PROPERTIES OF YIELD CURVES AND FORWARD CURVES FOR AFFINE TERM STRUCTURE MODELS

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Abstract

The term structure of interest rates plays the key role in pricing of bonds. Therefore its properties are interesting for many financial analysts. However in literary sources usually a sketchy description of properties of term structure occurs. In this paper an attempt of detailed description of every possible shapes of term structure is made for the class of affine interest rate models because for these models the solutions in closed form are attainable. As basis the general model (GM) with arbitrary lower boundary for interest rate is taken. The results for well-known models the CIR model and the Vasiček model are obtained as particular cases. It is found that there is four modes for yield curve shapes. The empirical evidences are presents that are based on the parameter estimates for 13 different models of real time series of yield interest rates.

Keywords

Affine term structure model, yield interest rate, forward interest rate, yield curve, forward curve, shape of term structure, CIR model, Vasiček model.

Introduction

It is known that the term structure of interest rates plays the key role in pricing of bonds. The term structure of interest rates is the set of yields to maturity, at a given time, on bonds of different maturities. The yield curve is a plot of the term structure, that is the graphical description of the relationship between the yield on bonds of the same credit quality but different maturities on some particular date. The yield curve defines the prices of discount bonds and also coupon bonds, since a coupon bond is simply a portfolio of discount bonds. Modeling the yield curve is bounded up with modeling the forward rate curve. The forward rate curve is a graph of rates of return for discount bonds on some future date. The sense of this curve will below be given more precisely.

The objective of this paper is an investigation of properties of the yield curves and the forward rate curves for the affine one-factor models of the term structure. As it is known the affine models of the term structure are occurred if the short interest rate follows the stochastic process described by a stochastic differential equation

$$dr(t) = (\alpha\, r(t) + \beta\,)\, dt + \sqrt{\gamma r(t) + \delta}\, dW(t), \qquad \gamma\, r(0) + \delta\, > 0, \qquad (1)$$

where α, β, γ and δ are constants, and $W(t)$ is a standard Wiener process. It is supposed that the values of parameters α, β, γ and δ are such that a stationary solution of equation (1) exists. Then this equation my be rewritten in the more convenient form

$$dr(t) = \; k\,(\theta - \; r(t))\; dt + \sqrt{2kD\frac{r(t) - x}{\theta - x}}\, dW(t), \;\; r(0) > x. \qquad (2)$$

Parameters of equation (2) have specific practical interpretation: θ – the stationary expectation of short interest rate $r(t)$, D – the stationary variance of $r(t)$; x is parameter that has a sense of a lower boundary for the process $r(t)$: $r(t) \geq x$ for every t (by Feller (1951) this boundary is inaccessible if $(\theta - x)^2 > D$); k – parameter that determines the velocity of transition to stationary regime for process (1); the other interpretation of parameter k: it determines correlation coefficient of process (1) in the form $\rho(\tau) = \mathrm{E}[(r(t) - \theta)(r(t + \tau) - \theta)]/D = \exp\{- k|\tau|\}$. The relations between parameters of equations (1) and (2) are set obviously by compare:

$$k = -\,\alpha\; > 0, \;\; \theta \; = -\,\frac{\beta}{\alpha} > 0, \;\; D = \frac{\gamma\beta - \alpha\delta}{2\alpha^2} > 0, \;\; x = -\,\frac{\delta}{\gamma} < \theta. \qquad (3)$$

Assume that the no arbitrage conditions are held for the short rate process (1) (that is (2) also). In this case the market price of risk $\lambda(r)$ is determined by equivalent relations (Ilieva, 2001):

$$\lambda(r) = \frac{\xi}{\sqrt{\gamma}}\sqrt{r + \frac{\delta}{\gamma}} \; = \frac{\eta}{\sqrt{\delta}}\sqrt{\frac{\gamma}{\delta}r + 1} \; = -\,\lambda\sqrt{2kD\frac{r - x}{\theta - x}}\,, \qquad (4)$$

where λ is parameter that determines value of risk premium, $\lambda \geq 0$, and parameters ξ and η are connected by the relation $\eta\gamma - \xi\delta = 0$.

Then on current time point t, when $r(t) = r$, the price $P(t, r, T)$ of zero coupon bond that pays on maturity date T the one money unity is determined by formula

$$P(t, r, T) = \exp\{A(T - t) - rB(T - t)\}. \qquad (5)$$

In future for short the time up to maturity will be designed by $\tau \equiv T - t$. The interest rate models allowing to present the bond price $P(t, r, T)$ in form (5) relate to class of affine term structures of interest rates. The term structure functions $A(\tau)$ and $B(\tau)$ solve equations

$$\frac{dB}{d\tau} = 1 - \left(k + \lambda \frac{2kD}{\theta - x}\right) B(\tau) - \frac{kD}{\theta - x} [B(\tau)]^2, \quad B(0) = 0, \tag{6}$$

$$\frac{dA}{d\tau} = -\left(k\theta + \lambda x \frac{2kD}{\theta - x}\right) B(\tau) - \frac{kDx}{\theta - x} [B(\tau)]^2, \quad A(0) = 0. \tag{7}$$

The solutions of these equations are

$$B(\tau) = \left(\frac{\varepsilon}{e^{\varepsilon\tau} - 1} + V\right)^{-1}, \tag{8}$$

$$A(\tau) = x\,[B(\tau) - \tau] - \frac{(\theta - x)^2}{D} [v\tau - \ln(1 + vB(\tau))], \tag{9}$$

where for short designation it is assumed

$$\varepsilon \equiv \sqrt{(\alpha + \xi)^2 + 2\gamma} = \sqrt{\left(k + \lambda \frac{2kD}{\theta - x}\right)^2 + \frac{4kD}{\theta - x}},$$
$$v \equiv \frac{1}{2}\left(\varepsilon - k - \lambda \frac{2kD}{\theta - x}\right), \quad V \equiv \frac{1}{2}\left(\varepsilon + k + \lambda \frac{2kD}{\theta - x}\right). \tag{10}$$

The properties of the term structure functions $A(\tau)$ and $B(\tau)$ in form (8) – (9) are studied in details by Ilieva (2000).

The yield to maturity $y(t, T)$ of zero coupon bond in framework of the affine term structure has a form

$$y(t, T) \equiv -\frac{\ln P(t, r, T)}{T - t} = y(\tau) \equiv \frac{rB(\tau) - A(\tau)}{\tau}. \tag{11}$$

For determinacy note that here and below the term structure functions $A(\tau)$ and $B(\tau)$ are functions of one argument only at case in question when parameters of equation (1) are constants.

The forward rate $f(t, T, T')$ determines the bond yields between dates T и T' when $t < T < T'$ by information about yield that is available at time t:

$$f(t, T, T') = \frac{1}{T' - T} \ln\left[\frac{P(t,r,T)}{P(t,r,T')}\right] = \frac{r[B(\tau') - B(\tau)] - A(\tau') + A(\tau)}{\tau' - \tau}, \qquad (12)$$

where $\tau' = T' - t$. When $T' \to T$, that is $\tau' \to \tau$, the forward rate (12) turns into instantaneous forward rate

$$f(t, T) \equiv -\frac{\partial \ln P(t,r,T)}{\partial T} = f(\tau) \equiv r\frac{dB(\tau)}{d\tau} - \frac{dA(\tau)}{d\tau}, \qquad (13)$$

that is used more often as it connected with yield to maturity rather simple relations

$$y(t, T) = \frac{1}{T - t}\int_t^T f(t,s)ds = y(\tau) = \frac{1}{\tau}\int_0^\tau f(s)ds \qquad (14)$$

and conversely

$$f(t, T) = \frac{\partial[(T-t)y(t,T)]}{\partial T} = f(\tau) = y(\tau) + \tau\frac{dy(\tau)}{d\tau}. \qquad (15)$$

Therefore usually an idiom „the forward rate“ means the instantaneous forward rate.

Properties of yield curves and forward curves

In this section we will investigate mutual properties of the forward rate and yield to maturity as functions of the time to maturity τ in framework the affine term structure for different values of parameters r, x and λ: $r = r(t)$ is parameter of state at time t; x is parameter of the short rate model; λ is parameter of yield rate model. From the practical point of view there is sense to consider the properties of functions $f(\tau)$ и $y(\tau)$ only for nonnegative time to maturity $\tau \geq 0$, nonnegative values of short rate $r \geq 0$, nonnegative parameter of risk premium $\lambda \geq 0$ and in case when the Feller condition for inaccessibility of lower boundary by process $r(t)$ is held: $\theta - x > \sqrt{D}$.

Note that in the case $x = -\infty$ the equation (2) generate the Vasiček model (Vasiček, 1977), in the case $x = 0$ the equation (2) generate CIR model (Cox, Ingersoll and Ross, 1985). The model with arbitrary value of boundary x, that is general model, will refer as GM model. The term structure functions $A(\tau)$ and $B(\tau)$ for GM model was found by Medvedev and Cox (1996); the detailed analysis of GM model is contained in Ilieva (2000, 2001).

In order to obtain the close form of function $y(\tau)$ that determines a dependence of yield on the time to maturity it is sufficient to substitute the functions (8) and (9) into (11). This results in to relation

$$y(\tau) = x + (r - x)\frac{B(\tau)}{\tau} + \frac{k(\theta - x)}{V}\left(1 - \frac{\ln(1 + vB(\tau))}{\tau v}\right), \tag{16}$$

Here it is relevant to note that function $B(\tau)$ depending also on parameters x and λ plays the key role in determination both the functions $A(\tau)$ and $y(\tau)$ and the function $f(\tau)$ (see below). According to (8) function $B(\tau)$ is monotonically increasing and such that

$$B(0) = 0 \le B(\tau) \le B(\infty) = V^{-1}, \ 0 \le \tau \le \infty. \tag{17}$$

$$B(\tau) = \tau - \frac{1}{2}(k + 2\lambda vV)\tau^2 + O(\tau^3) \qquad \text{for small values of } \tau. \tag{18}$$

Using definition (13) for forward rate and equations (6) and (7) for functions $A(\tau)$ and $B(\tau)$ it is possible to obtain following relation for $f(\tau)$

$$f(\tau) = r + [k(\theta - x) - (V - v)(r - x)]B(\tau) - vV(r - x)[B(\tau)]^2. \tag{19}$$

The functions $y(\tau)$ and $f(\tau)$ that are determined by formulae (16) and (19) respectively will be named further the yield curve and the forward curve. Note that the forward curve in different forms for the Vasiček model and the CIR model were obtained by Schlögl and Sommer (1997) and there are cited some properties for these forward curves. In some literary sources there are information about mutual behavior of yield and forward curves. For example in Hull (1989, p. 83,84), Bodie, Kane and Marcus (1996, p. 437), Campbel, Lo and MacKinlay (1997, p. 398), Kortanek and Medvedev (2001, p. 201) the mutual forms of yield and forward curves are presented on some time periods of finite duration. However by these plots in full measure to present the nature of curve modifications to set impossible. From these plots it can seem that with increasing of the time to maturity the spread between yield and forward curves will be increase. Here it will be shown that it is impossible at least for affine term structure models.

Below the properties of yield and forward curves are formulated. The proofs of properties are given in Appendix.

Property 1. The yield curve $y(\tau)$ and the forward curve $f(\tau)$ take the same values on limiting times to maturity $\tau = 0$ and $\tau = \infty$:

$$f(0) = y(0) = r, \tag{20}$$

$$f(\infty) \equiv f^*(x) = y(\infty) \equiv y^*(x) = \frac{k}{V}\theta + \left(1 - \frac{k}{V}\right)x. \qquad (21)$$

Because $0 < k/V < 1$ then $x < f(\infty) \equiv f^*(x) = y(\infty) \equiv y^*(x) < \theta$. From this it follows in particular that as $\tau \to \infty$ limiting values of yield curve and forward curve always less of the stationary expectation θ of short interest rate $r(t)$.

Note also that from the definitions of the yield to maturity and the forward rate following limiting relations follow as $T \to t$

$$y(t, T) \to y(t, t) = r(t), \quad f(t, T) \to f(t, t) = r(t)$$

that are equivalent the equations (20).

Property 2. If the time to maturity τ is small then the yield curve $y(\tau)$ and the forward curve $f(\tau)$ can be presented in the forms

$$y(\tau) = r + \tfrac{1}{2}\,[(\theta - r)(k + 2\lambda vV) - 2\lambda kD]\tau + O(\tau^2) \equiv$$

$$\equiv r + \tfrac{1}{2}\,[k(\theta - x) - (V - v)(r - x)]\tau + O(\tau^2), \qquad (22)$$

$$f(\tau) = r + [(\theta - r)(k + 2\lambda vV) - 2\lambda kD]\tau + O(\tau^2) \equiv$$

$$\equiv r + [k(\theta - x) - (V - v)(r - x)]\tau + O(\tau^2). \qquad (23)$$

These formulae indicate that for $\lambda = 0$ (the term structure model is risk neutral) if $r > \theta$ then the yield curve $y(\tau)$ and the forward curve $f(\tau)$ have a negative slope (decrease) in neighborhood of value $\tau = 0$; if $r < \theta$ then these curves have a positive slope (increase) in this neighborhood. Moreover the forward curve $f(\tau)$ varies doubly rather than the yield curve $y(\tau)$. Formulae (22) – (23) justify also equation (20).

The Vasiček model ($x = -\infty$) is often attacked because it allows the negative values of short rates $r(t)$. This can result in to negative values of the yield curve $y(\tau)$ and the forward curve $f(\tau)$. At the same time the CIR model ($x = 0$) guarantees that both the short rates $r(t)$ and the curves $y(\tau)$ and $f(\tau)$ are nonnegative. Therefore it is interesting to clear up what is a minimal value of boundary x at GM model in order to the yield curve $y(\tau)$ and the forward curve $f(\tau)$ would be nonnegative. For example the necessary conditions for it can be following: a positive slope of the curves $y(\tau)$ and $f(\tau)$ in neighborhood of point ($\tau = 0$, $r = 0$) and a positive size of the limiting value $f^*(x) = y^*(x)$ that is determined by (21).

Property 3. The limiting value $y^*(x)$ of the yield curve $y(\tau)$ (and $f^*(x)$ of the forward curve $f(\tau)$ too) as $\tau \to \infty$ that is determined by equation (21) is a monotonic increasing function of boundary x and on interval $[-\infty, \theta]$ takes values

$$y^*(-\infty) = \theta - (1 + 2\lambda k)D/k \le y^*(x) \le y^*(\theta) = \theta. \quad (24)$$

Thus if $k\theta > (1 + 2\lambda k)D$ then the limiting value $y^*(x)$ is positive for every $x < \theta$. In the case when $k\theta \le (1 + 2\lambda k)D$ the limiting values of curves $f^*(x) = y^*(x) \ge 0$ for

$$x \ge x^* \equiv -\theta \ \frac{k\theta + \sqrt{k\theta D + (\lambda k D)^2} - \lambda k D}{D - k\theta + 2\lambda k D}. \quad (25)$$

Property 4. The necessary conditions in order to the yield curve $y(\tau)$ and the forward curve $f(\tau)$ take only nonnegative values for $0 \le \tau \le \infty$, that is

$$f(\infty) = y(\infty) \ge 0 \quad (26)$$

and for $r = 0$

$$\left.\frac{df(\tau)}{d\tau}\right|_{\tau=0} > 0, \quad \left.\frac{dy(\tau)}{d\tau}\right|_{\tau=0} > 0, \quad (27)$$

are held for every $x < \theta$ if $k\theta \ge (1 + 2\lambda k)D$ and for $\theta > x \ge x^*$ if $k\theta < (1 + 2\lambda k)D$. Here x^* is determined by equation (25).

Thus the yield curve $y(\tau)$ and the forward curve $f(\tau)$ take only positive values at the Vasiček model ($x = -\infty$) too if $k\theta \ge (1 + 2\lambda k)D$.

The many authors analyzing the yield curves and the forward curves have noted that these curves can be humped, that is the curves can have maximums. We will found the conditions of existing the maximums of these curves and determine the characteristics of these maximums.

As already it was said the affine term structure function $B(\tau)$ have the key role for determination the yield curves and the forward curves and them properties. It follows from (8) that $B(\tau)$ is monotonic increasing on interval $[0, \infty]$. Note also that from (8) we have that the inverse function $B(\tau)$ is found in form

$$\tau(B) = [\ln(1 + vB) - \ln(1 - VB)]/\varepsilon \,. \quad (28)$$

In future it is convenient to consider the forward curve $f(\tau)$ and the yield curve $y(\tau)$ as the composite functions that depend on time to maturity τ only by the affine term structure function $B(\tau)$, that is $y(\tau) \equiv Y(B(\tau))$ and $f(\tau) \equiv F(B(\tau))$. First, it is convenient because the possible values of function $B(\tau)$ are situated into a finite interval (17) therefore the properties of functions $Y(B)$ and $F(B)$ are visually displayed by graphical charts on finite interval. Second, as CIR (1979) have suggested, the function $B(\tau)$ may be regarded as a duration measure because, like standard duration, it is equal to minus the semi-elasticity of the bond price with respect to an interest rate (in this case short rate): $[\partial P/\partial r]/P = -B(\tau)$.

It is obtained from the equations (16), (19) and (28) that

$$Y(B) \equiv x + \frac{k(\theta - x)}{V} + \varepsilon \frac{(r-x)B - k(\theta - x)\ln(1+vB)/vV}{\ln(1+vB) - \ln(1-VB)} \tag{29}$$

and

$$F(B) \equiv r + [k(\theta - x) - (V - v)(r - x)]\, B - vV(r - x)\, B^2. \tag{30}$$

Property 5. The forward curve $F(B)$ is a concave function.

If (positive) parameter r in (30) is in accord with inequalities

$$\frac{k}{V+v} \le \frac{r-x}{\theta - x} \le \frac{k}{V-v} \tag{31}$$

then the forward curve $F(B)$ on the interval $0 \le B \le V^{-1}$ have maximum in point

$$B^* = \frac{1}{2vV}\left(k\frac{\theta - x}{r-x} - V + v\right) \tag{32}$$

In this case the maximum value of the forward curve $F(B)$ is determined by formula

$$F(B^*) = r + \frac{[k(\theta - x) - (V - v)(r - x)]^2}{4vV(r-x)}.$$

If the parameter r meets the inequality $\frac{r-x}{\theta - x} < \frac{k}{V+v}$ then the forward curve $F(B)$ strongly increase on the interval $0 \le B \le V^{-1}$.

If the parameter r meets the inequality $\frac{r-x}{\theta - x} > \frac{k}{V-v}$ then the forward curve $F(B)$ strongly decrease on the interval $0 \le B \le V^{-1}$.

Note that earlier the concavity property of the forward rate has been registered by Brown and Schaefer (1994).

Corollary. If the value of short rate r meets the equality

$$\frac{r-x}{\theta - x} = \frac{k}{V-v}$$

then the maximum of forward curve is suited at point $B = 0$ (that is $\tau = 0$). For larger values of short rate r the forward curve has no maximum and its largest value locates at point $B = 0$ also. It follows from this that the forward rates are largest for the short end of term structure.

If the value of short rate r meets the equality

$$\frac{r-x}{\theta-x}=\frac{k}{V+v}$$

then the maximum of forward curve is suited at point $B = V^{-1}$ (that is $\tau = \infty$). For lesser values of short rate r the forward curve has no maximum and its largest value locates at point $B = V^{-1}$ also. It follows from this that in this case the forward rates are largest for the long end of term structure.

The analysis of features of changing of the yield curve $y(\tau) \equiv Y(B(\tau))$ as function of $B(\tau)$ is more complicated.

Property 6. If the value of short rate r meets the inequality

$$\frac{r-x}{\theta-x}\geq\frac{k}{v}\ln\left(1+\frac{v}{V}\right)$$

then the yield curve $Y(B)$ is a concave function on interval $0 \leq B \leq V^{-1}$.

If the value of short rate r meets the inequality

$$\frac{r-x}{\theta-x}\leq\frac{k}{v+V}$$

then the yield curve $Y(B)$ is a convex function on interval $0 \leq B \leq V^{-1}$.

If the value of short rate r meets the inequalities

$$\frac{k}{v+V}<\frac{r-x}{\theta-x}<\frac{k}{v}\ln\left(1+\frac{v}{V}\right)$$

then the yield curve $Y(B)$ has a point of inflexion B_i on interval $0 \leq B \leq V^{-1}$. In this case the yield curve $Y(B)$ is the concave function on the interval $0 < B < B_i$ and it is the convex function on the interval $B_i < B < V^{-1}$.

Note that for the limiting value of the yield curve (21) as $\tau \to \infty$ it is possible to write the inequality

$$\frac{y(\infty)-x}{\theta-x}\equiv\frac{Y(V^{-1})-x}{\theta-x}=\frac{k}{V}>\frac{k}{v}\ln\left(1+\frac{v}{V}\right)>\frac{r-x}{\theta-x}.$$

Therefore in this case the yield curve $Y(B)$ increases on interval $0 < B < V^{-1}$.

Property 7. The yield curve $Y(B)$ has a maximum on interval $0 < B < V^{-1}$ if r meets the inequalities

$$\frac{k}{v}\ln\left(1+\frac{v}{V}\right)<\frac{r-x}{\theta-x}<\frac{k}{V-v}.$$

In this case the yield curve $Y(B)$ crosses the forward curve $F(B)$ at some point B_0 (that is $Y(B_0) = F(B_0)$) and this point B_0 is a point of maximum of yield curve $Y(B)$. Furthermore

$$\begin{aligned} Y(B) < F(B) \quad &\text{if} \quad 0 < B < B_0; \\ Y(B) > F(B) \quad &\text{if} \quad B_0 < B < V^{-1}. \end{aligned}$$

In other words the yield curve $Y(B)$ crosses the forward curve $F(B)$ at point B_0 of its maximum. From this it follows in particular that if a maximum of the yield curve $Y(B)$ exists then the maximum value always less than the maximum value of the forward curve $F(B)$, that is $Y(B_0) < F(B^*)$, as $B^* < B_0$.

Let us say that the yield curve $Y(B)$ has a mode A if its shape is concave decreasing. In this case the inequality

$$\frac{r-x}{\theta-x}>\frac{k}{V-v} \tag{33}$$

is valid. The yield curve $Y(B)$ has a mode B if its shape is concave and it has maximum. In this case the inequality

$$\frac{k}{v}\ln\left(1+\frac{v}{V}\right)<\frac{r-x}{\theta-x}<\frac{k}{V-v} \tag{34}$$

is valid. The yield curve $Y(B)$ has a mode C if it is increasing and it has a point of inflexion. In this case the inequality

$$\frac{k}{v+V}<\frac{r-x}{\theta-x}<\frac{k}{v}\ln\left(1+\frac{v}{V}\right) \tag{35}$$

is valid. The yield curve $Y(B)$ has a mode D if its shape is convex increasing. In this case the inequality

$$\frac{r-x}{\theta-x}\le\frac{k}{v+V} \tag{36}$$

is valid.

The main properties of the yield curve $Y(B)$ and the forward curve $F(B)$ are tabulated in the Table 1.

Table 1

The shapes of the yield curve $Y(B)$ and the forward curve $F(B)$ in dependence on value the interest rate r

<table>
<tr><td></td><td colspan="4">THE PERFORMANCE OF INEQUALITIES</td></tr>
<tr><td></td><td>(36)</td><td>(35)</td><td>(34)</td><td>(33)</td></tr>
<tr><td>$F(B)$</td><td>concave, increases</td><td colspan="2">concave, has maximum at point B^*</td><td>concave, decreases</td></tr>
<tr><td rowspan="2">$Y(B)$</td><td rowspan="2">convex, increases</td><td rowspan="2">has a point of inflexion, increases</td><td colspan="2">concave</td></tr>
<tr><td>has maximum at point B_0, $B_0 > B^*$</td><td>decreases</td></tr>
<tr><td colspan="3">$Y(B) < F(B)$</td><td>there is intersection at point B_0, $B_0 > B^*$</td><td>$Y(B) > F(B)$</td></tr>
</table>

On figure 1 as example all four modes represented for values of parameters close agreement to real: k = 0,03; θ = 0,06; D = 0,002; x = – 0,05; r = 0,07 (mode A), r = 0,03 (mode B), r = 0,014 (mode C), r = 0,005 (mode D).

In the inequalities (33) – (36) there is one the random parameter $r = r(t)$. The rest parameters are constants. Ilieva (2000) showed that the process $r(t)$ has the stationary probability density function $p(r)$, which is a density of shifted gamma distribution with a shift parameter $(-x)$, a form parameter $(q + 1)$, and a scale parameter c_0, that is

$$p(r) \;=\; \frac{c_0^q\,(r-x)^{q-1}}{\Gamma(q)}\;e^{-c_0(r-x)}, \quad x < r < \infty, \tag{37}$$

where $q = \dfrac{(\theta - x)^2}{D}$, $c_0 = \dfrac{\theta - x}{D} > 0$. However in the inequalities (33) – (36) the modes are determined by random variable $\zeta = (r - x)/(\theta - x)$. The probability density function $p_\zeta(z)$ for ζ will be the ordinary gamma density with the same form parameter q and the scale parameter $c_1 = (\theta - x)^2/D$, that is

$$p_\zeta(z) = \frac{c_1^q\, z^{q-1}}{\Gamma(q)}\;e^{-c_1 z}, \quad 0 < z < \infty. \tag{38}$$

Therefore it is possible to speak about the probabilities of modes A – D for some yield curve. In next section of paper we will compute such probabilities of

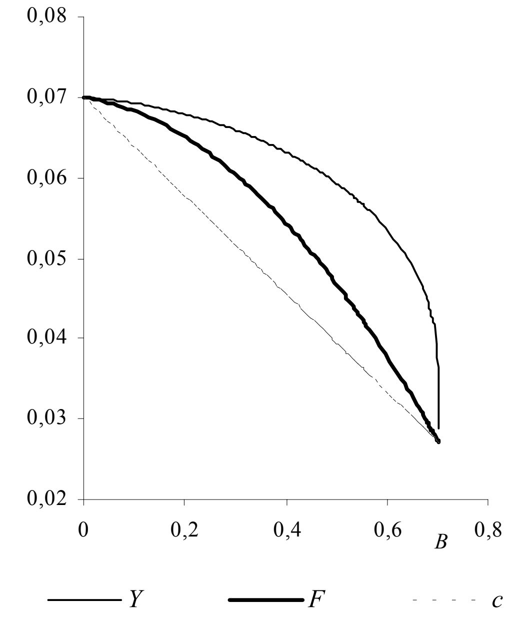

Figure 1a. *F* – the forward curve *F*(*B*), *Y* – the yield curve *Y*(*B*) of mode A, *c* – the straight-line segment *c*(*B*).

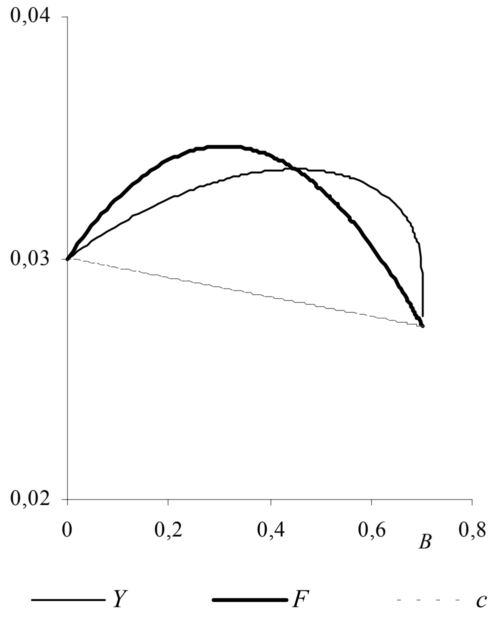

Figure 1b. *F* – the forward curve *F*(*B*), *Y* – the yield curve *Y*(*B*) of mode B, *c* – the straight-line segment *c*(*B*).

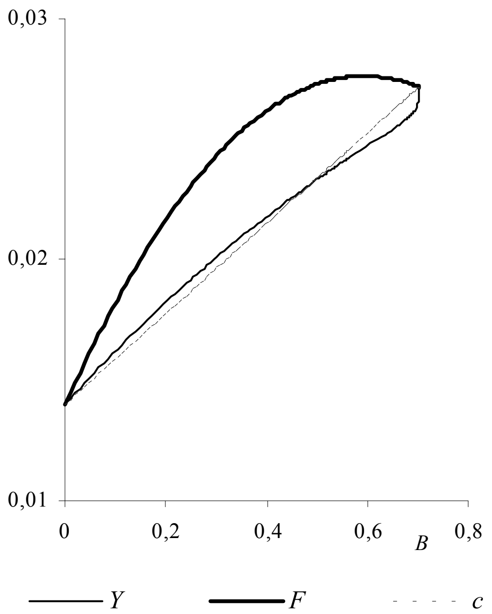

Figure 1c. *F* – the forward curve *F*(*B*), *Y* – the yield curve *Y*(*B*) of mode C, *c* – the straight-line segment *c*(*B*).

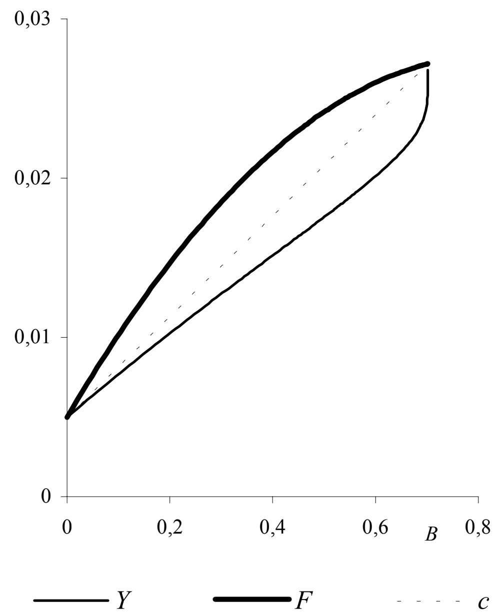

Figure 1d. *F* – the forward curve *F*(*B*), *Y* – the yield curve *Y*(*B*) of mode D, *c* – the straight-line segment *c*(*B*).

these modes for the empirical estimates of parameters that were obtained by different authors.

Property 8 (Vasiček model). The Vasiček model implies that $x = -\infty$. In this case from (10) we have that $v = 0$, $V = k$. Then for inequalities (33) – (36) we have

$$\frac{k}{v+V} = \frac{k}{v}\ln\left(1+\frac{v}{V}\right) = \frac{k}{V-v} = 1.$$

It means that the yield curve $Y(B)$ for Vasiček model lacks the modes B and C. In other words the Vasiček model does not produce a humped yield curves. The probability of modes A and D are identical, that is

$$\text{Prob[mode A]} = \text{Prob[mode D]} = 0{,}5.$$

Note that some authors observe the diversity of the yield curve shapes. For example Fabozzi and Fabozzi (1995, p. 801–2) represent also four modes – normal, rising, falling and humped – and explain these modes by economic motives. Model CIR (1985) implies that the yield curve attains all four modes: monotonic rising (modes C and D), humped (mode B), and monotonic declining (mode A).

Consider now the impact of x – the parameter that has a sense of a lower boundary for the process $r(t)$ – on the yield curve more detailed. For case when the lower boundary is inaccessible ($\theta - x > \sqrt{D}$) we have that $\sqrt{D} < \theta - x < \infty$. On the Property 3 if the inequality $k\theta > (1 + 2\lambda k)D$ is valid then the yield curve $Y(B)$ and the forward curve $F(B)$ are positive on the interval $0 \le B \le V^{-1}$ for all $(\theta - x) \in (\sqrt{D}, \infty)$. As the empirical data indicate (see next section) this inequality usually is valid. Therefore we will examine only this case.

From the inequalities (33) – (36) it follows that the boundaries of the modes are depended on x only through v and V that are determined by (10). Then one may write $v = v(\theta - x)$ and $V = V(\theta - x)$, $(\theta - x) \in (\sqrt{D}, \infty)$. From (10) one can see that the functions $v(\theta - x)$ and $V(\theta - x)$ monotonic decrease with rise of its argument and at limiting points are determined by the expressions

$$\begin{aligned} v(\sqrt{D}) &= \frac{k}{2}\left(\sqrt{(1+2\lambda\sqrt{D})^2 + 4\frac{\sqrt{D}}{k}} - 1 - 2\lambda\sqrt{D}\right), \quad v(\infty) = 0, \\ V(\sqrt{D}) &= \frac{k}{2}\left(1+2\lambda\sqrt{D} + \sqrt{(1+2\lambda\sqrt{D})^2 + 4\frac{\sqrt{D}}{k}}\right), \quad V(\infty) = k. \end{aligned} \tag{39}$$

Property 9. From relations (39) one can conclude that for small values of parameters k and D the variation interval the functions $v(\theta - x)$ and $V(\theta - x)$ will be small too and the impact of variation of parameter x on the yield curve $Y(B)$ and the forward curve $F(B)$ in this case will be weak. In addition the main effect of variation of parameter x tells only at beginning of interval $\sqrt{D} < \theta - x < \infty$.

Finally examine the effect of parameter $\lambda \geq 0$ that determines value of risk premium. This parameter influences on the yield curve $Y(B)$ and the forward curve $F(B)$ only through v and V that are determined by relations (10).

Property 10. From relations (10) one can see that the effect of increasing of parameter λ is practically identical to the effect of decreasing of the parameter $(\theta - x)$. The distinction is only in a fact that the parameter λ can be arbitrarily small whereas the parameter $(\theta - x)$ is restricted from below. Therefore the spreads of $v(\theta - x)$ and $V(\theta - x)$ are finite on interval $\sqrt{D} < \theta - x < \infty$ whereas the spreads of $v(\lambda)$ and $V(\lambda)$ are infinite on interval $0 < \lambda < \infty$. With increasing of parameter λ the yield curve $Y(B)$ and the forward curve $F(B)$ monotonic decrease if all other parameters are fixed.

In conclusion note that the parameter λ in the GM model considered here differs from this parameter in the CIR model usually considered. A relation between them is following: $\lambda \equiv \lambda_{GM} = -\lambda_{CIR}\theta/2kD$. This means that the risk premium (or term premium) in framework of the GM model is computed by formula

$$2kD\frac{r - x}{\theta - x}B(\tau)\lambda_{GM}.$$

Empirical evidences

In order to verify the shape of the yield curve $Y(B)$ and the forward curve $F(B)$ for real model it is necessary to know the parameter values for this model. There are quite a few papers where the empirical results for the estimates of parameters of the affine term structure models are described. Most often as the short interest rate model it takes the CIR model. Some authors examined several models including the Vasiček model. As a rule the estimates of parameters for CIR model and Vasiček model were close agreement. Author knows only one paper – Ilieva (2001) – where the parameters of GM model were estimated.

The Table 2 presents a report of results that are obtained for 13 different models of real time series of yield interest rates. These results are contained in ten well-known papers (see References). Some authors did not estimate the parameter of market risk λ. In that case the according cells in Table 2 are empty and under calculations it is assumed that $\lambda = 0$ that is such results correspond to the risk-neutral market.

Table 2

THE CIR MODEL PARAMETER ESTIMATES

Parameter estimate source	k	θ	σ	λ	D
CKLS (1992)	0,2339	0,0808	0,0854		0,00126
Sun (1992)	1,1570	0,0520	0,1223		0,00034
Gibbons & Ramaswamy (1993), I	12,4300	0,0154	0,4900	– 6,0800	0,00015
Gibbons & Ramaswamy (1993), II	8,8076	0,0085	0,0000	– 5,6083	0,00000
Gibbons & Ramaswamy (1993), III	14,4477	0,0264	0,5459	– 6,0101	0,00027
Chen & Scott (1993)	0,4000	0,0600	0,3000		0,00675
Pearson & Sun (1994)	0,8762	0,0311	0,1707	– 0,1282	0,00052
Ait-Sahalia (1996)	0,8922	0,0905	0,1809	– 0,0789	0,00166
Duffie & Singleton (1997), I	0,5440	0,3740	0,0230	– 0,0360	0,00018
Duffie & Singleton (1997), II	0,0030	0,2580	0,0190	– 0,0040	0,01552
Bali (1999)	0,0317	0,0642	0,0265		0,00071
Ait-Sahalia (1999)	0,0219	0,0721	0,0667		0,00732
Ilieva (2001)	0,1674	0,0638	0,0160		0,00005

The CIR model was set as $dr(t) = k(\theta - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t)$, $D = \sigma^2\theta/2k$. Real risk premium is $-\lambda r B(\tau)$. The sample periods, the financial instruments and the data character have been determined from following literary sources:

CKLS – Chan, Karolyi, Longstaff, and Sanders – (1992): the annualized one-month U.S. Treasury bill yield from June 1964 to December 1989 (306 observations).

Sun (1992): 182 monthly observations of U.S. Treasury prices from November 1971 to December 1986.

Gibbons and Ramaswamy (1993), I: monthly data from 1964 to 1989 on U.S. Treasury bill returns.

Gibbons & Ramaswamy (1993), II: monthly data from 1964 to 1976 on U.S. Treasury bill returns.

Gibbons and Ramaswamy (1993), III: monthly data from 1976 to 1989 on U.S. Treasury bill returns.

Chen and Scott (1993): weekly U.S. Treasury data (from Singleton, 2001).

Pearson and Sun (1994): 181 monthly prices of ten U.S. Treasury bills, notes and bonds from Dec 1971 to Dec 1986 (equally weighted bond portfolios).

Ait-Sahalia (1996): the 7-day Eurodollar deposit spot rate, daily from 1 Jun 1973 to 25 Feb 1995 (5505 observations).

Duffie and Singleton (1997), I: weekly data from 4 January 1988 to 28 October 1994 U.S. Treasury bond (zero prices).

Duffie and Singleton (1997), II: weekly data from 4 January 1988 to 28 October 1994 U.S. Treasury bond (yields).

Bali (1999): annualized one-month U.S. Treasury bill yield from June 1964 to December 1996 (390 observations).

Ait-Sahalia (1999): the Federal Reserve System funds data monthly from January 1963 to December 1998.

Ilieva (2001): annualized daily two-years U.S. Treasury note yield from 2 January 1991 to 1 October 1996 (1439 observations). In this paper the GM model was tested. The estimate of the lower boundary: $x = -0{,}01998$. It is one of ten U.S. Treasury bills, notes and bonds yields data examined there.

Table 3 present the useful numerical data that are necessary for computation of the yield curve $Y(B)$ and the forward curve $F(B)$ and their characteristics.

Table 3

USEFUL INFORMATION ABOUT MODELS EXAMINED

This table report useful information about the models tested in literary sources listed in Table 2. On basis of the parameter estimates presented by Table 2 here the next magnitudes have been computed. For that CIR models, where the parameter λ was estimated, it was produced recalculation to a form $\lambda_{GM} = -\lambda\theta/2kD$ for using in the GM model. In this model the risk premium (term premium) is calculate by formula $2kDB(\tau)\lambda_{GM}(r - x)/(\theta - x)$ in a dependence on the short rate r and the term to maturity τ. This table present too the magnitudes v and V that are important for representation of many characteristics the GM model including the yield curve $Y(B)$ and the forward curve $F(B)$. These magnitudes are calculated by the expressions (10). Then limiting values of the yield curve $y(\tau)$ (and the forward curve $f(\tau)$) as the term to maturity $\tau \to \infty$ are tabled: $y(\infty) = f(\infty) = \lim_{\tau\to\infty} y(\tau) \equiv \lim_{\tau\to\infty} Y(B(\tau))$ (see formula (21)). Farther the limiting values of the term structure function $B(\tau)$ as the term to maturity $\tau \to \infty$ are presented: $B(\infty) = \lim_{\tau\to\infty} B(\tau) = V^{-1}$ (see formula (17)). This quantity is a maximum value of argument the yield curve $Y(B)$ and the forward curve $F(B)$ for respective graphical charts. The parameter estimates for the CIR models, which are presented in Table 2, can be used as possible parameters for analysis the GM models. For this case in the last column there are the minimal values of $y^*(x) \equiv y(\infty)$ on x, that is $y^*_{min} \equiv y^*(-\infty) = \lim_{x\to-\infty} y^*(x)$ (see formula (24)). These data show that for the real values of the model parameters the limiting values of the yield curves $y(\tau)$ (and the forward curves $f(\tau)$) as the term to maturity $\tau \to \infty$ for practically all models remain positive for all values $x \in (-\infty, \theta - \sqrt{D})$ of inaccessible lower boundary for the interest rate process $r(t)$. Only two models out of 13 are some exclusions: 1) Model Duffie and Singleton (1997), II. In this case $y^*(x) < 0$ if $x < -0{,}0615$; and 2) Model Ait-Sahalia (1999). In this case $y^*(x) < 0$ if $x < -0{,}0625$.

Data source	λ_{GM}	v	V	$y(\infty)$	$B(\infty)$	y^*_{min}
CKLS (1992)		0,015	0,249	0,076	4,023	0,075
Sun (1992)		0,006	1,163	0,052	0,860	0,052
Gibbons & Ramaswamy (1993), I	25,32	0,006	18,516	0,010	0,054	0,008
Gibbons & Ramaswamy (1993), II		0,000	8,808	0,009	0,114	0,009
Gibbons & Ramaswamy (1993), III	20,17	0,007	20,465	0,019	0,049	0,015
Chen & Scott (1993)		0,092	0,492	0,049	2,034	0,043
Pearson & Sun (1994)	4,40	0,014	1,019	0,027	0,982	0,026
Ait-Sahalia (1996)	2,41	0,017	0,988	0,082	1,012	0,081
Duffie & Singleton (1997), I	68,05	0,000	0,580	0,351	1,723	0,349
Duffie & Singleton (1997), II	11,08	0,010	0,017	0,045	57,526	-5,260
Bali (1999)		0,009	0,040	0,050	24,771	0,042
Ait-Sahalia (1999)		0,037	0,059	0,027	16,844	-0,262
Ilieva (2001)		0,001	0,168	0,064	5,953	0,064

Table 4 presents the mode specifications for models with parameters determined by Table 2.

Table 4

PROBABILITIES OF MODES FOR YIELD CURVES

This table reports the features of the yield curves for the CIR models if the parameters of these models are equal to the estimates presented in Table 2. The first three columns show the bounds that separate the different shapes of the yield curves $Y(B)$ on the modes. These bounds are determined by equalities: $T1 \equiv k/(v + V)$, $T2 \equiv k \ln(1 + v/V)/v$, $T3 \equiv k/(V - v)$. For the determination of the shape mode of the yield curve $Y(B)$ it is convenient to use the (random) variable $\zeta = (r - x)/(\theta - x)$. This random variable has the probability density function of the gamma distribution (38). In according to (33) – (36) the yield curve $Y(B)$ has the mode D if $\zeta <$ T1, the yield curve $Y(B)$ has the mode C if T1 $< \zeta <$ T2, the yield curve $Y(B)$ has the mode B if T2 $< \zeta <$ T3, and the yield curve $Y(B)$ has the mode A if T3 $< \zeta$. The probabilities of these random events are presented in the four last columns.

DATA SOURCE	Mode boundaries			Probabilities of modes			
	T1	T2	T3	D	C	B	A
CKLS (1992)	0,888	0,914	1,000	0,453	0,025	0,080	0,442
Sun (1992)	0,989	0,992	1,000	0,535	0,003	0,009	0,453
Gibbons & Ramaswamy (1993), I	0,671	0,671	0,672	0,422	0,000	0,000	0,578
Gibbons & Ramaswamy (1993), II	1,000	1,000	1,000	0,500	0,000	0,000	0,500
Gibbons & Ramaswamy (1993), III	0,706	0,706	0,706	0,378	0,000	0,000	0,622
Chen & Scott (1993)	0,686	0,746	1,000	0,583	0,021	0,073	0,323
Pearson & Sun (1994)	0,848	0,854	0,872	0,511	0,004	0,011	0,474
Ait-Sahalia (1996)	0,888	0,896	0,919	0,458	0,007	0,022	0,513
Duffie & Singleton (1997), I	0,936	0,937	0,938	0,036	0,001	0,003	0,960
Duffie & Singleton (1997), II	0,108	0,135	0,429	0,001	0,001	0,084	0,914
Bali (1999)	0,646	0,711	1,000	0,201	0,062	0,292	0,445
Ait-Sahalia (1999)	0,226	0,286	1,000	0,282	0,045	0,329	0,344
Ilieva (2001)	0,993	0,995	1,000	0,490	0,006	0,019	0,485

From Table 4 one could say that the modes B and C for real models are unlikely occurred. This property is similar to property of the Vasiček model where these modes are absent.

Thus for real one-factor models of the affine class of term structure only two shape modes of the yield curve $Y(B)$ can practically to occur. The yield curve is concave decreases when the interest rate r is rather high: $r > x + k(\theta - x)/(V - v)$; and the yield curve is convex increase when the short interest rate r is rather small: $r < x + k(\theta - x)/(V + v)$. The intermediate values of the interest rate r are unlikely occurred since the values of magnitude v are very small (see Table 3).

On figures 2 – 5 the examples of the yield curves $Y(B)$ and the forward curves $F(B)$ are presented. For these examples the results of CKLS (1992) and Bali (1999) are chosen (see Table 2). Figures 2 and 3 show the yield curves $Y(B)$ and the forward curves $F(B)$ for the CKLS (1992) parameters of models for different values of spot rate r and barrier x. Figures 3 and 4 show the yield curves $Y(B)$ and the forward curves $F(B)$ for the Bali (1999) parameters of models for different values of spot rate r and barrier x.

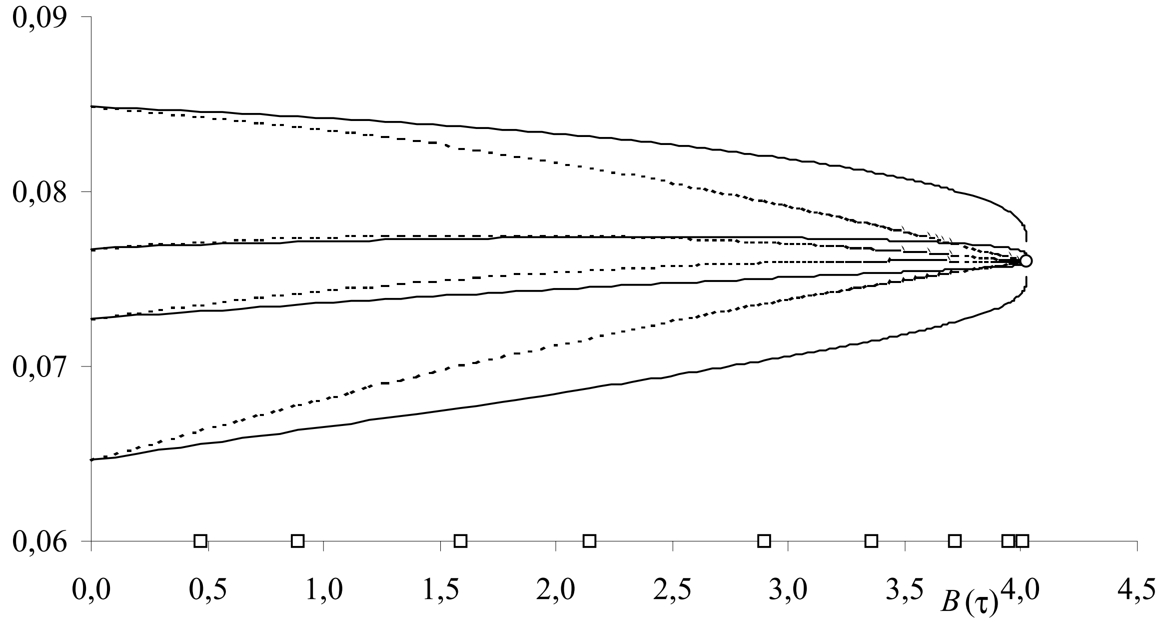

Figure 2. The yield curves $Y(B)$ (solid line) and the forward curves $F(B)$ (dashed line) for the CKLS (1992) parameters of models for different values of spot rate r = 0,065; 0,073; 0,077; 0,085; and barrier x = 0 (CIR model case). The limiting point of curves is shown by circle. The markers on axis B mean terms to maturity τ = 0,5; 1; 2; 3; 5; 7; 10; 20; 30 years.

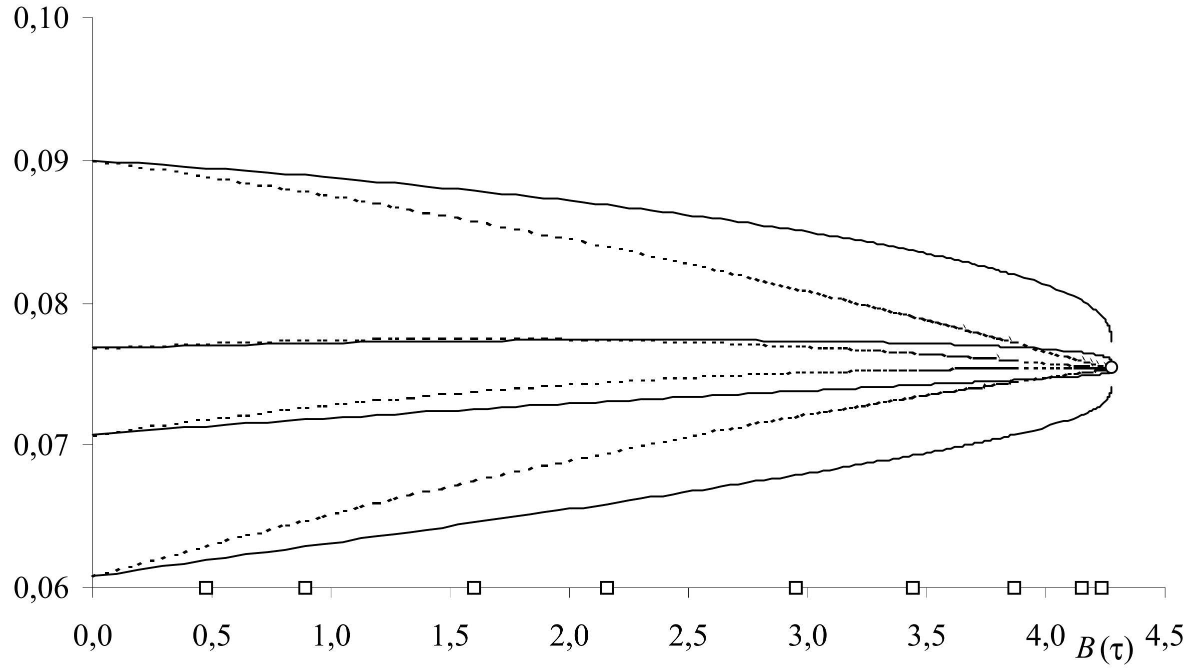

Figure 3. The yield curves $Y(B)$ (solid line) and the forward curves $F(B)$ (dashed line) for the CKLS (1992) parameters of models for different values of spot rate r = 0,061; 0,071; 0,077; 0,09; and barrier x = – 1000 (quasi Vasiček model case). The limiting point of curves is shown by circle. The markers on axis B mean terms to maturity τ = 0,5; 1; 2; 3; 5; 7; 10; 20; 30 years.

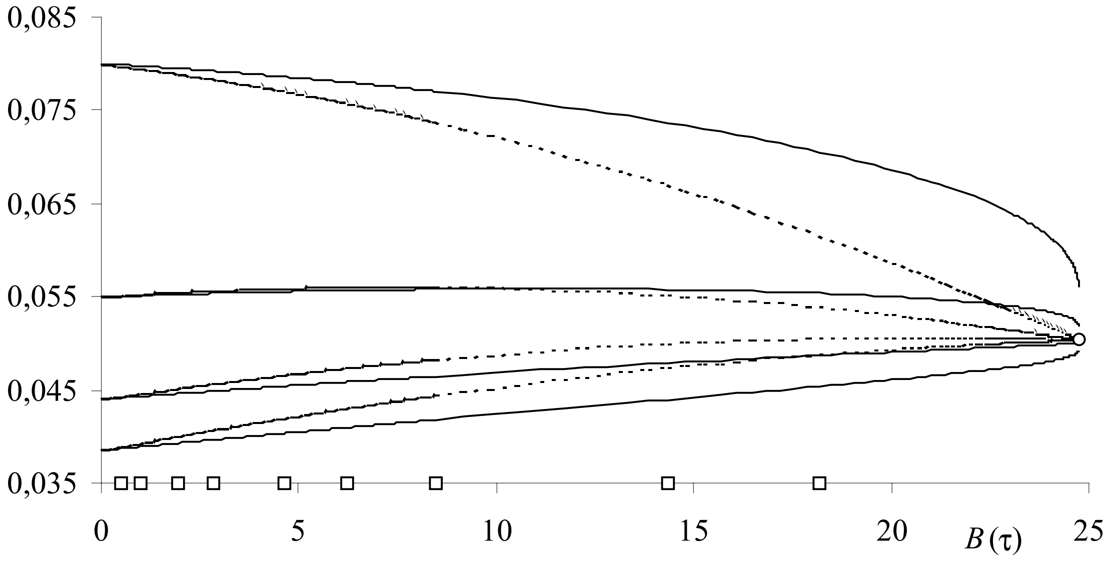

Figure 4. The yield curves $Y(B)$ (solid line) and the forward curves $F(B)$ (dashed line) for the Bali (1999) parameters of models for different values of spot rate $r = 0,0385$; $0,044$; $0,055$; $0,08$; and barrier $x = 0$ (CIR model case). The limiting point of curves is shown by circle. The markers on axis B mean terms to maturity $\tau = 0,5$; 1; 2; 3; 5; 7; 10; 20; 30 years.

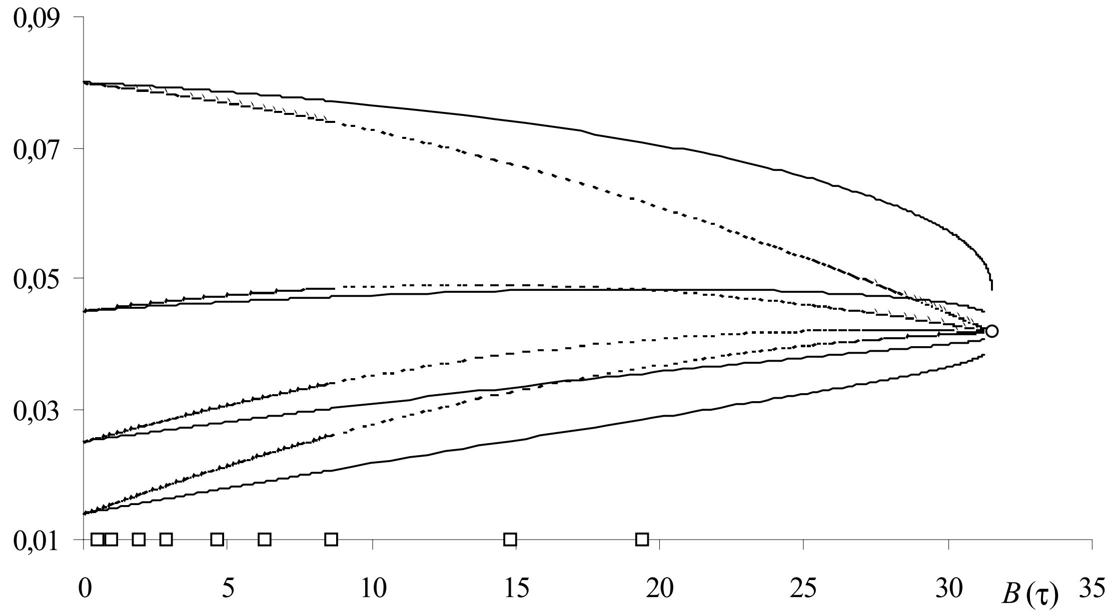

Figure 5. The yield curves $Y(B)$ (solid line) and the forward curves $F(B)$ (dashed line) for the Bali (1999) parameters of models for different values of spot rate $r = 0,014$; $0,025$; $0,045$; $0,08$; and barrier $x = -1000$ (quasi Vasiček model case). The limiting point of curves is shown by circle. The markers on axis B mean terms to maturity
$\tau = 0,5$; 1; 2; 3; 5; 7; 10; 20; 30 years.

Figures 2 – 5 confirm the data of Table 1 for the empirical results. One can see that the yield curves monotonic decrease if the spot rate is rather high (inequality (33) is fulfilled) and the yield curves monotonic increase if the spot rate is rather low (inequalities (35) and (36) are fulfilled). The forward curves tend to its limiting values as $\tau \to \infty$ faster than the yield curves. Therefore the yield curves are situated higher than the forward curves if the spot rate is rather high (inequality (33) is fulfilled) and the yield curves are situated lower than the forward curves if the spot rate is rather low (inequalities (35) and (36) are fulfilled). The yield curves and the forward curves can intersect for intermediate values of spot rates (inequality (34) is fulfilled).

Figure 6 shows an influence on the yield curves and the forward curves of parameter x – the lower boundary of the spot rate process $r(t)$. In this case the GM model is considered with parameters k = 0,2339, θ = 0,0808, D = 0,00126, the same as CKLS (1992) (see Table 2), $\lambda = 0$. The graphs of the yield curves and the forward curves are presented for r = 0,06 and several values of term to maturity τ = 0; 2; 10; 20 years. From these graphs one can see a week dependence the yield rate and the forward rate on x. The graphs are flat except for a small interval of values of parameter x near limiting values $x = \theta - \sqrt{D}$ = 0,045. Such behavior of the yield curves and the forward curves via parameter x confirm the Property 9. This means too that for used set of parameters such models as the CIR model and the Vasiček model are practically equivalent.

Figure 7 shows an influence on the yield curves and the forward curves of parameter λ – a parameter that determines the risk premium value. In this case also as in previous case the GM model is considered with parameters k = 0,2339, θ = 0,0808, D = 0,00126, the same as CKLS (1992) (see Table 2), the short rate r = 0,06, the term to maturity τ = 2 years, and the parameter x takes several values, x = 0; 0,03; 0,05. As it was maintained in the Property 10 with increasing of parameter λ the yield curve $Y(B)$ and the forward curve $F(B)$ monotonic decrease.

Conclusion

Thus in this paper an attempt of detailed description of every possible shapes of term structure is made for the class of the one-factor affine interest rate models for that the solutions in closed form are attainable. As basis the general model (GM) with arbitrary lower boundary for short interest rate is taken. It was formulated ten Properties of the yield curves and the forward curves. The results for well-known models the CIR model and the Vasiček model are obtained as particular cases. In particular it is found that there is four modes for yield curve shapes. The empirical evidences are presents that are based on the parameter estimates for 13 different models of real time series of yield interest rates. The numerical results are presented in form of tables and graphical charts.

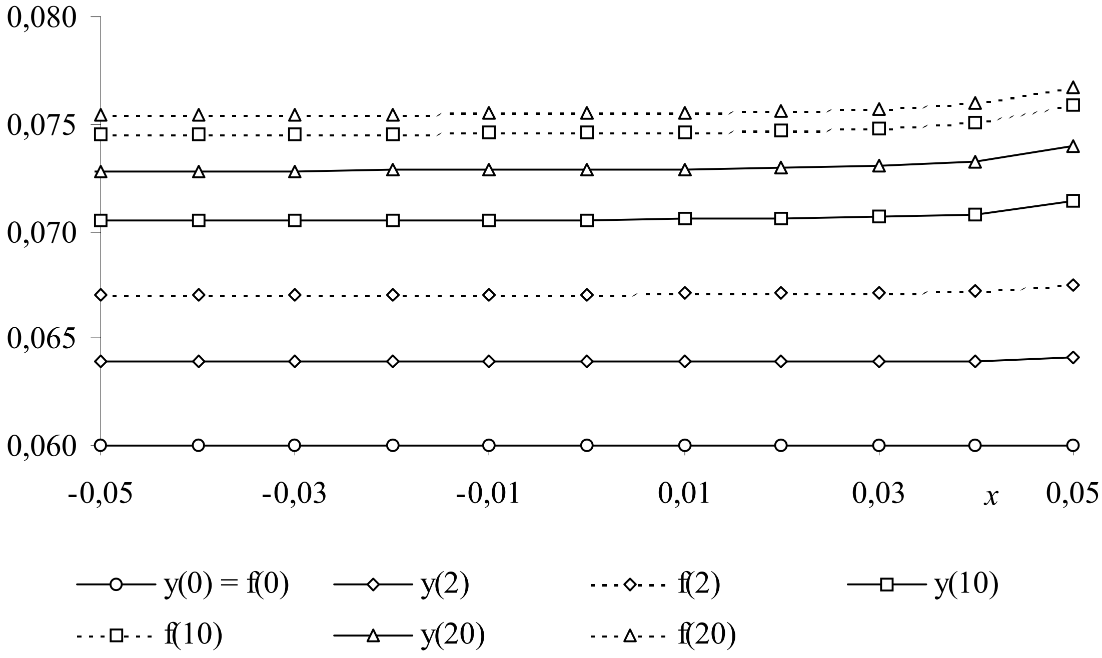

Figure 6. The yield curves (solid line) and the forward curves (dashed line) for the CKLS (1992) parameters of models for the spot rate $r = 0,06$ as functions of parameter x. The designations $y(\tau)$ and $f(\tau)$ are pointing to the curves for different terms to maturity $\tau = 0; 2; 10; 20$ years.

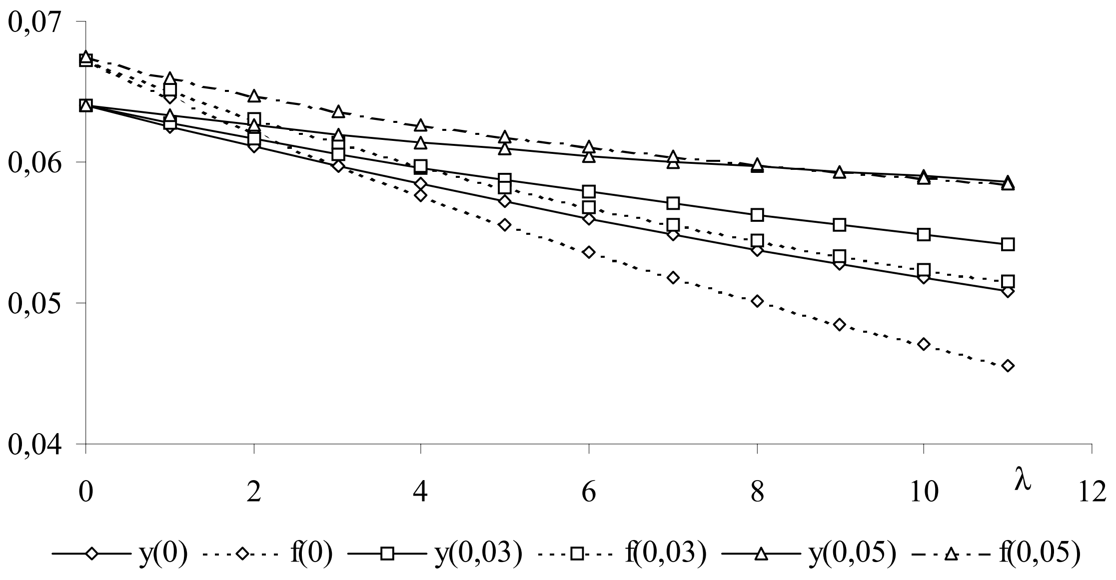

Figure 7. The yield curves (solid line) and the forward curves (dashed line) for the CKLS (1992) parameters of models as functions of parameter λ for the spot rate $r = 0,06$ and the term to maturity $\tau = 2$. The designations $y(x)$ and $f(x)$ are pointing to the curves for different barriers $x = 0; 0,03; 0,05$.

APPENDIX

Proof of Property 1. The equation (20) follows from (16) and (19) if take in account that $B(\tau) \to 0$ and $B(\tau)/\tau \to 1$ follow from (18) as $\tau \to 0$.

The equation (21) for $y(\tau)$ takes place because $B(\tau)/\tau \to 0$ follows from (17) as $\tau \to \infty$. Under proof (21) for the forward rate it is convenient to present (19) according to (13) in the form

$$f(\tau) = r\left(1 - \left(k + \lambda\frac{\sqrt{2kD}}{\theta - x}\right)B(\tau) - \frac{kD}{\theta - x}[B(\tau)]^2\right) +$$

$$+\left(k\theta + \lambda x\frac{\sqrt{2kD}}{\theta - x}\right)B(\tau) + \frac{kxD}{\theta - x}[B(\tau)]^2 = r\frac{dB(\tau)}{d\tau} - \frac{dA(\tau)}{d\tau}.$$

It follows from properties of functions $B(\tau)$ determined by formula (8) that the derivative $dB/d\tau \to 0$ as $\tau \to \infty$. Then from equation (6) one can see that the multiplier under r in relation for $f(\tau)$ tends to zero as $\tau \to \infty$, that is as $\tau \to \infty$

$$f(\tau) \to \left. -\frac{dA(\tau)}{d\tau}\right|_{\tau=\infty}.$$

To determine $dA/d\tau$ one can use the relation (9). This result in to an expression

$$-\frac{dA(\tau)}{d\tau} = x + \frac{(\theta - x)^2}{D}v - \frac{dB(\tau)}{d\tau}\left[x + \frac{(\theta - x)^2}{D}\frac{v}{2 + vB(\tau)}\right].$$

Last term tends to zero as $\tau \to \infty$ because in this case $dB/d\tau \to 0$ and the function $B(\tau)$ remains limited. This establishes the remaining equality in (21).

Proof of Property 2. The form of the term structure function $B(\tau)$ in neighborhood of value $\tau = 0$ is determined by the relation (18). The expressions (22) and (23) one can obtain if to use in the representations (16) and (19) of functions $y(\tau)$ and $f(\tau)$ the relation (18) and the denotation (10).

Proof of Property 3. Let us show that $f^*(x)$ monotonic increases. For short designate $\vartheta \equiv \theta - x$, $\omega \equiv 2\lambda kD$ and

$$G(\vartheta) \equiv 2D + \omega + k\vartheta - \sqrt{(\omega + k\vartheta)^2 + 4kD\vartheta}.$$

Then according to (10) and (21) function $f^*(x)$ can be expressed in the form

$$f^*(x) \equiv f^*(\theta - \vartheta) = \theta - \vartheta\ G(\vartheta)/2D.$$

The derivative of this function

$$\frac{df^*(x)}{dx} = -\frac{df^*(\theta-\vartheta)}{d\vartheta} = \frac{4D(D+\omega)-(2D+\omega)G(\vartheta)}{2D\sqrt{(\omega+k\vartheta)^2+4kD\vartheta}}$$

is positive for every $D > 0$, $k > 0$ and $\omega \geq 0$ if is held the inequality

$$4D(2D + \omega) > (2D + \omega)\ G(\vartheta),$$

which in its turn is equivalent to inequality that easily to verify

$$(2D + \omega)\sqrt{(\omega+k\vartheta)^2+4kD\vartheta} > \omega^2 + (2D + \omega)\ k\vartheta.$$

Thus the function $f^*(x)$ monotonic increases as x increases. The right-hand equality in (24) is verified by simple substitution $x = \theta$ to (21). The left-hand inequality in (24) one can obtain if to compute the limit $\lim_{x\to-\infty} f^*(x)$. For the proof the expression (25) it is necessary to solve the transcendental equation

$$\theta - v\ G(\vartheta)/2D = 0.$$

The analysis shows that this equation may be transformed to quadratic

$$(D - k\theta + \omega)\vartheta^2 - \theta\ (2D + \omega)\vartheta + \theta^2 D = 0$$

and required solution $\vartheta^* = \theta - x^*$ is lesser root of this quadratic equation that is determined by expression (25).

Proof of Property 4. The first necessary condition (26) will be held according to Property 3. Now it remains to clear up for that x the inequalities (27) are held. For this one considers the representations (22) and (23). In order to the functions $y(\tau)$ and $f(\tau)$ increase in the neighborhood of value $\tau = 0$ it is necessary to fulfill the demand: the coefficients under τ in (22) and (23) for $r = 0$ must be positive. It turned out that for both representations this demand reduces to the same inequality

$$\theta\ [k + 2\lambda kD/(\theta - x)] - 2\lambda kD > 0$$

that is

$$k\theta - 2\lambda kD + \frac{2\lambda kD}{\theta - x} > 0. \tag{A.1}$$

When $k\theta \geq 2\lambda kD$ this inequality is held for every $x < \theta$. Under this in the case $k\theta \geq D + 2\lambda kD$ the first necessary condition (26) is held. However in the case $2\lambda kD \leq k\theta \leq D + 2\lambda kD$ the first necessary condition (26) is broken. Therefore in according to Property 3 the condition of the mutual fulfillment of inequalities (26) and (27) for this case will be inequality $x \geq x^*$. Finally in the case $k\theta < 2\lambda kD$ the inequality (A.1) is held for

$$x > x^{**} \equiv -\frac{\theta^2}{2\lambda D - \theta}.$$

Now for the proof of Property it is sufficient to show that $x^* \geq x^{**}$, that is

$$\frac{\theta}{2\lambda D - \theta} \geq \frac{k\theta + \sqrt{k\theta D + (\lambda kD)^2} - \lambda kD}{D - k\theta + 2\lambda kD}.$$

This inequality is held if

$$\sqrt{k\theta D + (\lambda kD)^2} \leq \frac{\theta D}{2\lambda D - \theta} + \lambda kD.$$

For this it is necessary that $k\theta \leq D + 2\lambda kD$. However this inequality just is the condition of determination of value x^* by the expression (25) and the necessary condition (26) is held.

Proof of Property 5.

The first statement of Property 5 follows from the fact that the second derivative of forward curve $F(B)$ (see the presentation (30)) is always negative. This is a sufficient condition of the concavity.

The necessary condition of existence of maximum $F'(B^*) = 0$ gives the formula (32). However the maximum will exist in fact if only $B^* \in (0, V^{-1})$, that is

$$0 \leq B^* = \frac{1}{2vV}\left(k\frac{\theta - x}{r - x} - V + v\right) \leq V^{-1}.$$

This results in to the inequality (31).

Proof of Property 6. For this let us introduce an intermediary function – secant $c(B)$:

$$c(B) = r + [k(\theta - x) - V(r - x)]B, \quad 0 \leq B \leq V^{-1}. \qquad \text{(A.2)}$$

The secant $\{c(B), 0 \le B \le V^{-1}\}$ is a straight-line segment that connects the point (0, r) (the initial point of the curves $Y(B)$ and $F(B)$ as $B = 0$) with the point $(V^{-1}, x + v(\theta - x)^2/D)$ (the limiting point of curves $Y(B)$ and $F(B)$ as $B = V^{-1}$).

In according to Property 5 the forward curve is the concave function therefore the inequality $F(B) > c(B)$ for every $B \in (0, V^{-1})$.

The shape of the yield curve $Y(B)$ (see the presentation (29)) essentially depends on the value of parameter r.

If the yield curve $Y(B)$ is the concave function on the interval $0 < B < V^{-1}$ then at every point of this interval the inequality $Y(B) > c(B)$ must be fulfilled and on the ends of interval the equalities $Y(0) = c(0)$ and $Y(V^{-1}) = c(V^{-1})$ are held. In order to these conditions be fulfilled it is necessary to be valid the inequalities

$$\left.\frac{dY(B)}{dB}\right|_{B=0} > \left.\frac{dc(B)}{dB}\right|_{B=0}, \quad \left.\frac{dY(B)}{dB}\right|_{B\uparrow V^{-1}} < \left.\frac{dc(B)}{dB}\right|_{B=V^{-1}}. \tag{A.3}$$

Here the functions $Y(B)$ and $c(B)$ are determined by expressions (29) and (A.2) respectively. Computing the derivatives and demanding the fulfillment of inequalities (A.3) give the conditions on values of parameter r stated in Property 6.

The explicit form of first inequality (A.3) is found rather simple. Representing the expression (29) with accuracy $O(B^2)$ for small B we obtain that

$$Y(B) = r + \tfrac{1}{2}\,[k(\theta - x) - (V - v)(r - x)]B + O(B^2). \tag{A.4}$$

From this and from (A.2) one finds as $B \to 0$ that the first inequality (A.3) reduces to the form

$$k(\theta - x) - (V - v)(r - x) > 2[x - r + v(\theta - x)^2 / D]\, V$$

and this is equivalent to inequality

$$r > x + k(\theta - x)/(v + V), \tag{A.5}$$

if to use the property $vV = kD/(\theta - x)$.

It is more complicated to obtain the second inequality in (A.3) in explicit form because in the point $B = V^{-1}$ the derivative of the yield curve $Y(B)$ is unbounded in absolute value. Therefore for proof of Property it is sufficient to know only the sign of derivative but not its value. The derivative of expression (29) with respect to B has a form

$$\frac{dY(B)}{dB} = \varepsilon \frac{r - x - \frac{(\theta - x)^2}{D}\frac{v}{1 + vB}}{[(\ln(1 + vB) - \ln(1 - VB)]} -$$

$$-\varepsilon^2 \frac{(r - x)B - (\theta - x)^2 \ln(1 + vB)/D}{[\ln(1 + vB) - \ln(1 - VB)]^2 (1 - VB)(1 + vB)}. \tag{A.6}$$

Note that as $B \to V^{-1}$

$$[\ln(1 + vB) - \ln(1 - VB)] \to \infty, \quad [\ln(1 + vB) - \ln(1 - VB)]^2 (1 - VB) \to 0.$$

Therefore the first term in (A.6) will tend to zero and the second term will increase unbounded in absolute value and will be negative or positive in dependence on a sign of the numerator of second term of equation (A.6) at the point $B = V^{-1}$

$$(r - x)/V - (\theta - x)^2 \ln(1 + v/V)/D.$$

If the function $Y(B)$ is concave then this expression must be positive that results in to inequality

$$r > x + V(\theta - x)^2 \ln(1 + v/V)\big/D. \tag{A.7}$$

Thus if the yield curve $Y(B)$ is concave then it is necessary simultaneously to fulfill the inequalities (A.5) and (A.7). That is

$$\frac{r - x}{\theta - x} > \frac{k}{v + V}, \quad \frac{r - x}{\theta - x} > \frac{k}{v}\ln\left(1 + \frac{v}{V}\right). \tag{A.8}$$

Note that for every $z \in (0, 1)$ the inequality

$$\ln(1 + z) - \frac{z}{1 + z} = \sum_{k=2}^{\infty} \frac{k - 1}{k}(-z)^k > 0$$

is valid. Assuming $z = v/V$ in this case we obtain that

$$\frac{k}{v}\ln\left(1 + \frac{v}{V}\right) > \frac{k}{v + V}. \tag{A.9}$$

It means that two inequalities (A.8) reduce only to one (second) inequality and first statement of Property 6 is proved.

If the yield curve $Y(B)$ is the convex function on the interval $0 < B < V^{-1}$ then at every point of this interval the inequality $Y(B) < c(B)$ must be fulfilled and on the ends of interval the equalities $Y(0) = c(0)$ and $Y(V^{-1}) = c(V^{-1})$ are held. It means that both of the inequalities (A.3) must be fulfilled in reverse side that is both of the inequalities (A.8) also must be fulfilled in reverse side. In this case two inverse inequalities (A.8) reduce only to one (first) inverse inequality and second statement of Property 6 is proved.

At last if first inequality in (A.3) (or equivalent (A.8)) is fulfilled and second inequality is inverse then the curve $Y(B)$ and the secant $c(B)$ intersect into interval $0 < B < V^{-1}$ and the yield curve $Y(B)$ have a point of inflexion B_i. Then the yield curve $Y(B)$ is the concave function on the interval $0 < B < B_i$ because the yield curve intersects the secant top-down. And the yield curve $Y(B)$ is the convex function on the interval $B_i < B < V^{-1}$. This proves third statement of Property 6.

Proof of Property 7. If the yield curve $Y(B)$ is the concave function of B on interval $0 < B < V^{-1}$ then it can have a maximum on this interval. From Property 6 we have that the yield curve $Y(B)$ is a concave function on interval $0 \le B \le V^{-1}$ if the value of short rate r meets the inequality

$$\frac{r-x}{\theta-x} \ge \frac{k}{v}\ln\left(1+\frac{v}{V}\right).$$

Thus function $Y(B)$ has a maximum if the derivative $Y'(B) > 0$ at the point $B = 0$ and the derivative $Y'(B) < 0$ at point $B = V^{-1}$. From Property 6 the second condition in this case is held. From (A.4) the first condition will be held if

$$k(\theta - x) - (V - v)(r - x) > 0, \text{ that is } \frac{r-x}{\theta-x} < \frac{k}{V-v}.$$

Note that for every $z \in (0, 1)$ the inequality

$$\frac{z}{1-z} - \ln(1+z) = \sum_{k=1}^{\infty}\left(\frac{2k+1}{2k}z^{2k} + \frac{2k}{2k+1}z^{2k+1}\right) > 0$$

is valid. From this assuming $z = v/V$ in this case we obtain that

$$\frac{k}{V-v} > \frac{k}{v}\ln\left(1+\frac{v}{V}\right).$$

Therefore there are really the points B of existence of maximum $Y(B)$ and the yield curve $Y(B)$ has the maximum if r meets the inequalities

$$\frac{k}{v}\ln\left(1+\frac{v}{V}\right)<\frac{r-x}{\theta-x}<\frac{k}{V-v}. \qquad \text{(A.10)}$$

This proves the first statement of Property 7.

On the other hand in order to the function $Y(B)$ has a maximum at some point B_0 it is necessary that $Y'(B_0) = 0$. Note that in this case $B_0 = B(\tau_0)$ where τ_0 meets an equation $y'(\tau_0) = 0$. From (11) and (13) we have for $\tau = \tau_0$

$$y'(\tau) = -\frac{rB(\tau) - A(\tau)}{\tau^2} + \frac{1}{\tau}[rB'(\tau) - A'(\tau)] = -\frac{1}{\tau}\,y(\tau) + \frac{1}{\tau}f(\tau) = 0,$$

that is $y(\tau_0) = f(\tau_0)$. However this is equivalent to equality $Y(B_0) = F(B_0)$. So the maximum of the yield curve $Y(B)$ is reached at the point B_0 of intersection of curves $Y(B)$ and $F(B)$. This proves the second statement of Property 7.

Because from Property 5 and (A.9) the forward curve $F(B)$ have a maximum at point B^* on interval (A.10) and also the functions $Y(B)$ and $F(B)$ are convex and from Property 2 $F'(0) > Y'(0)$ then it is necessary that the intersection will be on ascending branch of forward curve, that is $B^* < B_0$. This proves the last third statement of Property 7.

Proof of Property 8. The GM model changes into the Vasiček model when the parameter $x \to -\infty$. In this case the probability density functions of gamma distribution (37) and (38) change into the density function of normal distribution (see Ilieva, 2000). Because the bound between the remaining modes is equal 1 and the expectation of the random variable $\zeta = (r - x)/(\theta - x)$ is equal 1 too then in view of symmetry of normal distribution

$$\text{Prob[mode A]} = \text{Prob}[\zeta > 1] = \text{Prob[mode D]} = \text{Prob}[\zeta < 1] = 0{,}5.$$

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