Pricing pension funds guarantees using a copula approach

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Abstract

In this paper we present a model for the pricing of a defined contribution pension fund with the guarantee of a minimum rate of return depending on two risky assets: a financial portfolio and the consumer price index. Risk free rates are supposed to be deterministic.

The dependence between the portfolio and the consumer price index is modelled using a copula approach and the pricing is made via Monte Carlo simulation; some useful algorithms are described.

An application and a comparative static analysis are presented.

Key words: pension funds, minimum guarantees, copula functions, options on the minimum or maximum, stochastic simulation.

1. INTRODUCTION[1]

Pension funds with a minimum value guarantee show some structural similarities with other insurance products, like equity (or index) - linked life insurance policies with an asset value guarantee whose features and mathematical properties have been analyzing since the 70's. In fact Brennan & Schwartz (1976) in their work "The pricing of equity-linked life insurance policies with an asset value guarantee" recognize for the first time the presence of an embedded option in an ELPAVG ("equity linked life insurance policy with an asset guarantee) contract and use the

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option theory to price the single and periodic premium of this kind of insurance products. In particular they find an explicit formula for the pricing of the single premium using the valuation model of Black & Scholes (1973) while they apply the finite difference equation numerical method to value the periodic one.
From them on, quite all the works analyzing this kind of insurance contracts (even more complex that the first one) used the contract decomposition proposed by Brennan and Schwartz to evaluate the implicit options and the price of the consequent premia.
Delbaen (1990), for instance, applies the martingale theory developed by Harrison & Kreps (1979) (instead of the Black & Scholes formula) to evaluate the periodic premium of policies with a minimum guarantee, while Bacinello & Ortu (1993) analyze the case of an insurance contract in which the minimum guarantees are endogenous, i.e. they are not fixed as data of the model, but depend on the premium (premia) paid. Moreover Bacinello & Ortu (1993) analyze insurance contracts not explicitly connected to a minimum amount guarantee but to a minimum number of units of the fund that must be bought each time the periodic premium is paid.
Besides these models describing maturity guarantees, which are binding only at the expiration of the contract, there is an increasing literature analysing multiperiod guarantees (see, for example, Hipp (1996)) with the contract period divided into several sub periods with a binding guarantee for each sub period. In this case, if the profits don't reach a fixed minimum amount each period, they must be integrated to fulfill the gap.
The evaluation of this kind of guarantees is again related to the pricing of the option embedded in the contract which is, this time, a forward option, and again this not necessarily bring to an explicit formula for the single and periodic premium.
The preceding models have a characterizing property: they all assume deterministic interest rates. Models assuming a stochastic interest rate have been examined by Bacinello & Ortu (1993), Nielsen & Sandmann (1995), Persson & Aase (1997), Micocci & Pellizzari & Perrotta (2002).

The purpose of this work is to propose a model useful for the pricing of defined contribution pension plans that provide the guarantee of a minimum rate of return when this guarantee depends on two risky assets: a financial portfolio and a consumer price index.
The model of valuation uses the traditional paradigms of mathematical finance (no arbitrage, risk neutral valuations) but introduce copula functions to model the dependence between the two sources of uncertainty (risk).
The article has the following structure: in section 2 we describe the mathematical background of copula functions and we introduce the non-parametric measures of association; we present also a list of the most used copulas and their main characteristics; section 3 introduces the economic framework and defines the pension contract to evaluate; section 4 presents an application and a comparative

static analysis together with some algorithms necessary to perform a Monte Carlo stochastic simulation; section 5 concludes and outlines further research.

2. COPULA FUNCTIONS: MAIN DEFINITIONS AND PROPERTIES

Abe Sklar introduced copula functions on 1959 in the framework of "Probabilistic metric Spaces". From 1986 on copula functions are intensively investigated from a statistical point of view due to the impulse of Genest and MacKay's work "The joy of copulas" (1986).
Nevertheless, applications in financial and (in particular) actuarial fields are revealed only in the end of the 90's. We can cite for example the papers of Frees and Valdez (1998) in actuarial direction and Embrechts for what concerns financial applications (Embrechts *et al.*, 2001, 2002).
Copula functions allow to model efficiently the dependence structure between variates, that's why they assumed in this last years an increasingly importance as a tool for investigating problems such as risk measurement in financial and actuarial applications.

Definition 2.1 *A bidimensional copula ("2-copula") is a function C that satisfies the following properties:*

(i) domain $[0,1]\times[0,1]$

(ii) $C(0,u)=C(u,0)=0$
$C(u,1)=C(1,u)=u$ for every $u\in[0,1]$

(iii) C is a function *2-increasing* that's to say

$C(v_1,v_2)+C(u_1,u_2)\ge C(v_1,u_2)+C(u_1,v_2)$
for every $(u_1,u_2)\in[0,1]\times[0,1];(v_1,v_2)\in[0,1]\times[0,1]$
such that $0\le u_1\le v_1\le 1$ and $0\le u_2\le v_2\le 1$.

Consequences.

- C is a distribution function with uniform marginals. Indeed, let's take two uniform variates U_1 and U_2 and construct the vector $U=(U_1,U_2)$. We then have:

  $$C(u_1,u_2)=\Pr\{U_1\le u_1,U_2\le u_2\}.$$

  From properties *(ii)* we get:

  $$\Pr\{U_1\le 0,U_2\le u\}=\Pr\{U_1\le u,U_2\le 0\}=0.$$

  Moreover:

  $$\Pr\{U_1\le 1,U_2\le u\}=\Pr\{U_1\le u,U_2\le 1\}=u$$

  i.e. the marginals of the joint distribution are uniform.
  From property *(iii)* we get finally:

$$\Pr\{u_1 \le U_1 \le v_1, u_2 \le U_2 \le v_2\} =$$
$$= C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \ge 0$$

that means C is indeed a probability distribution.

- Consider now two one-dimensional probability distributions F_1 and F_2, and a bidimensional copula C. It is clear that
$$F(x_1,x_2) := C\left(F_1(x_1), F_2(x_2)\right)$$
represents a bidimensional distribution with marginals F_1 and F_2.
Indeed, $U_i := F_i(X_i)$ defines a uniform distribution:
$$\Pr\{U_i \le u\} = \Pr\{F_i(X_i) \le u\} = \Pr\left\{X_i \le F_i^{-1}(u)\right\} = F_i(F_i^{-1}(u)) = u\ .$$
Besides marginals are:
$$C(F_1(x_1), F_2(\infty)) = C(F_1(x_1), 1) = F_1(x_1)$$
$$C(F_1(\infty), F_2(x_2)) = C(1, F_2(x_2)) = F_2(x_2)\ .$$

Fortunately the last result can be inverted, this conduces to the following fundamental theorem demonstrated by Sklar:

Theorem 2.1 *Let F be a bidimensional distribution, with marginals F_1 and F_2. Then there exists a 2-copula C such that*
$$F(x_1,x_2) = C(F_1(x_1), F_2(x_2)).$$
If the marginals F_1 and F_2 are continuous, then the copula C is unique.

The previous representation is called *canonical representation* of the distribution. Sklar's theorem is then a powerful tool to construct bidimensional distributions by using one-dimensional ones, which represent the marginals of the given distribution. Dependence between marginals is then characterized by the copula C. Note moreover that the construction of multidimensional non-gaussian models is particularly hard. An approach using copulas permits to simplify this problem, moreover one can construct multidimensional distributions with different marginals.

Remarks.

- The canonical representation can be written equivalently. Consider two continuous distributions G_1 and G_2 and let $Y_i = G_i^{-1}(U_i)$. The distribution G of $Y = (Y_1, Y_2)$ *will be:*
$$G(y_1,y_2) = C_{(X_1,X_2)}(G_1(y_1), G_2(y_2))$$
so that
$$C(x_1,x_2) = F(F_1^{-1}(G_1(x_1)), F_2^{-1}(G_2(x_2)))$$
with $U_i = F_i(X_i)$, F distribution of (X_1, X_2). This construction is called translation method.

- The definition of 2-*copula* can be generalized analogously to the *n-dimensional* case. The canonical form of an *n-dimensional* distribution takes the following form, according to Sklar's theorem:

  $$F(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n))$$

  where $F_1(x_1),...,F_n(x_n)$ are the n marginal distributions and C represents an *n-copula*.

2.1 Probability density

Suppose that the bivariate $X = (X_1, X_2)$ possesses a density function. We can then express it by means of the marginal density functions and the copula in the following manner:

$$f(x_1,x_2) = c(F_1(x_1),F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2)$$

with

$$c(u_1,u_2) = \frac{\partial C(u_1,u_2)}{\partial u_1 \partial u_2}.$$

The condition $C(v_1,v_2) - C(v_1,u_2) - C(u_1,v_2) + C(u_1,u_2) \geq 0$ leads to the positivity of the density $c(u_1,u_2) \geq 0$.
In the case of *n-dimensional* distributions, if the density function exists we will get analogously:

$$f(x_1,...,x_n) = c(F_1(x_1),...,F_n(x_n)) \cdot \prod_{i=1}^{n} f_i(x_i)$$

with:

$$c(u_1,...,u_n) = \frac{\partial^n C(u_1,...,u_n)}{\partial u_1 ... \partial u_n}.$$

The density of a copula can then be written as

$$c(u_1,u_2) = \frac{f(F_1^{-1}(u_1), F_2^{-1}(u_2))}{f_1(F_1^{-1}(u_1)) \cdot f_2(F_2^{-1}(u_2))}.$$

2.2 Copulas examples

We present here some important copulas.

2.2.1 The product copula

The product copula is $C^{\perp}(u_1,u_2) = u_1 \cdot u_2$ which density is $c^{\perp}(u_1,u_2) = 1$.
We deduce that a distribution constructed with this copula satisfies:

$$f(x_1,x_2) = f_1(x_1) \cdot f_2(x_2)$$

which characterizes independence between X_1 and X_2.

2.2.2 Gumbel Logistic Copula

The Gumbel Logistic copula is:

$$C(u_1,u_2)=\frac{u_1\cdot u_2}{u_1+u_2-u_1\cdot u_2}=F(F_1^{-1}(u_1),F_2^{-1}(u_2))$$

where $F(x_1,x_2)=(1+e^{-x_1}+e^{-x_2})^{-1}$ is the Gumbel logistic 2-distribution having marginals $F_1(x_1)=(1+e^{-x_1})^{-1}$ and $F_2(x_2)=(1+e^{-x_2})^{-1}$, moreover quantiles have the expression $F_1^{-1}(u_1)=\log u_1-\log(1-u_1)$ and $F_2^{-1}(u_2)=\log u_2-\log(1-u_2)$.
The density function is:

$$c(u_1,u_2)=\frac{2u_1\cdot u_2}{(u_1+u_2-u_1\cdot u_2)^3}.$$

2.2.3 Gumble-Barnett copula

Gumble-Barnett copula is:

$$C(u_1,u_2,\vartheta)=u_1\cdot u_2\cdot e^{-\vartheta\log u_1\log u_2}.$$

One easily verifies that $C(0,u,\vartheta)=C(u,0,\vartheta)=0$ and $C(1,u,\vartheta)=C(u,1,\vartheta)=u$.
Density is given by:

$$c(u_1,u_2,\vartheta)=[1-\vartheta-\vartheta(\log u_1+\log u_2)+\vartheta^2\log u_1\cdot\log u_2]\cdot e^{-\theta\log u_1\log u_2}.$$

2.2.4 Normal copula

The normal copula is given by:

$$C_\rho^{Ga}=\Phi_\rho^n(\Phi^{-1}(u_1),...,\Phi^{-1}(u_n))$$

where we supposed that $Z=(Z_1,...,Z_n)$ has normal distribution $N_n(\mu,\Sigma)$ with marginals $F(Z_i)$ where $Z_i\sim N(\mu_i,\Sigma_{ii})$ and ρ represents the linear correlation matrix corresponding to the covariance matrix Σ.
We denote Φ_ρ^n the multivariate normal distribution function with correlation matrix ρ and Φ^{-1} is the inverse of the standard univariate normal distribution.
The density of the normal copula is:

$$c(u,\rho)=|\rho|^{-\frac{1}{2}}\cdot\exp\left(-\frac{1}{2}\zeta^T\cdot(\rho^{-1}-I)\cdot\zeta\right)$$

where $\zeta=\left(\Phi^{-1}(u_1),...,\Phi^{-1}(u_n)\right)$.

2.2.5 Frank copula

Frank copula is given by:

$$C(u_1,u_2;\vartheta) = -\frac{1}{\vartheta}\log\left[1+\frac{(e^{-\vartheta u_1}-1)\cdot(e^{-\vartheta u_2}-1)}{e^{-\vartheta}-1}\right].$$

2.2.6 FGM copula (Farlie-Gumbel-Morgenstern)

The FGM copula is given by:

$$C_\vartheta(u_1,u_2) = u_1 \cdot u_2[1+\vartheta(1-u_1)\cdot(1-u_2)].$$

2.2.7 The *t* Student copula

Let the variate $Z=(Z_1,...,Z_n) \sim N_n(0,\Sigma)$ with non-degenerate marginals and let

$$X = \mu + \frac{\sqrt{\nu}}{\sqrt{S}} Z$$

where Z and $S \sim \chi_\nu^{\,2}$ are independent. We will say that X has a *t Student* distribution with degrees of freedom ν, main μ (if $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\Sigma$ (if $\nu > 2$).

If X_i has distribution G_i, then the distribution function of $G_1(X_1),...,G_n(X_n)$ is the t_ν copula $C^t_{\nu,\rho}$ where ρ is the linear correlation matrix associated to Σ.

The density of the *t copula* is:

$$c(u_1,...,u_n) = |\,\rho\,|^{-1/2}\,\frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^n\left(1+\frac{1}{\nu}\xi^T\rho^{-1}\xi\right)^{-\frac{\nu+n}{2}}}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n\Gamma\left(\frac{\nu}{2}\right)\prod_{i=1}^n\left(1+\frac{\xi_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

where Γ is the gamma function and $\zeta_i = t_\nu^{-1}(u_i)$.

For $\nu \to \infty$ we obtain the normal copula.

2.2.8 Archimedean copulas

Let ϕ be a continuous, decreasing and convex function $\phi:[0,1]\to[0,+\infty]$ with $\phi(1)=0$ and $\phi(u)+\phi(v)\le\phi(0)$. We define an Archimedean copula with generator ϕ in the following way:

$$C(u,v) = \phi^{-1}[\phi(u)+\phi(v)]$$

with $u,v \in [0,1]$.

If we take

$$\phi(t) = (-\log t)^\vartheta$$

with $\vartheta \in [1,+\infty)$ we get the Gumble copula.
Otherwise in the case

$$\phi(t) = \frac{t^{-\vartheta} - 1}{\vartheta}$$

with $\vartheta \in [-1,+\infty) \setminus \{0\}$ we get the Clayton copula:

$$C_\vartheta(u,v) = \max\left([u^{-\vartheta} + v^{-\vartheta} - 1]^{-1/\vartheta}, 0\right).$$

If we take $\phi(t) = -\log t$ we get the product copula $C^{\perp}$.
If we take:

$$\phi(t) = -\log \frac{e^{-\vartheta t} - 1}{e^{-\vartheta} - 1}$$

we get Frank copula.
If we take:

$$\phi(t) = -\log[1 - (1-u)^{\vartheta}]$$

we get Joe copula

$$C(u,v) = 1 - [u^{-\vartheta} + v^{-\vartheta} - u^{-\vartheta} \cdot v^{-\vartheta}]^{1/\vartheta}.$$

Finally Genest and MacKay show that a copula C is Archimedean if it admits partial derivatives and if it exists an integrable function $\xi : (0,1) \to (0,+\infty)$ such that:

$$\xi(v)\frac{\partial C(u,v)}{\partial u} = \xi(u)\frac{\partial C(u,v)}{\partial v}$$

for every $u,v \in [0,1] \times [0,1]$.
In such a case the generator of the copula is:

$$\phi(t) = \int_t^1 \xi(u)du$$

with $0 \le t \le 1$.
The density of the Archimedean copula is:

$$c(u,v) = -\frac{\phi''(C(u,v)) \cdot \phi'(u) \cdot \phi'(v)}{\left[\phi'(C(u,v))\right]^3}$$

moreover we can define multidimensional Archimedean copulas setting

$$C(u_1,...,u_n) = \phi^{-1}[\phi(u_1) + ... + \phi(u_n)]$$

with the additional condition for the generator ϕ:

$$(-1)^k \frac{d^k}{du^k}\phi^{-1}(u) \ge 0$$

for $k \ge 1$. We obtain for example the multidimensional Gumble copula

$$C(u_1,...,u_n) = Exp\left[-\left((-\log u_1)^{\vartheta} + ... + (-\log u_n)^{\vartheta}\right)^{1/\vartheta}\right]$$

2.3 Copulas properties

Proposition 2.3.1 *A copula C is uniformly continuous in its domain. Besides it can be shown that*

$$\left|C(v_1,v_2)-C(u_1,u_2)\right| \le \left|v_1-u_1\right| + \left|v_2-u_2\right| .$$

Proposition 2.3.2 *Partial derivatives* $\partial_1 C$ *and* $\partial_2 C$ *exist for every* $(u_1,u_2)\in[0,1]\times[0,1]$ *and they satisfy the following properties:*

$$0 \le \partial_1 C(u_1,u_2) \le 1 \text{ and } 0 \le \partial_2 C(u_1,u_2) \le 1 .$$

Proposition 2.3.3 *Let* X_1, X_2 *be two continuous variates with marginals* F_1 *and* F_2 *and copula* $C(X_1,X_2)$*. If* h_1, h_2 *are two strictly increasing functions on* $\operatorname{Im} X_1$ *and* $\operatorname{Im} X_2$ *then* $C(h_1(X_1),h_2(X_2)) = C(X_1,X_2)$*, in other words the copula function is invariant under strictly increasing transformations of the variates.*

Having described copulas and their properties, we shall now study some aspects linked to the dependence between variates.

2.4 Concordance order

Definition 2.4.1 *The distribution F belongs to the Fréchet class* $\Im(F_1,F_2)$ *if and only if the marginals of F are* F_1 *and* F_2.

The extremal distributions F^- and F^+ in $\Im(F_1,F_2)$ are defined as:

$$F^-(x_1,x_2) = \max\left\{F_1(x_1)+F_2(x_2)-1,0\right\}$$

$$F^+(x_1,x_2) = \min\left\{F_1(x_1),F_2(x_2)\right\}.$$

F^- and F^+ are also called Fréchet lower bound and Fréchet upper bound. We can associate to them the copulas

$$C^-(u_1,u_2) = \max\left\{u_1+u_2-1,0\right\}$$

$$C^+(u_1,u_2) = \min\left\{u_1,u_2\right\}.$$

The following relations hold

$$F^-(x_1,x_2) \le F(x_1,x_2) \le F^+(x_1,x_2)$$

for every $(x_1,x_2)\in\mathbb{R}^2$ and for every $F\in\Im(F_1,F_2)$ or in terms of copulas:

$$C^-(u_1,u_2) \le C(u_1,u_2) \le C^+(u_1,u_2) .$$

We define now a partial order relation for the set of copulas.

Definition 2.4.2 *We say that the copula* C_1 *is less then the copula* C_2 *(*$C_1 \prec C_2$*) if and only if*

$$C_1(u_1,u_2) \le C_2(u_1,u_2)$$

for every $(u_1,u_2)\in[0,1]\times[0,1]$.

The order "$\prec$" is called concordance order and corresponds to the first order stochastic domination for distribution functions. It turns out to be a partial order, indeed not every copulas can be confronted. The following still hold: $C^- \prec C \prec C^+$ and $C^- \prec C^\perp \prec C^+$. So that we can give the following:

Definition 2.4.3 *The copula C represents a positive (negative) dependence structure if* $C^\perp \prec C \prec C^+$ *(if* $C^- \prec C \prec C^\perp$ *respectively).*

Remark. A parametric copula $C(u_1,u_2,\vartheta)=C_\vartheta(u_1,u_2)$ is said to be totally ordered if we have $C_{\vartheta_2} \succ C_{\vartheta_1}$ for every $\vartheta_2 \geq \vartheta_1$ (positively ordered family) or $C_{\vartheta_2} \prec C_{\vartheta_1}$ (negatively ordered family).
We define besides the positive quadrant dependence ("PQD") in the following way:

Definition 2.4.4 *Two variates* X_1, X_2 *are called PQD if they satisfy:*

$$\Pr\{X_1 \leq x_1, X_2 \leq x_2\} \geq \Pr\{X_1 \leq x_1\}\cdot\Pr\{X_2 \leq x_2\}$$

for every $(x_1,x_2)\in\mathbb{R}^2$. *In terms of copulas:* $C(u_1,u_2)\succ C^\perp$.

We define analogously the negative quadrant dependence ("NQD") by assuming that $C(u_1,u_2)\prec C^\perp$.

2.5 Measure of dependence

We introduce now another dependence concept. Recall that:

$$C^-(u_1,u_2)=\max\{u_1+u_2-1,0\}$$

$$C^+(u_1,u_2)=\min\{u_1,u_2\}$$

with the relation $C^-(u_1,u_2)\leq C(u_1,u_2)\leq C^+(u_1,u_2)$.
If we denote $U\sim U(0,1)$ the following also hold:

$$C^-(u_1,u_2)=\Pr\{U\leq u_1, 1-U\leq u_2\}$$

$$C^+(u_1,u_2)=\Pr\{U\leq u_1, U\leq u_2\}.$$

One can prove the following:

Theorem 2.5.1 *Suppose that the bivariate (X,Y) has a copula* C^- *or* C^+. *So there exist two monotonous functions* $u,v:\mathbb{R}\to\mathbb{R}$ *and a variate Z such that (X,Y)=(u(Z),v(Z)) with u increasing and v decreasing in the case of the copula* C^- *; u and v decreasing in the case of a copula* C^+ *(the converse is true).*

Using this result we can introduce the following:

Definition 2.5.2 *If the couple (X,Y) admits copula C^+, the variates X and Y are called comonotonous; in the case of a copula C^- they are called countermonotonous.*

When the distributions F_1 and F_2 are continuous, the last theorem can be strengthened in the following manner:
$C = C^- \Leftrightarrow Y = T(X)$ with $T = F_2^{-1} \circ (1 - F_1)$ decreasing;
$C = C^+ \Leftrightarrow Y = T(X)$ with $T = F_2^{-1} \circ F_1$ increasing.
We conclude with a list of suitable properties, which should satisfy a good dependence measure between variates:

Definition 2.5.3 *A dependence measure δ is an application which associates to a couple of variates (X,Y) a real number $\delta(X,Y)$ such that:*
(i) $\delta(X,Y) = \delta(Y,X)$ symmetry;
(ii) $-1 \le \delta(X,Y) \le 1$ normalization;
(iii) $\delta(X,Y) = 1$ if and only if X, Y are comonotonous;
(iii') $\delta(X,Y) = -1$ if and only if X, Y are countermonotonous;
(iv) for every monotonous application $T : \mathbb{R} \to \mathbb{R}$ we have: $\delta(T(X),Y) = \delta(X,Y)$ for T increasing, $\delta(T(X),Y) = -\delta(X,Y)$ for T decreasing.

Linear correlation satisfies properties (i) and (ii); we shall see later on that rank correlation satisfies also properties (iii) and (iv).

Remark. We may want to introduce a property of the form $\delta(X,Y) = 0$ if and only if *X* and *Y* are independent, unfortunately it can be proved that such a property is incompatible with (iv).

2.6 Rank correlation

Definition 2.6.1 *Consider the variates X, Y with marginals F_1 and F_2 and joint distribution F. The Spearman's rank correlation ("Spearman's ρ") is defined as $\rho_S(X,Y) = \rho(F_1(X), F_2(Y))$ where ρ is the usual linear correlation. Let $(X_1, X_2), (Y_1, Y_2)$ two independent couples of variates from F, then Kendall's rank correlation ("Kendall's τ") is given by*

$$\rho_\tau(X,Y) = \Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - \Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\}.$$

We can assume that ρ_S is the correlation of the copula *C* associated to *(X,Y)*; both ρ_S and ρ_τ measure the monotonic dependence degree between *X* and *Y* (whereas linear correlation only measure the linear dependence degree).
We list some fundamental properties of ρ_S and ρ_τ.

Theorem 2.6.2 *Let X, Y continuous variates with continuous distributions F_1, F_2; joint distribution F and copula C. We then have:*
(1) $\rho_S(X,Y) = \rho_S(Y,X)$; $\rho_\tau(X,Y) = \rho_\tau(Y,X)$;
(2) if X and Y are independent, then $\rho_S(X,Y) = \rho_\tau(X,Y) = 0$;
(3) $\rho_S, \rho_\tau \in [-1,1]$;
(4) $\rho_S(X,Y) = 12\int_0^1\int_0^1 [C(u,v) - u\cdot v]dudv$;
(5) $\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$;
(6) given the strictly monotonous application $T:\mathbb{R}\to\mathbb{R}$, ρ_S and ρ_τ satisfy property (iv) of the last section;
(7) $\rho_S(X,Y) = \rho_\tau(X,Y) = 1$ if and only if $C = C^+$ if and only if Y=T(X) with T increasing;
(8) $\rho_S(X,Y) = \rho_\tau(X,Y) = -1$ if and only if $C = C^-$ if and only if Y=T(X) with T decreasing.

The rank correlation satisfy then properties (i), (ii), (iii) and (iv) of the last section.

Remarks.

- for Gumbel copula we have:
$$\rho_\tau = \frac{\vartheta - 1}{\vartheta}$$
- for Frank copula:
$$\rho_S = 1 - 12\vartheta^{-1}[D_1(\vartheta) - D_2(\vartheta)]$$
$$\rho_\tau = 1 - 4\vartheta^{-1}[1 - D_1(\vartheta)]$$
where $D_k(x)$ is the Debye function defined in the following way:
$$D_i(x) = \frac{i}{x^i}\int_0^x t^i\left(e^t - 1\right)^{-1} dt$$
which satisfies $D_1(-x) = D_1(x) + \frac{x}{2}$.
- for Archimedean copula we have:

$$\rho_\tau = 1 + 4\int_0^1 \frac{\phi(u)}{\phi'(u)}\,du$$

- finally for Clayton copula:

$$\rho_\tau = \frac{\vartheta}{\vartheta + 2}$$

The results in section 2 can be found in Roncalli (2000) and in Embrechts *et al.* (2001, 2002).

3. THE PENSION CONTRACT AND THE ECONOMIC FRAMEWORK

In this section we introduce our assumptions and notations concerning the economic framework and the pension contract to evaluate.
To this end, we assume initially independence between the financial and demographic components just quite like all the authors who model and price these kind of insurance products. We delay until a succeeding section to introduce definitions and notations concerning life contingencies.

3.1 Notations and assumptions

As usual in financial literature, we assume a perfectly competitive and frictionless market, no arbitrage and rational operators all sharing the same information revealed by a filtration.
In this economic framework, we introduce the following variables:

T	the expiration date of the contract
$r(t)$	the instantaneous risk-free interest rate; it is supposed to be deterministic
$x(t)$	the value of a stock index (or reference portfolio) at time t
$p(t)$	the value of the consumer price index at time t
$b(t)$	the benefit payable at time t
D	the reference capital invested at time t=0
$V_\tau(b(t))$	the market value at $\tau \le t$ of $b(t)$, payable at time t
$V_\tau(x(t))$	the market value at $\tau \le t$ of $x(t)$, payable at time t
$C(x, t-\tau, G)$	the market value at $\tau \le t$ of a European call option with strike price $G(t)$ written on x
$P(x, t-\tau, G)$	the market value at $\tau \le t$ of an European put option with strike price $G(t)$ written on x
$v(\tau, t)$	the price at $\tau \le t$ of a unitary zero coupon bond with maturity time t.

We now introduce the dynamics of the state variables characterizing our model.

Reference portfolio.
As in the Black & Scholes model, we assume that the index (or the reference portfolio) price $x(t)$ is driven by the following log-normal stochastic process:

$$\frac{dx(t)}{x(t)} = \mu_x dt + \sigma_x d\omega_x(t)$$

with $\mu_x, \sigma_x \in R^+$ and ω_x a standard Brownian motion.

Consumer price index.
We suppose that $p(t)$ is described, such as $x(t)$, by a lognormal stochastic process:

$$\frac{dp(t)}{p(t)} = \mu_p dt + \sigma_p d\omega_p(t)$$

with $\mu_p, \sigma_p \in R^+$ and ω_p a standard Brownian motion.

The dependence between $x(t)$ and $p(t)$.
As already said in section 1, we model the dependence between the two stochastic processes of $x(t)$ and $p(t)$ using copula functions; in particular we use Archimedean copulas to describe the dependence between ω_x and ω_p.

3.2 Definition and financial decomposition of the pension contract.

We consider a pension contract that pays at time t a benefit consisting in the reference capital increased by the greatest of the two variation rates: the return on a financial risky portfolio and the stochastic consumer price index.
We also assume that, the contractual features require the benefit payable at the end of the year $t=1,...,T$ if occurs one of the events provided by the regulations (death, invalidity, disability,...) or at maturity.
The independence between demographic and financial risks allows us to treat the benefit, as it should be paid with certainty at the end of a fixed year t introducing the demographic component only in a second time.
With these assumptions, the benefit $b(t)$ is given by

$$b(t) = D \cdot \max\left\{ \frac{x(t)}{x(0)}, \frac{p(t)}{p(0)}, h \right\}$$

with $0 \le h \le 1$ and assuming $x(0)=p(0)=H$:

$$b(t) = \frac{D}{H} \cdot \max\left\{ x(t), p(t), h \cdot H \right\}.$$

Knowing that:

$$\max\{x, y\} = x + \max\{y - x, 0\} = y + \max\{x - y, 0\}$$

$b(t)$ may be written using the "call decomposition[2]":

$$b(t) = h \cdot D + \frac{D}{H} \cdot \left\{ \max \left[\max(x(t), p(t)) - h \cdot H, 0 \right] \right\}$$

If $h=1$, i.e., if the fixed amount guaranteed is the whole reference capital, the last equation becomes

$$b(t) = D + \frac{D}{H} \cdot \left\{ \max \left[\max(x(t), p(t)) \right] - H, 0 \right\}.$$

So $b(t)$ is given by the sum of an amount D (the invested capital) and the payoff at time t of a call option on the maximum between x and p with strike price equal to H, and its value at time 0 is given by:

$$V_0(b(t)) = D \cdot v(0,t) + \frac{D}{H} \cdot C(x, p, t, H)$$

where $C(x,p,t,H)$ is the price in 0 of a European call option on the maximum between x and p, time to maturity t and strike price H.
So the price of the proposed pension contract depends on the value at time 0 of a z.c.b. with maturity t and on the pricing of the implicit European option whose evaluation will be the subject of the following section.

3.3 Mortality risk.

After computing the value of the guarantee for a fixed time t, we now introduce the mortality risk assuming uncertain the expiration date of the contract. To this end we preliminarily give some definitions and assumptions.
Let $\alpha(0,t)$ be the probability that the insurance contract will expire in $t=1,...,T$, for one of the causes provided for by the regulation of the fund (death, disability, inability, right to get the pension benefit and so on...). If, as usual in actuarial practice, we assume that a sufficient number of contracts are written so that the demographic risk is eliminated, the single premium will be computed as follows:

$$U = \sum_{t=1}^{T} \alpha(0,t) \cdot V_0(b(t)) = \sum_{t=1}^{T} \alpha(0,t) \cdot \left[D \cdot v(0,t) + \frac{D}{H} \cdot C(x, p, t, H) \right] =$$
$$= D \cdot \left(\sum_{t=1}^{T} \alpha(0,t) \cdot v(0,t) + \frac{1}{H} \cdot \sum_{t=1}^{T} \alpha(0,t) \cdot C(x, p, t, H) \right)$$

where $D \cdot \sum_{t=1}^{T} \alpha(0,t) \cdot v(0,t)$ is similar to the single premium of a traditional mixed insurance policy (a term insurance plus a pure endowment) while

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[2] $b(t)$ may assume a more general form if, according to De Felice & Moriconi (1999), we define

$$b(t) = D \cdot \max \left\{ \beta_x \frac{x(t)}{x(0)}, \beta_p \frac{p(t)}{p(0)}, \beta_h h \right\}$$

where β_x, β_p and β_h are supposed to be constant variables belonging to $\{0,1\}$ assuming a value equal to one only if the peculiar guarantee they refer to is considered in the contract.

$\frac{D}{H}\cdot\sum_{t=1}^{T}\alpha(0,t)\cdot C(x,p,t,H)$ represents the additional amount the insured must pay because of the presence of the guarantee in the contract.
Considering the presence of a bivariate risk neutral distribution with copulas, to price the option embedded in the contract a Monte Carlo approach must be used because no closed form solution to evaluate this peculiar derivative is known.

4. THE EVALUATION MODEL AND AN APPLICATION TO THE PENSION CONTRACT

According to the standard results in Harrison & Kreps (1979) and Harrison & Pliska (1981, 1983) and to the generalization of the option pricing in case of the bivariate risk neutral distribution proposed by Rapuch & Roncalli (2001) the price of the option embedded in the contract is given by:

$$C(x,p,t,H)=E_0^C\left(Y(t)e^{-\int_0^t r(u)du}\right)$$

where $E_0^C(.)$ is the date 0 expectation of $Y(t)$ taken under the bivariate risk neutral distribution with copula and

$$Y(t)=\left\{\max\left[\max\left(x(t),p(t)\right)\right]-H,0\right\}$$

is the payoff of the considered option.
Because of the presence of the copula function, it seems there is no analytic expression for $C(x,p,t,H)$ and numerical methods must be introduced.
In this section we present an application of this approach developed on US data concerning the dynamics of US stock markets and US inflation since 1970; the data have been obtained by Datastream on an yearly base.

4.1 A Monte Carlo approach

The evaluation of the option embedded in the guarantee proposed in the previous sections requires to price a derivative written on two assets, namely the stock index $x(t)$ and the consumer price index $p(t)$.
Both $x(t)$ and $p(t)$ follow a geometric brownian motion and their dependence structure is modelled by an Archimedean copula function.
If the lifespan of the option is discretized in n steps of length $\Delta t=T/n$ the standard way to generate random paths is by using the recursions:

$$x(t)=x(t-\Delta t)\cdot e^{\left(r(t-\Delta t)-0.5\sigma_x^2+\sigma_x\cdot\Lambda_1\cdot\sqrt{\Delta t}\right)}$$

$$p(t)=p(t-\Delta t)\cdot e^{\left(r(t-\Delta t)-0.5\sigma_p^2+\sigma_x\cdot\Lambda_2\cdot\sqrt{\Delta t}\right)}$$

where Λ_1 and Λ_2 are standard normal variates whose dependence structure is described by an Archimedean copula function.
To generate them the following algorithm can be used:

1. Generate U_1 and U_2 independent (0, 1) uniform random numbers;
2. Set $\Lambda_1 = F^{-1}(U_1)$ where F is the standard normal cdf;
3. Calculate Λ_2 as the solution of

$$U_2 = \frac{\phi^{-1(1)}\left(\phi\left(F(\lambda_1)\right)+\phi\left(F(\lambda_2)\right)\right)}{\phi^{-1(1)}\left(\phi\left(F(\lambda_1)\right)\right)}$$

If we simulate a large number M of bivariate data $\left(x_k(t), p_k(t)\right)$, the price of the option $\hat{C}(x,p,t,H)$ can be estimated by the sample mean:

$$\hat{C}(x,p,t,H) = e^{\int_0^T r(u)du} \frac{1}{M}\sum_{k=1}^{M}\max\left[\max\left[x_k(t), p_k(t)\right]-H, 0\right]$$

4.2 Estimating Archimedean copulas.

Schweizer & Wolff (1981) established that the value of the parameter α characterizing each family of Archimedean copulas can be related to the Kendall's measure of concordance τ. The relationships are shown in the table below.

Table 1
Relationship between α and τ

Family	τ
Gumbel (1990)	$1-\frac{1}{\alpha}$
Clayton (1978)	$\frac{\alpha}{\alpha+2}$
Frank (1979)	$1-\frac{4}{\alpha}\cdot\left(D_1(-\alpha)-1\right)$

where $D_i(.)$ is the value of the Debye function already described in section 2.6.
From calculation of the Kendall's measure of concordance of our bivariate data (US stock market returns and US inflation rate), we obtain τ equal to 0.341.
This value gives $\alpha = 1.51745$ for the Gumbel, $\alpha = 1.0349$ for the Clayton and $\alpha = 3.39839$ for the Frank.

Now for each of these different copulas we must verify how close it fits the data by comparison with the empirical sample.
This fit test can be made using a procedure developed by Genest & Rivest (1993) whose algorithm is well described by Frees & Valdez (1998).
The procedure has the following steps:

- identify an intermediate variable $Z_i = F(X_i, Y_i)$ that has distribution function $K(z)$;
- for Archimedean copulas this function is

$$K(z) = z - \left(\frac{d \ln \phi_\alpha(z)}{dz} \right)^{-1}$$

- define

$$Z_i = \frac{card\left\{ (X_j, Y_j) : X_j < X_i, Y_j < Y_i \right\}}{N-1}$$

  and calculate the empirical version of $K(z)$, $K_N(z)$;
- reply the procedure for each copula under examination and compare the parametric estimate with the non parametric one;
- choose the "best" copula by using an adequate criterion (like a graphical test and/or a minimum square error analysis).

From our data we obtain the following forms of $K(z)$ for the copulas under examination:

Table 2

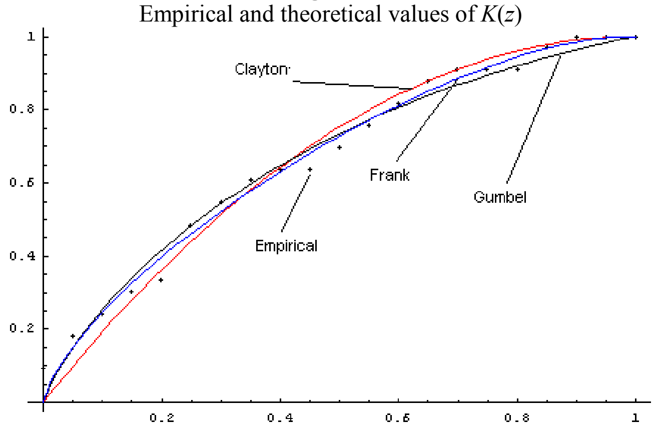
The function $K(z)$

Family	$K(z)$
Gumbel (1990)	$\frac{z \cdot (\alpha - \ln z)}{\alpha}$
Clayton (1978)	$\frac{z \cdot \left(1 + \alpha - z^{\alpha}\right)}{\alpha}$
Frank (1979)	$\frac{\alpha z - \left(\exp(\alpha z) - 1\right) \cdot \ln\left(\frac{\exp(-\alpha z) - 1}{\exp(-\alpha) - 1} \right)}{\alpha}$

The empirical version of $K_N(z)$ and the three $K(z)$ coming from the fitted Archimedean copulas are presented in figure 1 below.

Figure 1

Empirical and theoretical values of $K(z)$



The corresponding mean square errors for the three copulas are 0.1354% for the Frank, 0.2763% for the Clayton and 0.2047% for the Gumbel.
Using this statistics, it is evident both from the figure and from the errors that the Frank copula provides the best fit.

4.3 The value of the guarantee and a sensitivity analysis.

In this subsection we present a numerical application of the model described in the preceding sections and a sensitivity analysis of the guarantee value through the change of the kendall's τ and, in this way, of the generator of the best copula.
The application is made with the following parameters:

- $x = 45$;
- $T = 4$;
- $\Delta t = 1$;
- $D = H = 100$;
- $r(t) = r = 0.05$;
- $\sigma_x = 0.20$;
- $\sigma_p = 0.02$;
- $\tau = 0.341$;
- $\alpha = 3.39839$.

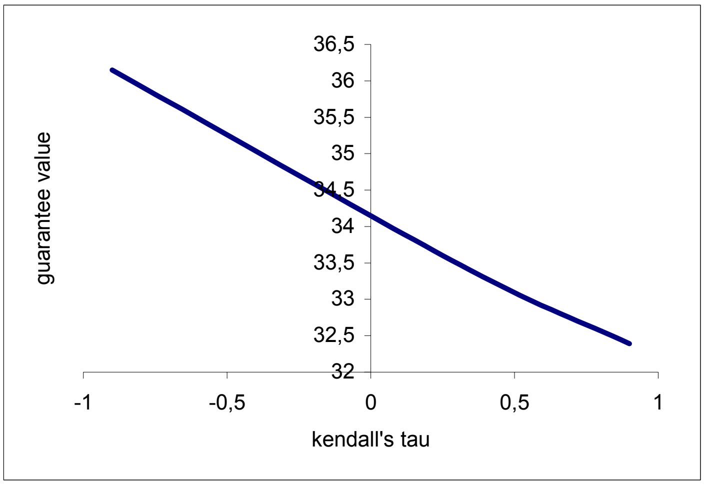
As already said the pension contract pays at time t a benefit consisting in the reference capital increased by the greatest of the two variation rates: the stochastic return of a financial risky portfolio and the stochastic consumer price index.
In other terms, using the previous symbols, the pension contract provides a benefit $b(T)$ at the maturity T if the insured is alive or the payment of $b(t)$ at time t if the insured dies before the maturity of the contract.
For the demographic technical bases we have used the mortality tables published by the Italian national Institute of Statistics (Istat) in 1996.
In the following tables we show the values of the options embedded in the described contract corresponding to the various dates; together we tabulate also the probabilities $\alpha(0,t)$ described in the section 3.2.

Table 3
Options value and probabilities

t	$C(x,p,t,H)$	$\alpha(0,t)$
1	12.71	0.003023
2	20.61	0.003382
3	27.61	0.003763
4	33.54	0.989832

The value of the options embedded in the contract is equal to 33.41.
It's important to analyse the sensitivity of the guarantees value to the variation of the kendall's τ (i.e. to the parameter of the copula's generator).
The following graph shows the dynamic of the value of the guarantees value through the change of τ.

Figure 2
The variation of the value of the guarantees through the change of kendall's τ



5. CONCLUSIONS

In this paper we propose a scheme useful to realize the pricing of defined contribution pension plans that provide a guarantee of a minimum rate of return when this guarantee depends on two risky assets: a financial portfolio and the consumer price index. The scheme considers that the dependence between the two risky assets can be expressed and modelled through an Archimedean bivariate copula and is based on the Monte Carlo method.

We have performed the numerical simulation to evaluate the values of the options embedded in the pension contract for a male aged 45 and a maturity of 4 years using a step of discretization of 1 year. Also a sensitivity analysis has been conducted.

In the paper there are also some algorithms useful to generate random numbers from a risk neutral Archimedean copula.

In conclusion we want to highlight that copula functions can represent useful tools to realize more refined risk management strategies for the financial risk managers of pension funds always following the traditional scheme of risk neural valuations.

Further lines of research can be found in creating algorithms able to generate pseudo-random numbers from multivariate risk neutral copulas and, in this way, considering also other forms of risk like, for example, a stochastic risk free interest rate.

REFERENCES

[1] Amin K.I., Jarrow R.A.: *Pricing options on risky assets in a stochastic interest rate economy,* Mathematical finance, Vol. 2, n.4, 217-237 (1992).

[2] Bacinello A.R., Ortu F.: *Pricing equity-linked life insurance with endogenous minimum guarantee,* Mathematics and Economics, 12, 245-257 (1993).

[3] Bacinello A.R.: *Valuation of contingent claims characterizing particular pension schemes*, Insurance: Mathematics and Economics, 27, 177-188 (2000).

[4] Boyle P.: *Options: a Monte Carlo approach,* Journal of Financial Economics, 4, 323-338 (1977).

[5] Black F., Scholes M.: *The pricing of options and corporate liabilities*, Journal of Political Economy, 81 (May-June 1973), 637-654 (1973).

[6] Brennan M.J., Schwartz E.S.: *The pricing of equity-inked life insurance policies with an asset value guarantee,* Journal of Financial Economics, n.3 (1976).

[7] Cherubini U., Luciano E.: *Value at risk, trade off and capital allocation with copulas* (2001), Economic Notes, 30 (2).

[8] Delbaen F.: *Equity linked policies,* Bulletin Association Royal Actuaries Belge, 33-52 (1990).

[9] Durrleman V., Nikeghbali A., Roncalli T.: *Which copula is the right one?* Groupe de recherche opérationnelle, Crédit Lyonnais (2000).

[10] Embrechts P., Hoing A., Juri A.*: Using copulae to bound the value at risk for functions of dependent risks*, Working paper, www.math.ethz.ch/~baltes/ftp/papers.html (2001).

[11] Embrechts P., McNeil A., Straumann D.: *Correlation and dependence in risk management: properties and pitfalls,* Risk Management: Value at risk and beyond, ed. M.A.H. Dempster, Cambridge University Press: 176, 2–23 (2002).

[12] Frees E., Valdez E.: *Understanding relationships using copulas*, North American Actuarial Journal, 2: 1-25 (1998).

[13] Genest C., MacKay J.: *The joy of copulas: bivariate distributions with uniform marginals*, The American Statistician; 40: 280-283 (1986).

[14] Genest C., Rivest L.: *Statistical inference procedure for bivariate Archimedean copulas*, Journal of the American statistical association, 88, 1034-1043 (1993).

[15] Harrison J.M., Kreps J.M.: *Martingales and arbitrage in multi-period security markets,* Journal of Economic Theory, 20, 381-408 (1979).

[16] Hipp C.: *Options for guaranteed index-linked life insurance,* Proceedings of VI AFIR International Colloquium, Vol. 2, Nurnberg, Germany, 1463-1483 (1996).

[17] Klugman S.A., Parsa R., *Fitting bivariate loss distributions with copulas,* Insurance: Mathematics and Economics, 24: 139-148 (1999).

[18] Micocci M., Pellizzari P., Perrotta A.: *Pension scheme with a minimum guarantee: the pricing in the case of two underlying variables and stochastic interest rates,* AMASES Colloquium (2002).

[19] Nielsen J.A., Sandmann K.: *Equity-linked life insurance: a model with stochastic interest rate,* Insurance, Mathematics and Economics, 16, 225-253 (1995).

[20] Pennacchi G.G.: *The value of guarantees on pension fund returns,* The Journal of Risk and Insurance, Vol. 66, n. 2, 219-237 (1999).

[21] Persson S.A., Aase K.K.: *Valuation of the minimum guaranteed return embedded in life insurance product,* Journal of Risk and Insurance, 64, 4, 599-617 (1997).

[22] Rapuch G., Roncalli T.: *Some remarks on two asset options pricing and stochastic dependence of asset prices*, Crédit Lyonnais (2001).

[23] Roncalli T.: *Copules et aspects multidimensionnels du risque*, Crédit Lyonnais (2000).

[24] Schweizer B., Wolff E.F., *On non parametric measures of dependence for random variables*, The annals of statistics, 9: 879-885 (1981).

[25] Sklar A., *Fonction de répartition à n dimensions et leurs marges,* Publ. Inst. Stat. Université de Paris (1959).

[26] Stulz R. M.: *Options on the minimum or the maximum of two risky assets,* Journal. of Financial Economics, Vol. 10, n. 1,: 161-185 (1982).