An optimization approach to the dynamic allocation of economic capital

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Motivation (1)

- Artzner (1999, NAAJ):
  Tail conditional expectation,

$$\mathrm{TCE}_\alpha(X) = \mathbb{E}[X|X > \mathrm{VaR}_\alpha(X)]$$

- Overbeck (2000, WP), Panjer (2001, WP):
  Proportional tail conditional expectation,

$$\mathrm{PTCE}_\alpha(X_i) = \mathbb{E}[X_i|X > \mathrm{VaR}_\alpha(X)]$$

Motivation (2)

- Four problems:
  1. Given $\mathrm{TCE}_\alpha(X)$ the allocation to the subsidiaries can be performed better than by $\mathrm{PTCE}_\alpha(X)$.
  2. In case the dependence structure is unknown, $\mathrm{PTCE}_\alpha(X)$ cannot be computed in general.
  3. $\mathrm{TCE}_\alpha(X)$ will typically not be held physically.
  4. Imposing bounds on the risk realization may provide unjustified incentives to split-up conglomerates.

Outline

- Optimal allocation for an exogenously given amount of capital
- Optimal amount of economic capital
- Allocating the cost of risk-bearing
- Dynamic extension
- Application

The capital allocation problem for an exogenously given amount of capital (1)

- Notation:
  $X = \sum_{i=1}^{n} X_i$,
  u,
  $\pi(\cdot)$,
  $\rho(X_i) = u_i$
- Two inequalities:
  $(\sum_{i=1}^{n} X_i - u)_+ \leq_1 \sum_{i=1}^{n}(X_i - u_i)_+, \qquad u = \sum_{i=1}^{n} u_i$
  $\pi((\sum_{i=1}^{n} X_i - u)_+) \leq \pi(\sum_{i=1}^{n}(X_i - u_i)_+), \quad u = \sum_{i=1}^{n} u_i$

The capital allocation problem for an exogenously given amount of capital (2)

- Risk residual: $(X_i - u_i)_+$
- Optimization problem:
  $\min_{\rho(\cdot)} \pi(\sum_{i=1}^{n}(X_i - \rho(X_i))_+), \text{ s.t. } \sum_{i=1}^{n} \rho(X_i) = u$
- Coexisting problem:
  $\max_{(X_1,\ldots,X_n)\in\Gamma} \pi((X_1 + \ldots + X_n - u)_+)$
- For $\pi(\cdot) = \mathbb{E}[\cdot]$ and strictly increasing d.f.'s:
  1. $u_i^* = F_{X_i}^{-1}(F_{X^c}(u))$, in which $X^c = X_1^c + \ldots + X_n^c$
  2. $(X_1^*, \ldots, X_n^*) = (X_1^c, \ldots, X_n^c)$

Bounds on the risk realization

- $(\sum_{i=1}^{n} X_i - u)_+ \leq_1 \sum_{i=1}^{n}(X_i - u_i)_+, \quad u = \sum_{i=1}^{n} u_i$
- Upper bounds M, M_i
- $Y = \min((X - u)_+, M - u)$,
  $Y_i = \min((X_i - u_i)_+, M_i - u_i)$
- Even if $M = M_1 + \ldots + M_n$,
  not in general $Y \leq_1 Y_1 + \ldots + Y_n$
- Many counterexamples in which
  $\mathbb{E}[Y] \geq \mathbb{E}[Y_1 + \ldots + Y_n], M = \mathrm{CTE}_\alpha(X), M_i = \mathrm{PCTE}_\alpha(X_i)$

The optimal amount of economic capital

- Optimization problem:
  $\min_u \pi_v((X-u)_+) + (r_c - r_f)u$
- Which valuation measure?
- Distortion risk measure:
  $\pi_g(X) = \int_0^\infty g(1-F_X(x))dx$ for non-negative r.v.
- $u^* = F_X^{-1}(1-g^{-1}(r_c - r_f)) = F_X^{*\,-1}(1-(r_c - r_f))$

Allocating the cost of risk-bearing (1)

- $\pi_v((X - u^*)_+) + (r_c - r_f)u^*$
  $\pi_v((X_i - u_i^*)_+) + (r_c - r_f)u_i^*$
- $\frac{u_i^*}{u^*}(\pi_v((X - u^*)_+) + (r_c - r_f)u^*)$
- $\frac{\pi_v((X-u^*)_+)}{\sum_{i=1}^{n} \pi_v((X_i-u_i^*)_+)}\pi_v((X_i - u_i^*)_+) + (r_c - r_f)u_i^*$
- $\pi_v(\cdot)$: Distortion risk measure
  $u^* = F_X^{-1}(1 - g^{-1}(r_c - r_f))$

Allocating the cost of risk-bearing (2)

- $\pi(\cdot)$: Expectation operator
  $u_i^* = F_{X_i}^{-1}(F_{X^c}(u^*))$
- Total cost of risk-bearing charged to subsidiary i:
  $\gamma\pi_v((X_i - u_i^*)_+) + (r_c - r_f + \kappa)u_i^*$

Allocating the cost of risk-bearing (3)

- Definitions γ and κ:

$$\gamma = \begin{cases} \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)}, & \text{if } \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))} \geq \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)} \\ \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))}, & \text{if } \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))} < \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)} \end{cases}$$

$$\kappa = \begin{cases} 0, \\ (\pi_v((X-u^*)_+) - \gamma\pi_v((X^c-u^*)_+))\frac{1}{u^*}, \end{cases}$$

- Monotonicity of the allocation of the cost of risk-bearing: $X_1 \leq_1 X_2 \Rightarrow \ \gamma\pi_v((X_1-u_1^*)_+) + (r_c - r_f + \kappa)u_1^* \leq \gamma\pi_v((X_2-u_2^*)_+) + (r_c - r_f + \kappa)u_2^*$

Allocating the cost of risk-bearing (4)

Proof monotonicity of the allocation of the cost of risk-bearing:

- $X_1 \leq_1 X_2 \Rightarrow$
  $u_1^* = F_{X_1}^{-1}(F_{X^c}(u^*)) \leq F_{X_2}^{-1}(F_{X^c}(u^*)) = u_2^*$
- Inequalities:
  $\gamma\pi_v((X_1 - u_1^*)_+) + (r_c - r_f + \kappa)u_1^* \leq$
  $\gamma g(1 - F_{X_1}(u_1^*))(u_2^* - u_1^*) + \gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_1^* \leq$
  $(r_c - r_f)(u_2^* - u_1^*) + \gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_1^* \leq$
  $\gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_2^*$
- The second inequality holds true since
  $\gamma g(1 - F_{X_1}(u_1^*)) = \gamma g(1 - F_{X^c}(u^*)) \leq$
  $g(1 - F_X(u^*)) = r_c - r_f$

Dynamic extension

- Notation:
  $T < \infty$,
  $\mathbf{X} = (X_1, \ldots, X_n)$,
  $\mathbb{F} = (\mathcal{F}_t : 0 \le t \le T), \quad \mathcal{F}_t \subset \mathcal{F}_T = \mathcal{F}$
- $\pi(\cdot|A)$, $A \in \mathcal{F}$

Application (1)

- $d\mathbf{X}_t = \mu dt + \boldsymbol{\Sigma} d\mathbf{W}_t,\ \mathbf{X}_0 = \mathbf{0}$
- $\sum_{i=1}^{n} dX_{t,i} = \tilde{\mu} dt + \tilde{\Sigma} d\tilde{W}_t,\ X_0 = 0$
- $X_{T,i} | \mathcal{F}_t \sim N(x_{t,i} + (T-t)\mu_i, (T-t)\sum_{j=1}^{m} \Sigma_{i,j}^2)$
- $(\sum_{i=1}^{n} X_{T,i}) | \mathcal{F}_t \sim N(x_t + (T-t)\tilde{\mu}, (T-t)\tilde{\Sigma}^2)$

Application (2)

- $\pi(\cdot)$: Expectation operator
  $\pi_v(\cdot)$: Distortion risk measure
- $u_t^* = F_{X_T|X_t}^{-1}(1 - g^{-1}(r_c - r_f)),\ t \in [0, T]$
  $u_{t,i}^* = F_{X_{T,i}|X_{t,i}}(F_{X_T^c|X_t}(u_t^*)),\ t \in [0, T],\ i = 1, \ldots, n$

Application (3)

- T=3
- $n = 4$
- 4-dimensional Wiener process:

$$\mu = \begin{pmatrix} -0.4 \\ 0.5 \\ 0.9 \\ -1.4 \end{pmatrix}, \ \mathbf{\Sigma} = \begin{pmatrix} 0.5 & 0.5 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.5 \\ 0 & 0.75 & 0.25 & -0.5 \\ 0.5 & 0 & -0.5 & 0.5 \end{pmatrix}$$

- $r_c = 0.07$, $r_f = 0.04$
- $g(s) = s^{1/1.25}$

Figure 1: Sample paths of the multivariate Wiener process

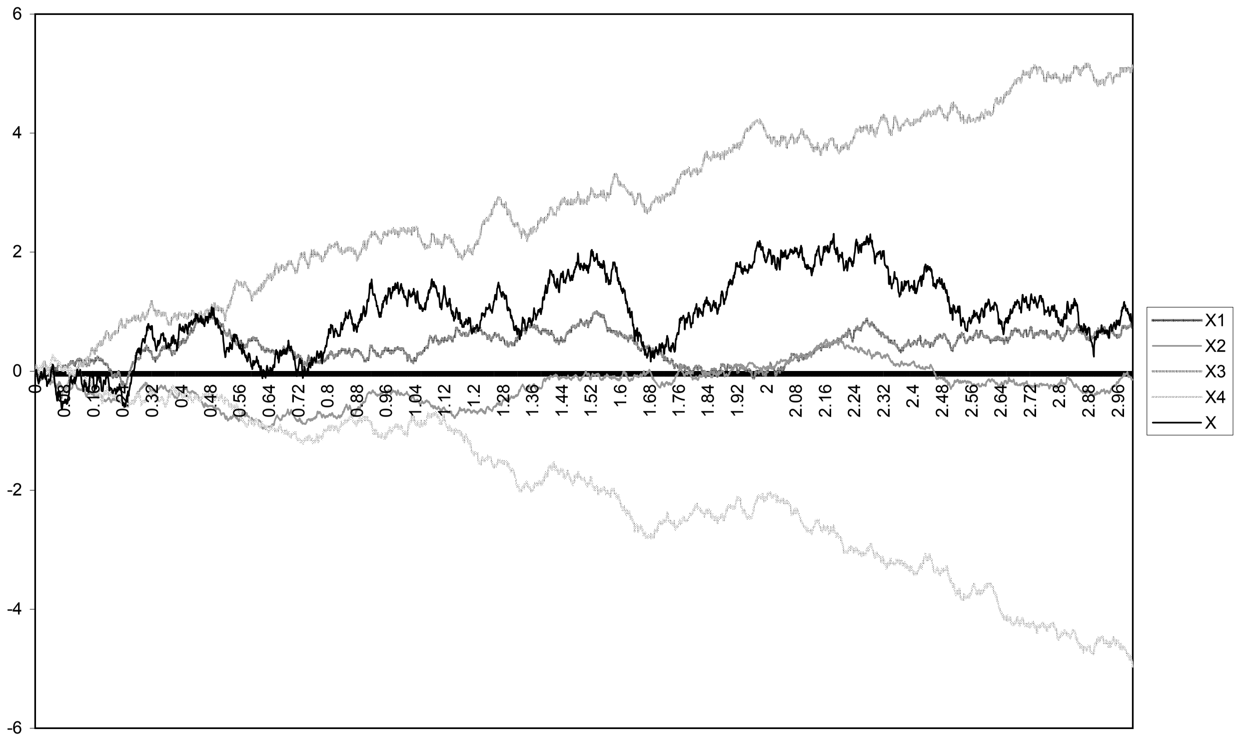

Figure 2: Dynamic allocation among the subsidiaries with $\pi_v(\cdot) = H_{g_{\text{ph}}}(\cdot)$, a=1.25, $r_c = 0.07$, $r_f = 0.04$ and $\pi(\cdot) = \mathbb{E}[\cdot]$