risk (not only interest rate risk but also stock price variation risk and foreign exchange risk) and credit risk.

At the center of the framework are stochastic differential equations (SDEs) which describe the dynamics of interest rate and default probability. We (i) run simulations with in the Monte-Carlo method and generate scenarios based on these basic equations, (ii) calculate asset values for a pre-specified future date (risk horizon) using no-arbitrage theory, and (iii) obtain the future value of the portfolio by summing up all asset values. The main features of this framework are: (i) we can take the correlation between the interest rate and the default probability into account, (ii) the theoretical values of bonds derived in this setting are consistent with those observed in the real market, (iii) we can incorporate the term structure of the default probability into the model. Moreover, we can obtain the distribution of the portfolio or each asset value in the future, and so we can calculate any risk measure (e.g. standard deviation, VaR, T-VaR etc.). Also, the expected return can be computed, so the risk-return analysis can be made in a consistent way.

This framework enables us to construct many types of models by changing the basic SDEs. We briefly present results for a Gaussian model, which is relatively easy to calculate.

Keywords: market risk, credit risk, hazard rates, conditional independence, VaR, risk contributions, SDEs

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1. Market Risk and Credit Risk

First, we briefly discuss market risk and credit risk as used in this paper.

The market risk of a portfolio consisting of many actively traded assets is measured through the risk horizon (a prespecified time in the future to evaluate the risk), such as 1 day or 10 days. One of the standard models used to measure the market risk today is RiskMetrics™. In this model, the return of each asset, or more precisely the risk factor, is assumed to have a multivariate Gaussian distribution. This assumption is thought to be a good approximation.

We call the situation in which an issuer of a financial product cannot implement the contract a *default*, and losses incurred by such default are said to represent *credit risk*. The credit risk includes not only direct losses from cash flows that cannot be received due to the issuer's default, but also indirect losses when the asset value plunges because the possibility of the counterparty's default in the future rises sharply. Default, the origin of credit risk, is a binomial event in that it either happens or not to happen. Hence, we cannot regard asset returns with credit risk as Gaussian distributed. Moreover, because the typical horizon for credit risk is at least 1 year, we have to distinguish the observed probability measure and the pseudo probability measure.¹

Because there are many difficult problems, we still don't have a standard model of credit risk measurement. Typical models are CreditMetrics™ and CREDITRISK⁺, but these can only measure the credit risk, and the integration of market risk and credit risk had been made in a tentative way. One of the reasons is that the theoretical framework of the integrative valuation of multiple risks had not been constructed clearly until Kijima and Muromachi. In this paper, we examine the framework of the integrative evaluation of the market risk and credit risk based on Kijima and Muromachi [17].

Technical documents and textbooks on this issue can help in understanding typical models in more detail.

2. Desired Properties of Risk Valuation Models

Based on the previous section, we list the desired properties of an integrative valuation model of market and credit risk.

- (1) All assets are evaluated by one consistent method, such as no-arbitrage pricing.
- (2) Portfolio effect, namely the effect due to the correlation among assets (especially in this case, the correlation between two default probabilities or the correlation between the default probability and the interest rate) can be taken into consideration.
- (3) The valuation is consistent with observable market prices.
- (4) Term structure of interest rates and default rates are included in the framework. The framework has the flexibility to incorporate empirical results into the model.
- (5) The framework can properly treat the distributions of each asset return as asymmetric and non-Gaussian.

(6) Any distribution that we desire can be obtained.

(7) The calculation load is not too heavy.

There are many sorts of assets, but the principles used to evaluate assets are not unified. For instance, marketable assets traded actively in a market are evaluated by market value. On the other hand, non-marketable assets such as loans are measured by contract value (book value). Calculating each asset's value in the simplest way makes it difficult to compute the risk of the whole portfolio. In order to avoid this difficulty, we should adopt a unified measure to some extent.

There is agreement regarding the above conditions for integrative valuation models. But there has been no actual model satisfying these conditions. However, Kijima and Muromachi [17] recently proposed a framework to evaluate interest rate risk and credit risk in an integrative way and provided a concrete model which satisfies most of the above conditions (we hereafter refer to this model as the KM model). Tanaka and Muromachi [19] extended the KM model by including stock price risk, mortality rate risk, and prepayment risk, and discussed its applicability to ALM of an insurance company.

In this paper, we add stock price risk and foreign exchange risk to the KM model. Then we overview integrated evaluation models and provide simple calculation results.²

This paper is organized as follows. In Section 3., we propose a framework of the integrated evaluation model. In Section 4., we provide a concrete model based on this framework. In Section 5., we show numerical results of the model. Section 6. presents our conclusions.

3. Framework

In this section, we develop a general integrative framework to evaluate market and credit risk based on the model Kijima and Muromachi proposed.

First, we explain the hazard rate which defines the default process. Then, we introduce the structure of basic equations which describe the whole system, present the valuation method based on this structure, and finally show a general procedure to calculate the distribution of future asset prices.

Hereafter, we denote the observed probability measure by P , and consider the probability space (Ω, \mathcal{F}, P) . $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$ is a filtration generated by the stochastic system in this model. We assume that there exists a unique risk-neutral probability measure \tilde{P} .

3.1. Hazard rates

Let $t = 0$ be the present time and τ_j denotes the default time of Firm j . We define $h_j(t)$ to be its hazard rate of default under the observed probability measure P . The hazard rate $h_j(t)$ represents the instantaneous rate that the default occurs at time t given no default before that

time. Therefore, $h_j(t)$ under condition $\tau_j > t$ is given by

$$h_j(t) = \lim_{dt \rightarrow 0} \frac{P_t \{t < \tau_j \leq t + dt | \tau_j > t\}}{dt} \quad (1)$$

where P_t denotes the conditional probability measure of P given the information \mathcal{F}_t . The hazard rate can be thought of as the conditional intensity that a firm which does not default at time t will default during the instantaneous period $(t, t + dt]$.

Suppose that $h_j(t)$ is a deterministic function of t . Then the probability that Firm j survives after time T , $T > t$, is given by

$$P \{\tau_j > T\} = \exp \left\{ - \int_t^T h_j(u) du \right\}. \quad (2)$$

The default process of Firm j is characterized by the stochastic property of τ_j , but it is sufficient to consider the term structure of $h_j(t)$, since we have the equation (2).

Next, we deal with the case that $h_j(t)$ is a random variable and \mathcal{F}_t -measurable. If we know the information \mathcal{F}_T , the survival probability is given by

$$P_T \{\tau_j > T\} = \exp \{-H_j(t, T)\}, \quad H_j(t, T) = \int_t^T h_j(u) du, \quad (3)$$

where $H_j(t, T)$ is a cumulative hazard rate. On the other hand, the survival probability under the condition $\tau_j > t$ is given by

$$P_t \{\tau_j > T\} = E_t \left[\exp \left\{ - \int_t^T h_j(u) du \right\} \right], \quad (4)$$

where $E_t[\cdot]$ is the conditional expectation operator given the history \mathcal{F}_t .

3.2. Basic stochastic differential equations

We now construct a model based on stochastic differential equations (SDEs) which describe the behaviors of default-free interest rates, hazard rates, stock prices and exchange rates. In this section, we first introduce these SDEs under the observed probability measure P . Next, we explain how to construct the corresponding SDEs under the pseudo-probability measure, such as the risk-neutral and the forward-neutral probability measure.

3.2.1. Interest rate processes and hazard rate processes

First, following Kijima and Muromachi [17], we introduce SDEs which interest rate and hazard rate processes follow. This framework is basically the same as Jarrow and Turnbull [9], though the expression is different.³

Suppose that there exist n firms. Let $r(t) = h_0(t)$ be the default-free spot rate and $h_j(t)$, $j = 1, \dots, n$, be the hazard rate of Firm j . We assume that the default-free spot rate and the hazard rate processes under the observed probability measure P follow the system of SDEs:⁴

$$dr(t) = \mu_0(r(t), t)dt + \sigma_0(r(t), t)dz_0(t), \quad t \geq 0, \quad (5)$$

$$dh_j(t) = \mu_j(\mathbf{h}(t), t)dt + \sigma_j(\mathbf{h}(t), t)dz_j(t), \quad t \geq 0, \quad j = 1, \dots, n, \quad (6)$$

where $(z_0(t), z_1(t), \dots, z_n(t))$ is a $(n + 1)$ -dimensional standard Wiener process under P with correlation

$$dz_j(t)dz_k(t) = \rho_{jk}(t)dt, \quad j, k = 0, 1, \dots, n. \quad (7)$$

We mentioned in the previous section that a hazard rate has its own term structure. By selecting parameter values in (6) which fits the observed term structure, we can reflect results of empirical analyses. We discuss this issue later.

Let $\beta_0(t)$ be the market price of risk associated with $z_0(t)$. Then for sufficiently large T^* ,

$$\tilde{z}_0(t) = z_0(t) + \int_0^t \beta_0(u)du, \quad 0 \leq t \leq T^* \quad (8)$$

is a standard Wiener process under the risk-neutral probability measure \tilde{P} .⁵ The SDE for $r(t)$ under \tilde{P} is given by

$$dr(t) = \tilde{\mu}_0(r(t), t)dt + \sigma_0(r(t), t)d\tilde{z}_0(t), \quad 0 \leq t \leq T^*, \quad (9)$$

where

$$\tilde{\mu}_0(r(t), t) = \mu_0(r(t), t) - \beta_0(t)\sigma_0(r(t), t).$$

Next, we consider the hazard rates under \tilde{P} . By Artzner and Delbaen, when τ_j has its hazard process $h_j(t)$ under the probability measure P , there exists an intensity process of τ_j under \tilde{P} which is equivalent to P . Following Kijima [13], we assume that there exist a risk-premia adjustment $\ell_j(t)$ which is a deterministic function⁶ and satisfies

$$\tilde{h}_j(t) = h_j(t) + \ell_j(t), \quad j = 1, \dots, n, \quad (10)$$

where $\tilde{h}_j(t)$ is the hazard rate under the risk-neutral probability measure \tilde{P} . Note that the behavior of the default-free spot rate and the hazard rate processes under the risk-neutral probability measure \tilde{P} is determined by (6)C(7)C(9) and (10).

3.2.2. Stock price processes

As in Jarrow and Turnbull [9], we model stock price processes in addition to the processes in 3.2.1..⁷

Let $S_j(t)$ denote the stock price of Firm j . We assume that $S_j(t)$ follows the following SDE:⁸

$$\begin{aligned} \frac{dS_j(t)}{S_j(t)} &= (\mu_{s,j}(t) + X_j(t)h_j(t))dt + \sigma_{s,j}(t)dz_{n+j}(t) + dX_j(t) \\ &= \begin{cases} (\mu_{s,j}(t) + h_j(t))dt + \sigma_{s,j}(t)dz_{n+j}(t), & t < \tau_j, \\ -1, & t = \tau_j, \end{cases} \quad j = 1, \dots, n, \end{aligned} \quad (11)$$

where $X_j(t) = 1_{\{\tau_j > t\}}$ and $(z_0(t), z_1(t), \dots, z_n(t), z_{n+1}(t), \dots, z_{2n}(t))$ is a $(2n + 1)$ -dimensional standard Wiener measure under P with correlation

$$dz_j(t)dz_k(t) = \rho_{jk}(t)dt, \quad j, k = 0, 1, \dots, 2n.$$

⁵T

In this case, under some regular conditions, there exists a market price of risk $\beta_{n+j}(t)$ such that

$$\tilde{z}_{n+j}(t) = z_{n+j}(t) + \int_0^t \beta_{n+j}(u) du, \quad j = 1, \dots, n$$

is a standard Wiener process under the risk-neutral probability measure \tilde{P} .⁹ Then, $S_j(t)$ under \tilde{P} follows the following SDE:

$$\begin{aligned} \frac{dS_j(t)}{S_j(t)} &= \left(r(t) + X_j(t)\tilde{h}_j(t) \right) dt + \sigma_{s,j}(t)d\tilde{z}_{n+j}(t) + dX_j(t) \\ &= \begin{cases} \left(r(t) + \tilde{h}_j(t) \right) dt + \sigma_{s,j}(t)d\tilde{z}_{n+j}(t), & t < \tau_j, \\ -1, & t = \tau_j, \end{cases} \quad j = 1, \dots, n. \end{aligned} \quad (12)$$

3.2.3. Exchange rate processes

We further model foreign exchange rate processes as in Amin and Jarrow [1]. Let us consider a default-free interest rate process, hazard rate processes, and stock price processes in the foreign country. Let $V(t)$ be the exchange rate, $r_f(t)$ the default-free spot rate, n_f the number of firms, $\tau_{f,j}$ the default time of Firm j and $h_{f,j}(t)$ its hazard rate, all in the foreign country. It is assumed that they follow the following SDEs:¹⁰

$$dr_f(t) = \mu_{f,0}(r_f(t), t)dt + \sigma_{f,0}(r_f(t), t)dz_{f,0}(t), \quad t \geq 0, \quad (13)$$

$$dh_{f,j}(t) = \mu_{f,j}(\mathbf{h}_f(t), t)dt + \sigma_{f,j}(\mathbf{h}_f(t), t)dz_{f,j}(t), \quad t \geq 0, \quad j = 1, \dots, n_f, \quad (14)$$

$$\begin{aligned} \frac{dS_{f,j}(t)}{S_{f,j}(t)} &= (\mu_{fs,j}(t) + X_{f,j}(t)h_{f,j}(t)) dt + \sigma_{fs,j}(t)dz_{f,n_f+j}(t) + dX_{f,j}(t) \\ &= \begin{cases} (\mu_{fs,j}(t) + h_{f,j}(t)) dt + \sigma_{fs,j}(t)dz_{f,n_f+j}(t), & t < \tau_{f,j}, \\ -1, & t = \tau_{f,j}, \end{cases} \quad j = 1, \dots, n_f, \end{aligned} \quad (15)$$

$$\frac{dV(t)}{V(t)} = \mu_V(t)dt + \sigma_V(t)dz_V(t), \quad (16)$$

where $X_{f,j}(t) = 1_{\{\tau_{f,j} > t\}}$, and $\mathbf{z}(t) = (z_0(t), \dots, z_{2n}(t), z_{f,0}(t), \dots, z_{f,2n_f}(t), z_V(t))$ is a $(2n + 2n_f + 3)$ -dimensional standard Wiener measure under P with correlation

$$dz_j(t)dz_k(t) = \rho_{jk}(t)dt, \quad j, k = 0, 1, \dots, 2n, \quad (17)$$

$$dz_{f,j}(t)dz_{f,k}(t) = \rho_{jk}^{ff}(t)dt, \quad j, k = 0, 1, \dots, 2n_f, \quad (18)$$

$$dz_j(t)dz_{f,k}(t) = \rho_{jk}^f(t)dt, \quad j = 0, 1, \dots, 2n, \quad k = 0, 1, \dots, 2n_f, \quad (19)$$

$$dz_j(t)dz_V(t) = \rho_{jV}(t)dt, \quad j = 0, 1, \dots, 2n, \quad (20)$$

$$dz_{f,j}(t)dz_V(t) = \rho_{jV}^f(t)dt, \quad j = 0, 1, \dots, 2n_f. \quad (21)$$

As in the stock processes, $\mu_V(t)$ and $\sigma_V(t)$ may depend on other random variables. Note that $dz_V(t)dz_V(t) = dt$.

Next we consider the transformation of measure. We assume that for $h_{f,j}(t)$ and $\tilde{h}_{f,j}(t)$, the relationship like (10) holds. From the calculation similar to the stock price process, we can derive

the following SDEs:

$$dr_f(t) = \tilde{\mu}_{f,0}(r_f(t), t)dt + \sigma_{f,0}(r_f(t), t)d\tilde{z}_{f,0}(t), \quad 0 \leq t \leq T^*, \quad (22)$$

$$\tilde{h}_{f,j}(t) = h_{f,j}(t) + \ell_{f,j}(t), \quad j = 1, \dots, n_f, \quad (23)$$

$$\begin{aligned} \frac{dS_{f,j}(t)}{S_{f,j}(t)} &= \left(r_f(t) + X_{f,j}(t)\tilde{h}_{f,j}(t) \right) dt + \sigma_{f,s,j}(t)d\tilde{z}_{f,n_f+j}(t) + dX_{f,j}(t) \\ &= \begin{cases} \left(r_f(t) + \tilde{h}_{f,j}(t) \right) dt + \sigma_{f,s,j}(t)d\tilde{z}_{f,n_f+j}(t), & t < \tau_{f,j}, \\ -1, & t = \tau_{f,j}, \end{cases} \quad j = 1, \dots, n_f, \end{aligned} \quad (24)$$

$$\frac{dV(t)}{V(t)} = (r(t) - r_f(t))dt + \sigma_V(t)d\tilde{z}_V(t), \quad (25)$$

where $\tilde{\mathbf{z}}(t) = (\tilde{z}_0(t), \dots, \tilde{z}_{2n}(t), \tilde{z}_{f,0}(t), \dots, \tilde{z}_{f,2n_f}(t), \tilde{z}_V(t))$ is a $(2n + 2n_f + 3)$ -dimensional standard Wiener process under \tilde{P} and the correlation structure is the same as $\mathbf{z}(t)$.

We now have the framework we need to construct for the risk measure model. In summary, the stochastic processes under the observed probability measure P follow (5)C(6)C(11)C(13)–(16) and those under the risk-neutral probability measure \tilde{P} follow (9)C(10)C(12)C(22)–(25). The correlation structure for $\mathbf{z}(t)$ and $\tilde{\mathbf{z}}(t)$ is given by (17)–(21).

This system of equations is so flexible that we can construct many types of models by changing the structure of equations. We should note that Equation (12), (24) and (25) are necessary only for the evaluation of stock/forex derivatives in the future time. For the portfolio consisting of only stocks in the home and foreign countries (not including stock/forex derivatives), we only use (11), (15) and (16) to calculate the future value, and (12), (24) and (25) are not needed.

3.3. Conditional independence

It is well known that the realization of the hazard rates $h_j(t)$ alone cannot determine the joint distribution of default times τ_j , since the joint distribution cannot be constructed from their marginal distributions except in the independent case. Hence, a further assumption is necessary for our purpose. In our model, we assume that τ_j are *conditionally independent* given the realization of the underlying stochastic processes.

Let $P_t \{ \tau_1 > t_1, \dots, \tau_n > t_n \}$, $t_j \geq t$, be the joint distribution of default times. The conditional independence means that given \mathcal{F}_T where $T \geq \max_j t_j$, the default times τ_j mutually independent, i.e.,

$$P_T \{ \tau_1 > t_1, \dots, \tau_n > t_n \} = \prod_{j=1}^n P_T \{ \tau_j > t_j \}. \quad (26)$$

By taking the unconditional expectation for both sides in (26), we then obtain

$$P \{ \tau_1 > t_1, \dots, \tau_n > t_n \} = E \left[\prod_{j=1}^n P_T \{ \tau_j > t_j \} \right] = E \left[\exp \left\{ - \sum_{j=1}^n \int_0^{t_j} h_j(u) du \right\} \right], \quad (27)$$

where we have used Equation (3). We can take the correlation structure of $h_j(t)$ into consideration when we assume the conditional independence in contrast to the ordinary independence, i.e.

$$P\{\tau_1 > t_1, \dots, \tau_n > t_n\} = \prod_{j=1}^n E \left[\exp \left\{ - \int_0^{t_j} h_j(u) du \right\} \right]$$

holds.

The reader who want to understand the conditional independence for more detail should consult the latter section which treats the simulation.

3.4. Valuation of present values

In this subsection, we briefly touch upon the general issue on the valuation of financial products, and then mention the valuation of defaultable discount bonds. The idea is based on the framework of Jarrow and Turnbull [9].

According to the no-arbitrage pricing theory in the finance literature, pricing of derivatives can be done by the risk-neutral method or the forward-neutral method. Consider a contingent claim which generates a cash flow X at the maturity date T . The price of this claim at time $t, t < T$, $p(t, T)$, can be written as

$$p(t, T) = \tilde{E}_t \left[e^{-\int_t^T r(s) ds} X \right] = v_0(t, Y) E_t^T[X], \quad (28)$$

where $\tilde{E}_t[\cdot]$ is a conditional expectation operator under the risk-neutral probability measure, $E_t^T[\cdot]$ is a conditional expectation operator under the forward-neutral probability measure, and $v_0(t, T)$ is the price at time t of discount bond free from default. For example, when the claim is an European call option of Stock j , we can calculate the price by substituting

$$X = (S_j(T) - K)^+,$$

where $S_j(T)$ is the stock price at the maturity date T , K is the exercise price, and $(A)^+ = \max(A, 0)$.

We provide a concrete example of a defaultable discount bond in order to make clear the idea of this pricing method. Consider the i -th discount bond which Firm j issued. For simplicity, we assume that a holder of the bond receives \$1 at the maturity T_i^j if the default does not occur until T_i^j , and δ_j if the default occurs. This assumption is consistent with Jarrow and Turnbull [9], when we consider the case that the bond holder receives $\delta_j v_0(\tau_j, T_i^j)$ at the default time τ_j when the default occurs. Our assumption can make the problem fairly simple because the future cash flow is not explicitly dependent on τ_j .

The cash flow of the bond holder at the maturity date T_j can be expressed as $X = \delta_j 1_{\{\tau_j \leq T_i^j\}} + 1_{\{\tau_j > T_i^j\}}$. Let $v_j(t, T)$ denote the timet price of the defaultable discount bond with maturity T issued by Firm j . Then, $v_j(t, T)$ is given by

$$\begin{aligned} v_j(t, T_i^j) &= \tilde{E}_t \left[\exp \left\{ - \int_t^{T_i^j} r(u) du \right\} \left\{ \delta_j 1_{\{\tau_j \leq T_i^j\}} + 1_{\{\tau_j > T_i^j\}} \right\} \right] \\ &= \delta_j v_0(t, T_i^j) + (1 - \delta_j) \tilde{E}_t \left[e^{-H_0(t, T_i^j) - \tilde{H}_j(t, T_i^j)} \right], \end{aligned} \quad (29)$$

where

$$\begin{aligned} v_0(t, T) &= \tilde{E}_t \left[e^{-H_0(t, T)} \right], \\ H_0(t, T) &= \int_t^T h_0(u) du, \\ \tilde{H}_j(t, T) &= \int_t^T \tilde{h}_j(u) du, \quad j = 0, 1, \dots, n. \end{aligned}$$

If $r(t)$ and $h_j(t)$ are independent, Equation (29) coincides

$$v_j(t, T_i^j) = v_0(t, T_i^j) \left[\delta_j + (1 - \delta_j) \tilde{P}_t \{ \tau_j > T_i^j \} \right],$$

which Jarrow and Turnbull [9] obtained. Here, $\tilde{P}_t \{ \tau_j > T_i^j \}$ is the survival probability of Firm j under the risk-neutral probability measure \tilde{P} conditional on \mathcal{F}_t . When $r(t)$ and $h_j(t)$ are not independent, we have to take the correlation between $H_0(t, T_i^j)$ and $\tilde{H}_j(t, T_i^j)$ into account in calculating the expectation.

The use of the forward-neutral probability measure makes the calculation simpler. Let P^T denote the forward-neutral probability measure. Then, the defaultable discount bond price is given by

$$\begin{aligned} v_j(t, T_i^j) &= v_0(t, T_i^j) E_t^{T_i^j} \left[\delta_j 1_{\{ \tau_j \leq T_i^j \}} + 1_{\{ \tau_j > T_i^j \}} \right] \\ &= v_0(t, T_i^j) \left[\delta_j + (1 - \delta_j) P_t^{T_i^j} \{ \tau_j > T_i^j \} \right], \end{aligned} \quad (30)$$

where P_t^T denotes the conditional probability measure given \mathcal{F}_t under P^T , which is equivalent to the risk-neutral probability measure \tilde{P} . We need not consider the correlation between the interest rate and the hazard rate in this method.

In (30), the survival probability $P_t^{T_i^j}$ can be calculated as follows. Let $h_j^T(t)$ be the risk-adjusted hazard rate process under P^T , and $h_j(t)$ be the observed hazard rate process. We assume that there exist a *risk premia adjustment* $\ell_j^T(t)$ satisfying

$$h_j^T(t) = h_j(t) + \ell_j^T(t) \quad (31)$$

In general, each risk premia adjustment $\ell_j^T(t)$ may depend on the whole history and the maturity T . However, we assume in our framework that $\ell_j^T(t)$ is a deterministic function of time t and independent of the maturity, and so we can denote it by $\ell_j(t)$. Then, the marginal survival probability under $P_t^{T_i^j}$ is given by

$$\begin{aligned} P_t^{T_i^j} \{ \tau_j > T \} &= E_t^{T_i^j} \left[\exp \left\{ - \int_t^T h_j^{T_i^j}(u) du \right\} \right] = P_t \{ \tau_j > T \} L_j(t, T), \quad t \leq T \leq T_i^j, \quad (32) \\ L_j(t, T) &= \exp \left\{ - \int_t^T \ell_j(u) du \right\}. \end{aligned}$$

At time t , we can observe the value of $v_0(t, s)$, $v_j(t, s)$ and $P_t \{ \tau_j > s \}$, $s \geq t$ in the market. Therefore, the risk premia adjustment $\ell_j^T(t)$ can be calculated as

$$\ell_j(s) = - \frac{\partial}{\partial s} \log \left\{ \frac{1}{P_t \{ \tau_j > s \}} \left(\frac{v_j(t, s)}{v_0(t, s)} - \delta_j \right) \right\}, \quad s \geq t. \quad (33)$$

$\beta_0(t)$, the market price of risk of $z_0(t)$, also can be estimated by the term structure of $v_0(t, s)$. We deal with this issue later.

When we obtain the functional form of $\ell_j^T(t) = \ell_j(s)$, we also obtain $h_j^T(s)$, the hazard rate under P^T . Hence, we can compute other prices of assets with the credit risk by the forward-neutral method. In general, we can evaluate asset prices more easily by the forward-neutral method than by the risk-neutral method. We have to take notice that the forward-neutral probability measure P^T depends on the maturity T .

Thanks to (33), $\ell_j(s)$ can be obtained and is consistent with the market data. Since any market price includes all risks evaluated by the market other than interest rate risk and credit risk, $\ell_j(s)$ obtained in this way comprehends effects by these risks. Using this $\ell_j(s)$ enables us to include implicitly the effect of other risks of the future in the model. This $\ell_j(s)$ can be said to be an very effective tool when we want to measure multiple risks in a integrative way, which is consistent with observable market values.

3.5. Distribution of future portfolio value

Let us consider the distribution of the future value of the portfolio which consists of discount bonds and stocks.

In what follows, we denote the risk horizon by $T, T > t$ and suppose that the our model satisfies the assumptions specified so far. Then the price at the risk horizon T of the i -th defaultable discount bond issued by Firm j with maturity T_i^j is given from (30) and (32) by

$$\begin{aligned} v_j(T, T_i^j) &= \begin{cases} v_0(T, T_i^j) \left[\delta_j + (1 - \delta_j) P_T \left\{ \tau_j > T_i^j \right\} L_j(T, T_i^j) \right], & \tau_j > T \\ v_0(T, T_i^j) \delta_j, & \tau_j \leq T \end{cases} \\ &= v_0(T, T_i^j) \left[\delta_j + (1 - \delta_j) P_T \left\{ \tau_j > T_i^j \right\} L_j(T, T_i^j) 1_{\{\tau_j > T\}} \right]. \end{aligned} \quad (34)$$

Here $v_j(T, T_i^j)$ is a random variable dependent on $v_0(T, T_i^j) C P_T \left\{ \tau_j > T_i^j \right\}$ and τ_j . On the other hand, the stock price issued by Firm j is writtem from (11) as

$$\begin{aligned} S_j(T) &= \begin{cases} S_j(t) + \int_t^T S_j(u) (\mu_{s,j}(u) + h_j(u)) du + \int_t^T S_j(u) \sigma_{s,j}(u) dz_{n+j}(u), & \tau_j > T \\ 0, & \tau_j \leq T, \end{cases} \\ &= \left[S_j(t) + \int_t^T S_j(u) (\mu_{s,j}(u) + h_j(u)) du + \int_t^T S_j(u) \sigma_{s,j}(u) dz_{n+j}(u) \right] 1_{\{\tau_j > T\}}. \end{aligned} \quad (35)$$

From (34) and (35), the portfolio value at the risk horizon T is given by

$$\begin{aligned} \pi(T) &= \sum_{j=0}^n \left(\sum_{i=1}^{N_j} w_i^j v_0(T, T_i^j) \left[\delta_j + (1 - \delta_j) P_T \left\{ \tau_j > T_i^j \right\} L_j(T, T_i^j) 1_{\{\tau_j > T\}} \right] \right. \\ &\quad \left. + w_s^j \left[S_j(t) + \int_t^T S_j(u) (\mu_{s,j}(u) + h_j(u)) du + \int_t^T S_j(u) \sigma_{s,j}(u) dz_{n+j}(u) \right] 1_{\{\tau_j > T\}} \right) \end{aligned} \quad (36)$$

where w_s^j denotes the number of $S_j(t)$ which the portfolio has, w_i^j denotes the number of $v_j(t, T_i^j)$, and N_j is the number of discount bonds issued by Firm j (or the default-free discount bonds for $j = 0$).

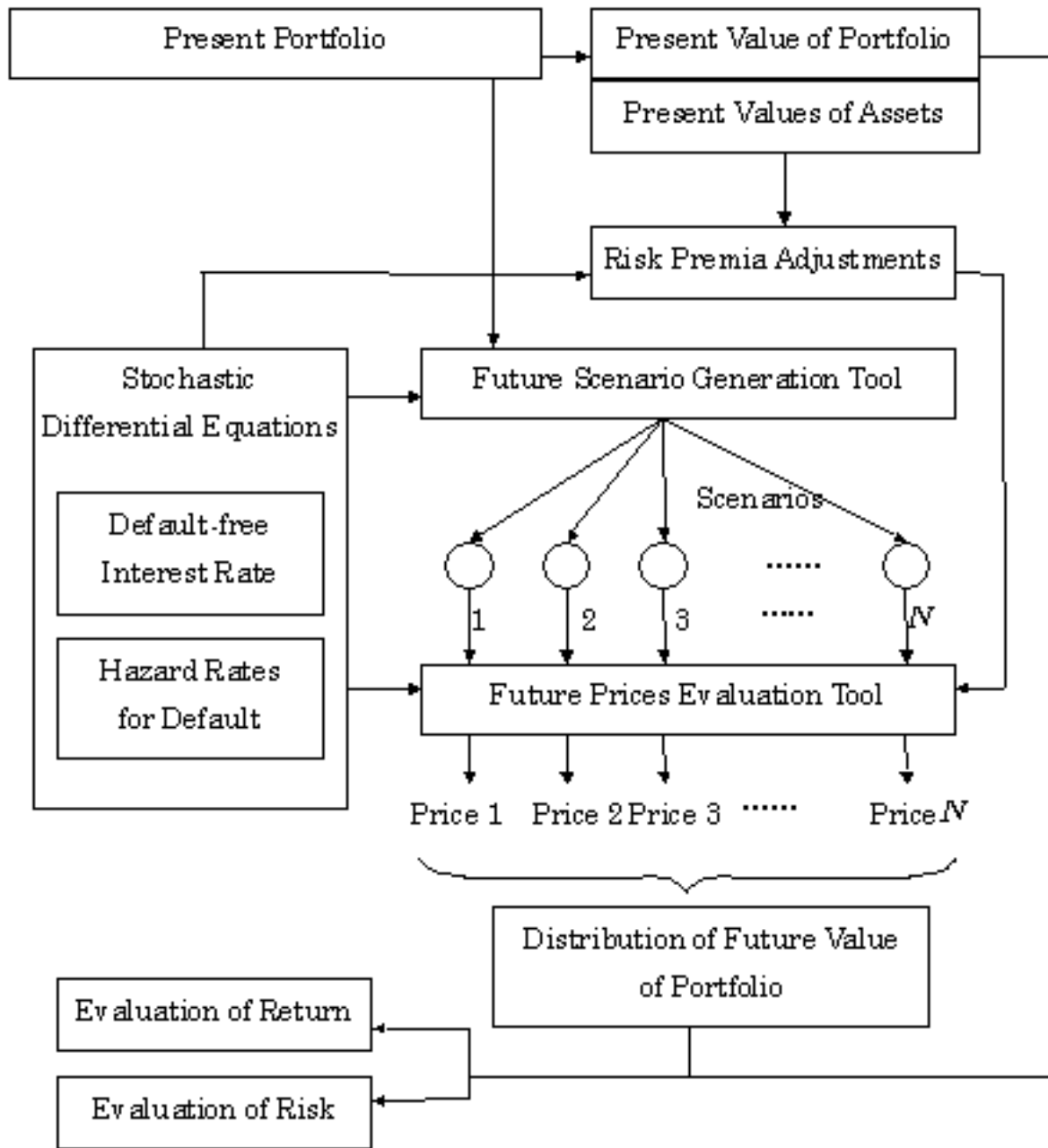
3.6. Global structure of the model and general calculation procedures

After obtaining the joint distribution of $v_0(T, T_i^j)$, $P_T \{ \tau_j > T_i^j \}$ and τ_j at T from the basic SDEs, we can calculate by (36) the distribution of the portfolio in principle. However, it seems very difficult to obtain analytically the distribution of the portfolio even when $v_0(T, T_i^j)$ and $P_T \{ \tau_j > T_i^j \}$ follow tractable distributions, because Equation (36) includes the indicator function 1_A . Hence, in this case, we compute the distribution of the portfolio by Monte-Carlo simulation.

We show the brief scheme in Figure 1. As mentioned above, this framework is flexible in setting the models of default-free interest rates, hazard rates, stock prices, foreign exchange rates. Therefore, we can flexibly construct a variety of models by changing the structure of SDEs.

As in the above procedure, we have to use the Monte-Carlo simulation in a nested way to obtain the distribution of the future value except for some special settings. That is, the generation of scenarios until the risk horizon is the first-step simulation and the evaluation of each scenario at the risk horizon is the second-step. When many scenarios are necessary at each step simulation, it takes a long time to compute and this framework becomes practically unrealistic. We show a Gaussian model which is an extension of Kijima and Muromachi [17] and makes the computation time remarkably shorter.

Figure 1: The Structure of the Model



4. The Gaussian Model

The Gaussian model enables us to obtain the closed-form solutions and save much time in calculating values.

4.1. Basic equations and their analytical solutions

4.1.1. Basic equations

Let $t = 0$ be the present time. We assume here that in our model the basic equations under the observed probability measure P follow the system of SDEs

$$dr(t) = (b_0(t) - a_0r(t)) dt + \sigma_0 dz_0(t), \quad t \geq 0, \quad (37)$$

$$dh_j(t) = (b_j(t) - a_j h_j(t)) dt + \sigma_j dz_j(t), \quad t \geq 0; \quad j = 1, \dots, n, \quad (38)$$

$$\frac{dS_j(t)}{S_j(t)} = \begin{cases} \mu_{s,j} dt + \sigma_{s,j} dz_{n+j}(t), & t < \tau_j, \\ -1, & t \geq \tau_j, \end{cases} \quad j = 1, \dots, n, \quad (39)$$

$$dr_f(t) = (b_{f,0}(t) - a_{f,0}r_f(t)) dt + \sigma_{f,0} dz_{f,0}(t), \quad t \geq 0, \quad (40)$$

$$dh_{f,j}(t) = (b_{f,j}(t) - a_{f,j}h_{f,j}(t)) dt + \sigma_{f,j} dz_{f,j}(t), \quad t \geq 0; \quad j = 1, \dots, n_f, \quad (41)$$

$$\frac{dS_{f,j}(t)}{S_{f,j}(t)} = \begin{cases} \mu_{fs,j} dt + \sigma_{fs,j} dz_{f,n_f+j}(t), & t < \tau_{f,j}, \\ -1, & t \geq \tau_{f,j}, \end{cases} \quad j = 1, \dots, n_f, \quad (42)$$

$$\frac{dV(t)}{V(t)} = \mu_V dt + \sigma_V dz_V(t). \quad (43)$$

where $a_0, a_j, a_{f,0}, a_{f,j}, \sigma_0, \sigma_j, \sigma_{f,0}, \sigma_{f,j}, \sigma_{s,j}, \sigma_{fs,j}, \sigma_V$ are non-negative constants, $\mu_{s,j}, \mu_{fs,j}, \mu_V$ are constants, $b_0(t), b_j(t), b_{f,0}(t), b_{f,j}(t)$ are deterministic function of time t and the correlations between $z_j(t)$ are constants. Let us further assume that the market price of risk $\beta_0(t), \beta_{f,0}(t)$ associated with $z_0(t), z_{f,0}(t)$ and the risk-premia adjustments $\ell_j(t), \ell_{f,k}(t)$ associated with $h_j(t), h_{f,k}(t), j = 1, \dots, n, k = 1, \dots, n_f$, are deterministic functions of time. Then, each variable follows under the risk-neutral probability \tilde{P} the following SDEs:

$$dr(t) = (\phi_0(t) - a_0r(t)) dt + \sigma_0 d\tilde{z}_0(t), \quad t \geq 0, \quad (44)$$

$$\tilde{h}_j(t) = h_j(t) + \ell_j(t), \quad j = 1, \dots, n, \quad (45)$$

$$\frac{dS_j(t)}{S_j(t)} = \begin{cases} [r(t) + \tilde{h}_j(t)] dt + \sigma_{s,j} d\tilde{z}_{n+j}(t), & t < \tau_j, \\ -1, & t \geq \tau_j, \end{cases} \quad j = 1, \dots, n, \quad (46)$$

$$dr_f(t) = (\phi_{f,0}(t) - a_{f,0}r_f(t)) dt + \sigma_{f,0} d\tilde{z}_{f,0}(t), \quad t \geq 0, \quad (47)$$

$$\tilde{h}_{f,j}(t) = h_{f,j}(t) + \ell_{f,j}(t), \quad j = 1, \dots, n_f, \quad (48)$$

$$\frac{dS_{f,j}(t)}{S_{f,j}(t)} = \begin{cases} [r_f(t) + \tilde{h}_{f,j}(t)] dt + \sigma_{fs,j} d\tilde{z}_{f,n_f+j}(t), & t < \tau_{f,j}, \\ -1, & t \geq \tau_{f,j}, \end{cases} \quad j = 1, \dots, n_f, \quad (49)$$

$$\frac{dV(t)}{V(t)} = [r(t) - r_f(t)] dt + \sigma_V d\tilde{z}_V(t). \quad (50)$$

Note that the default-free interest rate described as (44) is an extended Vasicek model which appeared in Hull and White. According to Inui and Kijima [8], the function $\phi_j(t)$ can be obtained

so that the model is consistent with the current term structure of the default-free interest rates observed in the market. That is to say,

$$\phi_0(t) = a_0 f_0(0, t) + \frac{\partial}{\partial t} f_0(0, t) + \frac{\sigma_0^2}{2a_0} (1 - e^{-2a_0 t}), \quad (51)$$

$$\phi_{f,0}(t) = a_{f,0} f_{f,0}(0, t) + \frac{\partial}{\partial t} f_{f,0}(0, t) + \frac{\sigma_{f,0}^2}{2a_{f,0}} (1 - e^{-2a_{f,0} t}), \quad (52)$$

where $f_0(0, t)$, $f_{f,0}(0, t)$ denote the forward rate of the default-free discount bond. The market prices of risk associated with $z_0(t)$ and $z_{f,0}(t)$ are given by

$$\beta_0(t) = \frac{b_0(t) - \phi_0(t)}{\sigma_0}, \quad \beta_{f,0}(t) = \frac{b_{f,0}(t) - \phi_{f,0}(t)}{\sigma_{f,0}},$$

respectively.

The hazard rate processes under P and \tilde{P} are formally the same as extended Vasicek model. From (38) and (45) and the assumption of the risk-premia adjustments, we obtain

$$\begin{aligned} d\tilde{h}_j(t) &= (\phi_j(t) - a_j \tilde{h}_j(t)) dt + \sigma_j d\tilde{z}_j(t), \quad j = 1, \dots, n, \\ \phi_j(t) &= b_j(t) + a_j \ell_j(t) + \frac{d\ell_j(t)}{dt}. \end{aligned} \quad (53)$$

Following Kijima [14], $\phi_j(t)$ can be computed so that the model is consistent with the current term structure of the defaultable interest rates observed in the market. That is to say, $\phi_j(t)$ is given by

$$\begin{aligned} \phi_j(t) &= a_j g_j(0, t) + \frac{\partial}{\partial t} g_j(0, t) + \frac{\sigma_j^2}{2a_j} (1 - e^{-2a_j t}) \\ &\quad + \rho_{0j} \sigma_0 \sigma_j \left(\frac{1 - e^{-a_0 t}}{a_0} + \frac{e^{-a_0 t} - e^{-(a_0 + a_j)t}}{a_j} \right), \end{aligned} \quad (54)$$

where

$$g_j(t, T) = -\frac{\partial}{\partial T} \log \left[\frac{v_j(t, T)}{v_0(t, T)} - \delta_j \right].$$

The market price of risk associated with $z_{n+j}(t)$ is given by

$$\beta_{n+j}(t) = \frac{\mu_{s,j} - r(t) - \tilde{h}_j(t)}{\sigma_{s,j}}.$$

Other market prices of risk associated with $z_{f,n_f+j}(t)$, $z_V(t)$ are given in the same way as above.

4.1.2. Analytical solutions

$h_j(t)$ in (38) and its cumulative hazard rates are easily solved as

$$\begin{aligned} h_j(t) &= h_j(0) e^{-a_j t} + \int_0^t b_j(u) e^{-a_j(t-u)} du + \sigma_j \int_0^t e^{-a_j(t-u)} dz_j(u), \quad t \geq 0, \\ H_j(t, T) &= h_j(t) B_j(t, T) + \int_t^T b_j(u) B_j(u, T) du + \sigma_j \int_t^T B_j(u, T) dz_j(u), \quad 0 \leq t \leq T, \end{aligned}$$

where

$$B_j(t, T) = \frac{1 - e^{-a_j(T-t)}}{a_j}.$$

Here $h_j(t)$ and $H_j(t, T)$ follow Gauss-Markov processes. Their means, variances and covariance are given by

$$\begin{aligned} m_j(t) &= E[h_j(t)] = h_j(0)e^{-a_j t} + \int_0^t b_j(u)e^{-a_j(t-u)} du, \\ s_j^2(t) &= V[h_j(t)] = \frac{\sigma_j^2}{2a_j} (1 - e^{-2a_j t}), \\ M_j(t, T) &= E[H_j(t, T)] = h_j(t)B_j(t, T) + \int_t^T b_j(u)B_j(u, T) du, \\ S_j^2(t, T) &= V[H_j(t, T)] = \frac{\sigma_j^2}{a_j^2} \left[(T-t) - 2\frac{1 - e^{-a_j(T-t)}}{a_j} + \frac{1 - e^{-2a_j(T-t)}}{2a_j} \right], \\ s_{jk}(t) &= \text{Cov}[h_j(t), h_k(t)] = \frac{\rho_{jk}\sigma_j\sigma_k}{a_j + a_k} (1 - e^{-(a_j+a_k)t}), \\ S_{jk}(t, T) &= \text{Cov}[H_j(t, T), H_k(t, T)] \\ &= \frac{\rho_{jk}\sigma_j\sigma_k}{a_j a_k} \left[(T-t) - \frac{1 - e^{-a_j(T-t)}}{a_j} - \frac{1 - e^{-a_k(T-t)}}{a_k} + \frac{1 - e^{-(a_j+a_k)(T-t)}}{a_j + a_k} \right], \\ C_{jk}(t) &= \text{Cov}[h_j(t), H_k(0, t)] = \frac{\rho_{jk}\sigma_j\sigma_k}{a_j} \left[\frac{1 - e^{-a_j t}}{a_j} - \frac{1 - e^{-(a_j+a_k)t}}{a_j + a_k} \right]. \end{aligned}$$

We can obtain the solution of (37)C(40)C(41)C(44)C(47)C(53) in the same manner as that of (38).

Next, we consider the stock price and the exchange rate. The solution of (39) is written as

$$S_j(t) = \left[S_j(0) \exp \left\{ \left(\mu_{s,j} - \frac{1}{2} \sigma_{s,j}^2 \right) t + \sigma_{s,j} z_j(t) \right\} \right] 1_{\tau_j > t}, \quad t \geq 0, \quad j = 1, \dots, n.$$

We can solve (42) in the same manner. Also, the solution of (43) is given by

$$V(t) = V(0) \exp \left\{ \left(\mu_V - \frac{1}{2} \sigma_V^2 \right) t + \sigma_V z_V(t) \right\}, \quad t \geq 0.$$

Finally, (46), (49) and (50) are also solvable. For instance, the solution of (46) is given by

$$\begin{aligned} S_j(t) &= S_j(0) \exp \left\{ \int_0^t (r(s) + \tilde{h}_j(s)) ds - \frac{1}{2} \sigma_{s,j}^2 t + \sigma_{s,j} \tilde{z}_{s,j}(t) \right\} \\ &= S_j(0) \exp \left\{ \tilde{H}_0(0, t) + \tilde{H}_j(0, t) - \frac{1}{2} \sigma_{s,j}^2 t + \sigma_{s,j} \tilde{z}_{s,j}(t) \right\}. \end{aligned}$$

$S_j(t)$ is a random variable dependent on $(\tilde{H}_0(0, t), \tilde{H}_j(0, t), \tilde{z}_{s,j}(t))$, which follows a 3-dimensional multivariate Gaussian distribution. These variables are only used in the evaluation of stock/foreign exchange derivatives and do not appear explicitly.

4.2. Term structure of hazard rates

We provide a concrete method which comprehends empirical results on the term structure of default probabilities.

In this model, we assume that the term structure of the hazard rates follows the Weibull distribution¹¹ with three parameters

$$m_j(t) = E[h_j(t)] = \lambda_j \gamma_j (t + m_j)^{\gamma_j - 1}, \quad t \geq 0, \lambda_j, \gamma_j > 0, m_j \geq 0. \quad (55)$$

The parameters λ_j, γ_j, m_j in (55) can be estimated from default data. Note that in this case, we have

$$b_j(t) = \frac{\partial}{\partial t} m_j(t) + a_j m_j(t) = \lambda_j \gamma_j [\gamma_j - 1 + a_j(t + m_j)] (t + m_j)^{\gamma_j - 2}$$

from the solution of $h_j(t)$.

4.3. Valuation formulas of simple instruments

We present some solutions of prices of simple products.

4.3.1. Discount bond

The time t price of the default-free discount bond is given by

$$v_0(t, T) = A_0(t, T) e^{-B_0(t, T)r(t)},$$

where

$$A_j(t, T) = \exp \left\{ \frac{1}{2} S_j^2(t, T) - \int_t^T \phi_j(u) B_j(u, T) du \right\}, \quad j = 0, 1, \dots, n.$$

The time t price of the defaultable discount bond with constant recovery rate δ_j is given from (29) by

$$\begin{aligned} v_j(t, T_i^j) &= v_0(t, T_i^j) \left[\delta_j + (1 - \delta_j) A_j(t, T_i^j) e^{S_{0j}(t, T_i^j)} e^{-B_j(t, T_i^j) \tilde{h}_j(t)} \mathbf{1}_{\{\tau_j > t\}} \right] \\ &= v_0(t, T_i^j) \left[\delta_j + (1 - \delta_j) P_t \left\{ \tau_j > T_i^j \right\} L_j(t, T_i^j) e^{S_{0j}(t, T_i^j)} \mathbf{1}_{\{\tau_j > t\}} \right] \end{aligned}$$

The forward-neutral method makes the expression of the price simpler. The time t price of the defaultable discount bond with recovery rate δ_j is given from (30) and (32) by

$$v_j(t, T_i^j) = v_0(t, T_i^j) \left[\delta_j + (1 - \delta_j) L_j(t, T) P_t \left\{ \tau_j > T_i^j \right\} \right]. \quad (56)$$

4.3.2. Fixed-rate coupon bond

The time t price of the defaultable coupon bond with recovery rate δ_j is given by

$$p_j(t, \mathbf{T}; C, \delta) = v_j(t, t_M; \delta) + C \sum_{j=1}^M v_j(t, t_j; \delta),$$

where C is the coupon rate, t_M is the maturity of the bond, and $\mathbf{T} = (t_1, t_2, \dots, t_M)$, $t_j > t$, $j = 1, \dots, M$ is the coupon payment dates. Especially, the time t price of the default-free coupon bond is given by

$$p_0(t, \mathbf{T}; C) = v_0(t, t_M) + C \sum_{j=1}^M v_0(t, t_j). \quad (57)$$

4.3.3. Floating-rate coupon bond

We consider the following floating-rate coupon bond. Let $\mathbf{T} = (t_1, t_2, \dots, t_M)$, $t_j > t$, $j = 1, \dots, M$ be the coupon payment dates, t_M be the maturity date, and δ be the recovery rate. The coupon rate in the period $(t_j, t_{j+1}]$ is $C(t_j, q)$ and $C(t, q)$ is

$$C(t, q) = \alpha R(t, q) + \beta, \quad R(t, q) = -\frac{\log v_0(t, t+q)}{q}, \quad q > 0,$$

where α and β are constants, and $R(t, q)$ denotes the default-free discount yield with the maturity q at time t . Then, we have the following relation,

$$C(t, q) = \alpha' r(t) + \beta',$$

where

$$\alpha' = \frac{\alpha}{q} B_0(t, t+q), \quad \beta' = \beta - \frac{\alpha}{q} \log A_0(t, t+q) \quad (58)$$

$t_0 < t$ denote the latest coupon date before t . If $t_0 = t$, the bond price is supposed to be ex-dividend.

When the bond is default-free, then the time t price of the floating-rate bond is given by

$$p_0(t, \mathbf{T}; \alpha, \beta) = v_0(t, t_M) + C(t_0, q)(t_1 - t_0)v_0(t, t_1) + \sum_{j=2}^M p_0(t, t_{j-1}, t_j; \alpha', \beta'), \quad (59)$$

where $C(t_0, q)$ is the last coupon rate observable at $t_0 \leq t$, and

$$p_0(t, t_1, t_2; \alpha', \beta') = p_0(t, t_1, t_2; \alpha, \beta, q)$$

denotes the price of the default-free discount bond whose holder receives $C(t_1, q)(t_2 - t_1)$ at t_2 , $t_2 > t_1$. Simple algebra yields

$$p_0(t, t_1, t_2; \alpha', \beta') = (t_2 - t_1)v_0(t, t_2) \left[\beta' + \alpha' \left\{ r(t)e^{-a_0(t_1-t)} + \int_t^{t_1} \phi_0(u)e^{-a_0(t_1-u)} du - \frac{\sigma_0^2}{a_0^2} \left(1 - e^{-a_0(t_1-t)} - \frac{e^{-a_0(t_2-t_1)} - e^{-a_0(t_1+t_2-2t)}}{2} \right) \right\} \right]$$

Next, when the floating-rate bond is defaultable, then the time t price of the bond is given by

$$p_j(t, \mathbf{T}; \alpha, \beta, \delta) = v_j(t, t_M; \delta) + C(t_0, q)(t_1 - t_0)v_j(t, t_1; \delta) + \sum_{j=2}^M p_j(t, t_{j-1}, t_j; \alpha', \beta', \delta),$$

where

$$p_j(t, t_1, t_2; \alpha', \beta', \delta) = p_j(t, t_1, t_2; \alpha, \beta, q, \delta)$$

denotes the price of the defaultable discount bond whose holder receives $C(t_1, q)(t_2 - t_1)$ at t_2 , $t_2 > t_1$. By some calculations, we obtain

$$p_j(t, t_1, t_2; \alpha', \beta', \delta) = p_0(t, t_1, t_2; \alpha', \beta') \left[\delta + (1 - \delta)P_t^{t_2} \{ \tau_j > t_2 \} \right] - (1 - \delta)(t_2 - t_1)\alpha'v_0(t, t_2)P_t^{t_2} \{ \tau_j > t_2 \} \times \frac{\rho_{0j}\sigma_0\sigma_j}{a_j} \left\{ \frac{1 - e^{-a_0(t_1-t)}}{a_0} - \frac{e^{-a_j(t_2-t_1)} - e^{-a_0(t_1-t) - a_j(t_2-t)}}{a_0 + a_j} \right\}.$$

4.3.4. Stock option

The time t price of the European call stock option of Firm j with maturity $T > t$ and exercise price $K > 0$ is given by

$$c(t, T; K) = S_j(t)\Phi(z) - Kv_j(t, T; \delta_j = 0)\Phi(z - \sigma_X),$$

where

$$\begin{aligned} z &= \frac{\log\left(\frac{S_j(t)}{Kv_j(t, T; \delta_j = 0)}\right)}{\sigma_X} + \frac{1}{2}\sigma_X, \\ \sigma_X &= \left(\sigma_0^2 \int_t^T B_0^2(u, T)du + \sigma_j^2 \int_t^T B_j^2(u, T)du + \sigma_{s,j}^2(T-t) \right. \\ &\quad + 2\sigma_0\sigma_j\rho_{0j} \int_t^T B_0(u, T)B_j(u, T)du + 2\sigma_0\sigma_{s,j}\rho_{0,n+j} \int_t^T B_0(u, T)du \\ &\quad \left. + 2\sigma_j\sigma_{s,j}\rho_{j,n+j} \int_t^T B_j(u, T)du \right)^{\frac{1}{2}}, \end{aligned}$$

and $v_j(t, T; \delta_j = 0)$ is the time t price of the discount bond issued by Firm j with maturity T with recover rate zero. We can obtain by some calculation

$$\begin{aligned} \log v_j(t, T; \delta_j = 0) &= -r(t)B_0(t, T) - \int_t^T \phi_0(u)B_0(u, T)du \\ &\quad - \tilde{h}_j(t)B_j(t, T) - \int_t^T \phi_j(u)B_j(u, T)du \\ &\quad + \frac{1}{2}\sigma_0^2 \int_t^T B_0^2(u, T)du + \frac{1}{2}\sigma_j^2 \int_t^T B_j^2(u, T)du \\ &\quad + \sigma_0\sigma_j\rho_{0j} \int_t^T B_0(u, T)B_j(u, T)du. \end{aligned}$$

The time t price of the European put stock option of Firm j with maturity $T > t$ and exercise price $K > 0$ is given by

$$p(t, T; K) = Kv_j(t, T; \delta_j = 0)\Phi(-z + \sigma_X) - S(t)\Phi(-z)$$

5. Numerical Examples

We briefly provide results on the distribution at the risk horizon of the portfolio consisting of n bonds. The risk horizon is set to be 1 year.

Before proceeding, we briefly explain the risk measures we use below.

- VaR: The difference between the expected value of the portfolio and 100(1 - α)-percentile point at the risk horizon. This can be obtained from the distribution of each price at the risk horizon.
- T-VaR: The difference between the expected value of the portfolio and the average below 100(1 - α)-percentile point at the risk horizon.

- RC (risk contribution): Risk quantity of each asset which is allocated based on the risk quantity of the whole portfolio (e.g. standard deviation, VaR, T-Var). This can be obtained by simulation results. The summation of each asset RC is equal to RC of the portfolio.
- RC ratio: The discounted present value of RC.
- RC measure: The proportion of RC ratio to the expected return. This measure is used for return/risk analysis.

Note that RC, RC ratio and RC measure are not commonly used terminologies.

5.1. Preconditions

The preconditions are basically the same as Kijima and Muromachi [17]. Table 1 shows the attributions of each bond (credit rating, face value, maturity, coupon rate, discovery rate).

Table 1: Attribution of each bond of the portfolio

Firm	Credit rating	Face value	Maturity (years)	Coupon rate	Recovery rate
A	Aaa	7	3	7.25%	0
B	Aa	1	4	8.0%	0
C	A	1	3	8.25%	0
D	Baa	1	4	9.0%	0
E	Ba	1	3	9.25%	0
F	B	1	6	10.25%	0
G	GB	1	2	7.0%	0
H	A	10	8	9.5%	0
I	Ba	5	2	9.0%	0
J	A	3	2	8.0%	0
K	A	1	4	8.5%	0
L	A	2	5	8.75%	0
M	B	0.6	3	9.75%	0
N	B	1	5	10.25%	0
O	B	3	2	9.5%	0
P	B	2	4	10.0%	0
Q	Baa	1	6	9.5%	0
R	Baa	8	5	9.25%	0
S	Baa	1	3	8.75%	0
T	Aa	5	5	8.25%	0

All coupon payments are done every six months and the cash flow generated during the two payment dates are assumed to be invested at a risk-free rate. Credit ratings consist of 6 grades

(Aaa, Aa, . . . , B). Parameters in the hazard rates are calculated by the estimation from the default data provided by Moody's Investors Service (Table 2).

Table 2: Parameters of hazard rates

	Aaa	Aa	A	Baa	Ba	B
λ_j	0.00005116	0.00023357	0.00028899	0.00153925	0.01249443	2.164202
γ_j	2.0142	1.5656	1.6963	1.4221	1.1998	0.1725
m_j	0.0	0.0	0.0	0.0	0.0	9.721
σ_j	0.00088458	0.00099323	0.00117139	0.00205425	0.00550477	0.00800367

We set $a_j = 0.2, j = 1, \dots, n$ in the basic SDEs. For term structures of default-free and defaultable bonds, forward rates $f_j(0, t)$ are given by quadratic functions

$$f_j(0, t) = c_0^j + c_1^j t + c_2^j t^2, \quad j = 0(\text{default free}), 1(\text{Aaa}), 2(\text{Aa}), 3(\text{A}), 4(\text{Baa}), 5(\text{Ba}), 6(\text{B}),$$

based on the data from JP Morgan's homepage, where credit ratings are inside the brackets. Concrete values used in this evaluation are given in Table 3.

Table 3: Term structure of forward rates

Credit rating (j)	c_0	c_1	c_2	Moody's rating
0	0.05218	0.0006693	-0.00004818	Treasury
1	0.05534	0.001304	-0.00006364	Aaa
2	0.05586	0.001386	-0.00005690	Aa
3	0.05650	0.001407	-0.00001768	A
4	0.05810	0.002199	-0.0001061	Baa
5	0.06465	0.005671	-0.0003572	Ba
6	0.07118	0.007465	-0.0002849	B

Recovery rates of defaultable bonds are all set to be 0.4.¹² Setting parameters in the default-free interest rates as $a_0 = 0.018$ and $b_0(t) = b_0 = 0.054a_0$, we make calculations by changing σ_0 as 0.0%, 0.5% and 1.0%. The interest rate and any hazard rate are uncorrelated, and the correlation between two hazard rates are given by

$$(\rho_{jk}) = \begin{pmatrix} 0.8 & 0.7 & 0.5 & 0.3 & 0.2 & 0.1 \\ 0.7 & 0.8 & 0.6 & 0.4 & 0.3 & 0.2 \\ 0.5 & 0.6 & 0.8 & 0.5 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.5 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.5 & 0.8 & 0.5 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.8 \end{pmatrix}$$

The order is as Aaa, Aa, A, Baa, Ba, B, both in column and row. The diagonal elements are all 0.8, which is the correlation between two firms in the same rating. We use 5000 sample paths.

5.2. Results

5.2.1. Integrative evaluation of credit risk and market risk

The distribution of the portfolio at time $T = 1$ is shown in Table 2. The result (a) is when we only consider the credit risk ($\sigma_0 = 0.00\%$), and (b) is the result when we consider the credit risk and the market risk integratively ($\sigma_0 = 1.0\%$). The distribution in (a) is multimodal. The highest peak is for the case that no default occurs, and the left three peaks next to the highest are for the case when one of the assets F, N, O, P, all of which belong to rating G, defaults. There are some peaks left to these three correspond to many kinds of defaults, such as the default of a bond with a large face value, multiple assets default, and so on. This multimodality is one of notable features of the credit risk.¹³ In (b), the shape of the distribution is unimodal, since peaks overlap due to the interest rate risk. Figure 4 presents risk measures for comparison. These results imply that the interest rate risk should not be ignored in the calculation of future distribution.

In Case (c), stocks are added to the portfolio ($\sigma_0 = 1.0\%$). The stocks added consist of those of Firms B, D, F, H, J and L. All initial prices are 2 and $\mu_S = 0.2$, $\sigma_S = 0.1$. Any two stocks are uncorrelated. In this case, the distribution has a fatter tail than that in (b). This means that adding stocks to a portfolio which consists only of bonds exposes the firm to much more market risks and also that the credit exposure due to specific firms becomes remarkable. That is why VaR and T-VaR, which is greatly affected by the credit risk, increases in this case even when the addition of stocks to the portfolio is not too much.

Table 4: Risk and return of portfolios

		(a)	(b)	(d)	(d)
Present value		59.299		71.299	59.285
Expectation		62.943	62.942	77.316	58.390
Expected return		6.1%	6.1%	8.4%	-1.5%
Standard deviation		1.107	1.990	2.364	1.140
VaR	90%	1.494	2.482	2.976	1.413
	95%	2.584	3.337	4.127	2.525
	99%	4.611	5.355	6.820	4.362
T-VaR	90%	2.778	3.753	4.645	2.772
	95%	3.577	4.642	5.806	3.593
	99%	5.954	6.919	8.701	5.872
Ratio to (a)					
Standard deviation		100.0%	179.7%	213.4%	102.9%
VaR	90%	100.0%	166.1%	199.2%	94.6%
	95%	100.0%	129.2%	159.7%	97.7%
	99%	100.0%	116.1%	147.9%	94.6%
T-VaR	90%	100.0%	135.1%	167.2%	99.8%
	95%	100.0%	129.8%	162.3%	100.5%
	99%	100.0%	116.2%	146.1%	98.6%

5.2.2. Hedge of interest rate risk

Next, we provide an example to hedge the interest rate risk in the portfolio consisting of 20 bonds as above by using interest-rate swap. The swaps which we use are the following.

- Swap A: Maturity 2 years, coupon payment every 6 months, 6-month floating rate and fixed rate (5.28%), notational principal 30.4
- Swap B: Maturity 4 years, coupon payment every 6 months, 6-month floating rate and fixed rate (5.32%), notational principal 30.0
- Swap C: Maturity 6 years, coupon payment every 6 months, 6-month floating rate and fixed rate (5.35%), notational principal 20.0

The present value of all these swaps is nearly zero and the duration of the portfolio equals approximately zero when these swaps are added to the original portfolio.

Table 3 shows the distributions of the portfolio to which Swap A, Swap B and Swap C are added in order. Swaps are evaluated as default-free. Figure 4 shows the result for Case (d), in which all swaps are included in the portfolio. Table 3 and 4 show clearly the effect of hedging by interest-rate swaps. In Case (d) where the duration is nearly zero, the shape is similar to (a) where the interest rate risk is not taken into account. This implication can also be seen from risk measures of (d).

5.2.3. Risk/return analysis of each asset

Finally, we provide results on the risk/return analysis of each asset.

Figure 4 shows the RC (risk contribution) of 95% T-VaR and the present values of RC. Comparing RC with T-VaR of each asset calculated separately, RC is half the T-VaR in average, 10% for some case, because of the diversification effect. Hereinafter, we use RC as the risk measure.

Figure 5 shows RC ratio and the expected returns about 95% T-VaR, which illustrates the relation between risk and return. The assets with 4% expected return all belong to grade B. This is because low coupon rates are set. Figure 6 provides RC measure. Roughly speaking, we regard high RC measure as of high quality. In this case, Bond G (default-free) and bonds with high rating and low face value are thought of as a high quality assets. Bonds with a high rating and high face value¹⁴ such as Bond H are low in RC measure. Assets of B are not good in terms of RC measure. The reason Bond I with grade Ba is low in RC measure is that the face value is high. These analyses imply that the RC measure of T-VaR reflects not only credit ratings and coupon rates but also magnitude of exposure, and that RC measure is an effective and practical tool.

6. Concluding Remarks

This paper extended Kijima and Muromach [17] and proposed a model to evaluate interest rate risk, stock price risk, foreign exchange risk in an integrative way. The Gaussian model in this paper

Figure 2: Distribution of future portfolios

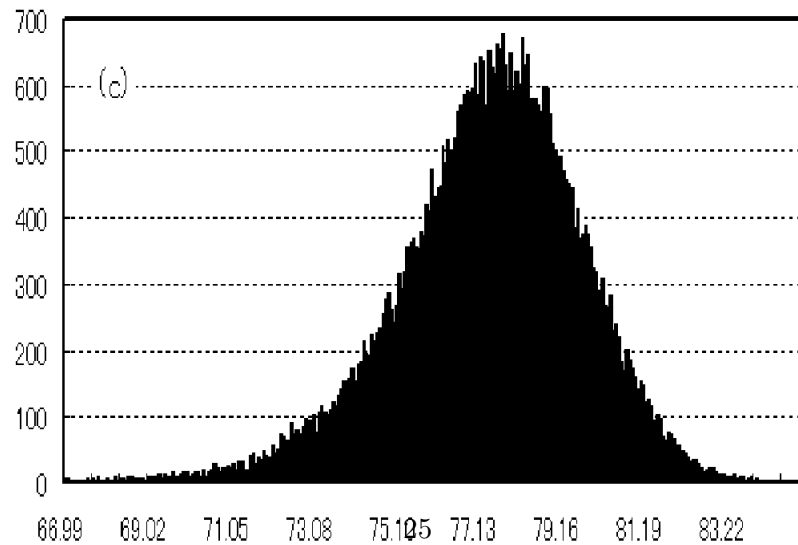
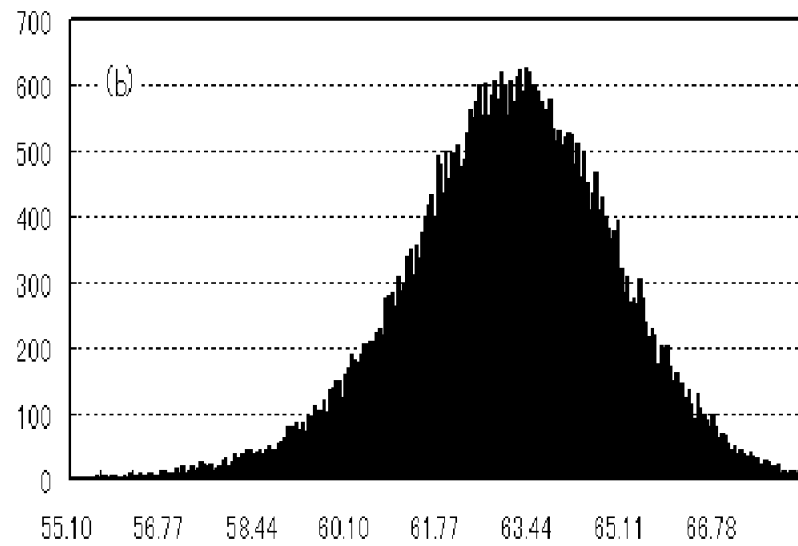
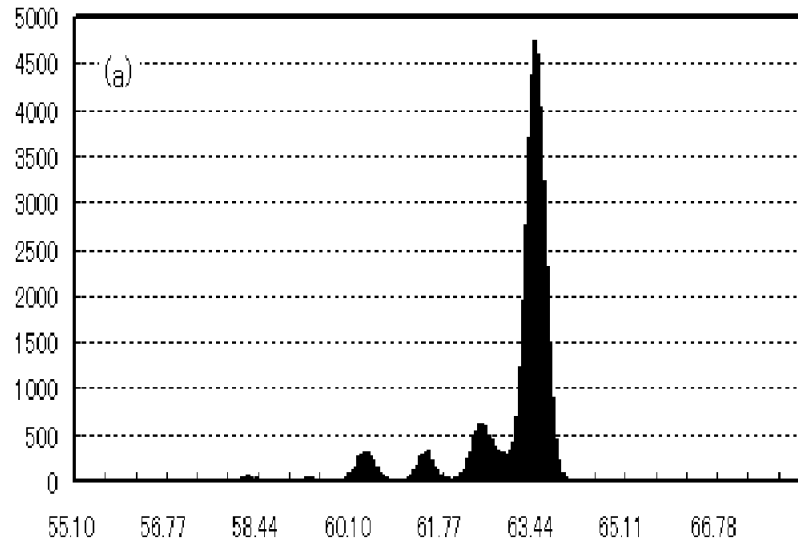


Figure 3: Distribution of future portfolios (hedged by interest swap)

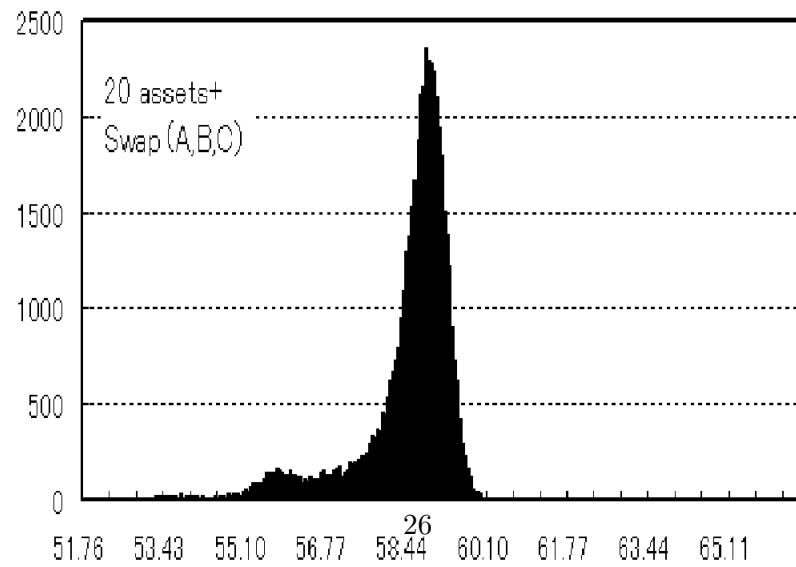
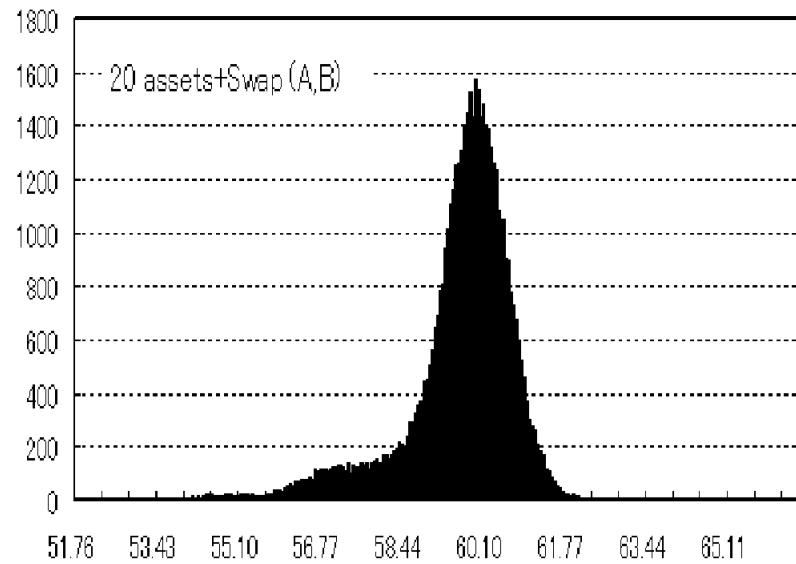
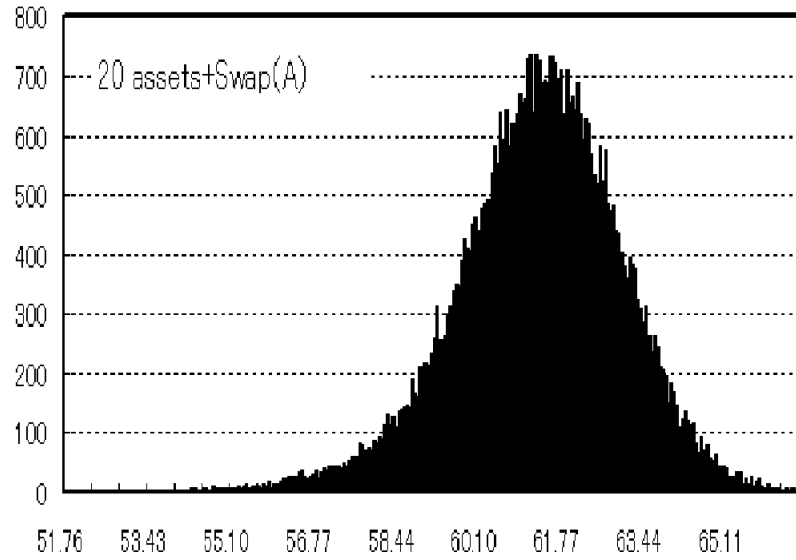


Figure 4: RC with respect to 95% VaR and present Value

(horizontal axis: RC, vertical axis: present value)

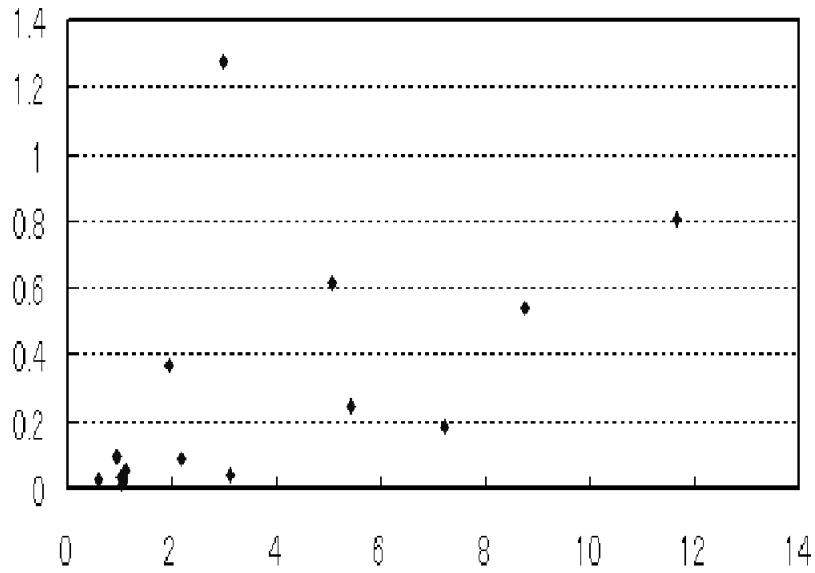


Figure 5: Risk/return of each asset

(horizontal axis: RC ratio w.r.t. 95%, vertical axis: expected return)

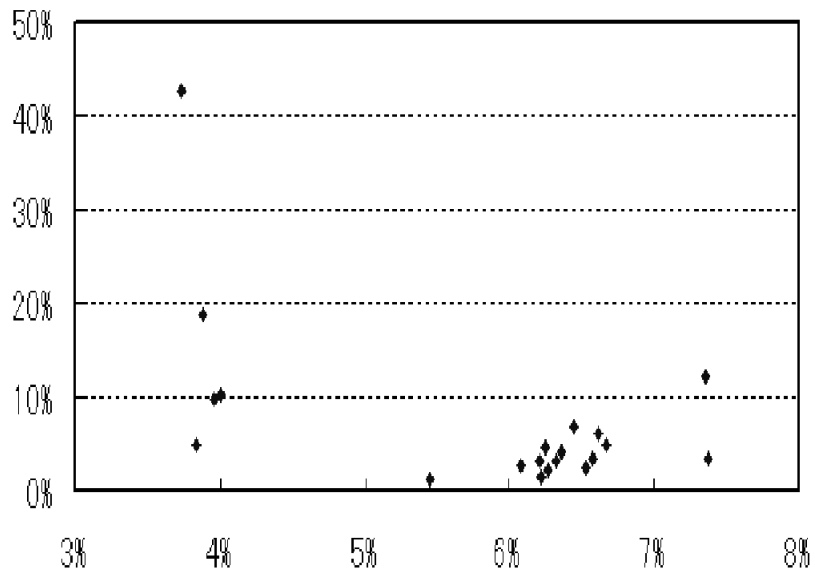
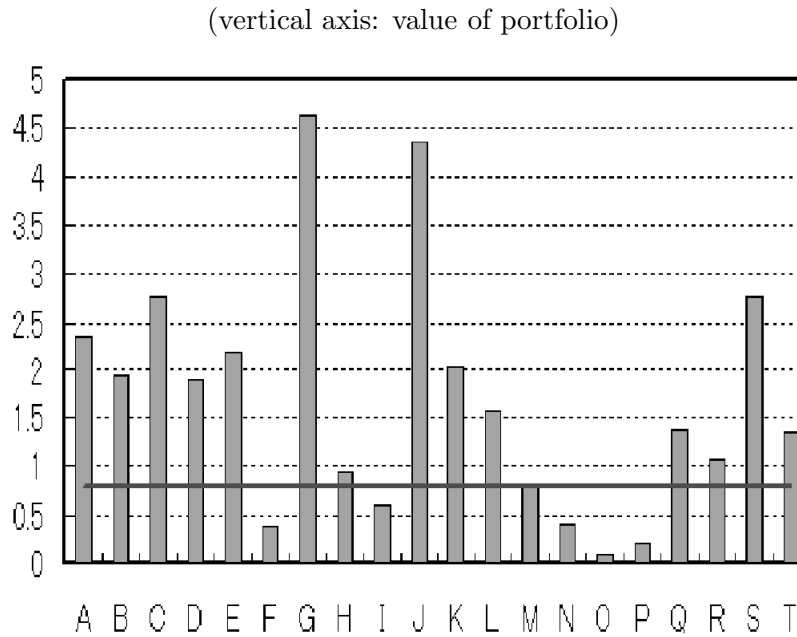


Figure 6: RC measure of each asset



has some drawbacks (there is a possibility that the interest rate and hazard rate turn negative, or that the relation between hazard rate and stocks is not fully expressed). However, there are many great advantages: we can integratively evaluate the market risk and the credit risk in a natural way, the method is simple and easy to use, the theoretical background is clear and consistent, and so on. These advantages make up for the drawbacks to a satisfactory extent. This model relates closely to the pricing model of default swaps proposed by Kijima [15] and in Kijima and Muromachi [16], and we can directly apply this model to testing the hedging strategies of credit risk by credit derivatives. Moreover, though omitted in this paper, it is possible to incorporate the effect of chain default into this evaluation.

We have introduced an integrative evaluation model of market and credit risk. However, there are many other important risks to be evaluated such as prepayment risks, liquidity risks, and so on. The next task is the integration of other risks to the two already dealt with.

Endnotes

¹ Refers to the risk-adjusted probability measures such as risk-neutral measure or forward-neutral measure. These probability measures are used only to evaluate asset prices.

² With Japan Financial Systems Research Institute and Sumitomo Computer Systems Corporation, we are now jointly developing a practical model based on the framework in this paper.

³ Kijima and Muromachi [17] set hazard rates as Ito processes with deterministic function parameters which are constant in Jarrow and Turnbull [9]. Another difference is that hazard processes of firms are modeled in KM, while forward rate processes of firms are modeled in JT.

⁴ Of course, we assume that all SDEs have a strong solution. For example, all parameters are assumed to satisfy Lipschitz and growth conditions. We adopt a 1-factor model for simplicity, but it is fairly easy to extend to multi-factor model.

⁵ We omitted details, but this transformation is guaranteed by the Girsanov theorem, which is often used in the pricing of derivatives. We implicitly assume that the sufficient condition for this theorem to hold such as (i) $\beta_0(t)$ is \mathcal{F}_t -measurable and continuous and (ii) the Novikov condition

$$E \left[\exp \left\{ \frac{1}{2} \int_0^{T^*} \beta_0^2(s) ds \right\} \right] < \infty$$

holds. Hereinafter, we omit this kind of statement unless it is needed.

⁶ $\ell_j(t)$ in (10) is assumed to include the market price of risk corresponding to $\beta_0(t)$ in (8). Namely, adding $\ell_j(t)$ means that the Wiener process $z_j(t)$ under P is transformed to the processes $\tilde{z}_j(t)$ under \tilde{P} . In general, $\ell_j(t)$ is \mathcal{F}_t -measurable as $\beta_0(t)$.

⁷ The following setting is an extension of Kijima and Muromachi [17]. Of course, all errors in this paper are the responsibility of NLI Research Institute.

⁸ More precisely, $\mu_{s,j}(t)$ and $\sigma_{s,j}(t)$ in (11) are $\mu_{s,j}(S_j(t), r(t), h_j(t), t)$ and $\sigma_{s,j}(S_j(t), r(t), h_j(t), t)$, respectively. (11) implies that the expected instantaneous return is $E[dS_j(t)/S_j(t)]/dt = \mu_{s,j}(t)$, where we have used $h_j(t)$ is the intensity of τ_j .

⁹ In a similar manner, we can obtain the sufficient conditions for $\beta_{n+j}(t)$ to exist or the concrete expression of β_{n+j} . We can also derive the equation corresponding to $r_j(t) = r(t) + (1 - \delta_j)\tilde{h}_j(t)$ in Duffie and Singleton [5], where $r_j(t)$ is an instantaneous spot rate of Firm j , and δ_j is the recovery rate of the discount bond issued by Firm j .

¹⁰ $\mu_V(t)$ and $\sigma_V(t)$ depend on other variables as in the stock price processes.

¹¹ Weibull distribution is a popular distribution in survival time analysis. Its hazard rate is given by $h(t) = \lambda\gamma t^{\gamma-1}$. λ and γ are called a scale parameter and a shape parameter, respectively. $h(t)$ is increasing in t if $\gamma > 1$, constant if $\gamma = 1$, and decreasing if $0 < \gamma < 1$. In this case, we adopt a 3-parameter Weibull distribution with shift parameter m_j .

¹² The recovery rates of bonds included in the portfolio can be set independently from the recovery rate of bonds which defines the forward rates.

¹³ If the amount of each asset is fairly uniform and the portfolio is more diversified, these small peak overlaps and the tail on the left side extends as a whole.

¹⁴ High face value is equivalent to high exposure because the loss amount is large when default occurs.

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