ON OPTIONED PORTFOLIO SELECTION UNDER OPTION STRATEGIES

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Abstract.

A contribution to the notorious unsolved optioned portfolio selection problem is made. We propose a class of fixed costs option strategies, for which optioned portfolio selection under various criteria could be implemented for practical purposes. Instead of mean-variance portfolio selection, we propose mean-CVaR portfolio selection, where CVaR is the conditional value-at-risk measure considered in the recent risk management literature.

Keywords : portfolio selection, option strategies, shortfall risk, conditional value-at-risk

1. Introduction.

Mean-variance portfolio selection, pioneered by Markowitz(1952/59/87/94), is one of the cornerstones of modern portfolio theory. An attempt to extend the method to portfolio selection under portfolio insurance has been proposed by the author(1996). This latter method requires investing either in a long put (holding simultaneously fund) or in a long call (holding simultaneously cash).

In general, however, optioned portfolios can be built using divers option strategies consisting of arbitrary linear combinations of long and short positions of puts and calls. Since the resulting portfolio return distribution of an optioned portfolio may be rather asymmetric and difficult to calculate explicitly (e.g. Bookstaber and Clarke(1983)), mean-variance analysis is usually not recommended in this situation (e.g. Scheuenstuhl and Zagst(1996)).

The construction of a general optioned portfolio selection method is among the notorious unsolved risk management problems both from a theoretical and practical point of view. Despite the many approaches to this problem, e.g. Bookstaber and Clarke(1984/85), Lee(1993), Albrecht, Maurer and Timpel(1995), Adam, Maurer and Möller(1996), Dert and Oldenkamp(1996), Scheuenstuhl and Zagst(1996), Janssen and Saib(1998), Adam and Maurer(2000), no satisfactory solution has been proposed, which has a universal potential for finance practice (like mean-variance portfolio selection).

In the present study, we propose a class of *fixed costs structured option strategies*, for which optioned portfolio selection under various criteria could be implemented for practical use. Instead of mean-variance portfolio selection, we propose mean-CVaR portfolio selection, where CVaR (=conditional value-at-risk) is a precise "coherent" measure of risk considered in the recent risk management literature.

2. A special class of option strategies.

Consider a market index or single stock with price process $S_t, t \in [0,T]$, T the planning horizon. Suppose an investor has bought m shares of stock at time $t = 0$, his investment amount being $m \cdot S_0$. The investor wishes to have at least (sometimes exactly) $n > m$ shares of the same stock at time *T*. For this, the investor can buy directly $n - m$ shares at any time $t \in [0,T]$. He is faced with the decision of choosing the "optimal" time of buying the shares at a possibly low price. To avoid this difficult decision, an investor can alternatively invest in options on the stock to achieve the same goal at time *T*. Several questions can be asked :

(Q1) When is an option strategy preferable to the direct-buy-stock strategy ?
(Q2) Does there exist a best option strategy to achieve the above goal ?
(Q3) Given a "simple" class of option strategies, is there a best option strategy to achieve the above goal ?

To answer the difficult questions (Q1) and (Q2), any sound scientific approach will require first satisfactory solutions to the more tractable question (Q3), which includes the construction of "simple" classes of option strategies, which are appropriate for direct investigation.

In order that our option strategies are practically feasible, we assume that the capital market is *liquid*, in particular every option position on the given stock at some standardized exercise prices $d_1,...,d_s$ is always available in the time interval $[0,T]$. Constrained by common investment management *prescriptions*, selling short puts may require that some cash is deposited, borrowed or otherwise secured, and selling short calls may require deposit of shares, cash borrowing, or any other security (e.g. Jarrow and Rudd(1983), Chap. 3). Furthermore, investment banks usually require a minimum volume on investment to trade short positions, which implies that our option strategies will be feasible for financial institutions but in general not for the ordinary private investor. Without loss of generality, we assume that all options are of *American type*, and may be exercised at any time up to the maturity date *T*.

Consider a relatively simple option strategy consisting of the following investment in options (all with the same maturity date *T*) :

p short puts with exercise price d_i, each at the option price $SP(i)$
q short calls with exercise price d_j, each at the option price $SC(j)$
n long calls with exercise price d_k, each at the option price $LC(k)$
$m-q$ long puts with exercise price d_r, each at the option price $LP(r)$

Intuitively, it is natural to assume that $d_i \le d_r$ (reduce costs when short puts are exercised) and $d_k \le d_j$ (reduce costs when short calls are exercised), and $d_i \le d_j$. The non-negative natural numbers *m*, *p*, *q*, *n* are assumed to satisfy the following constraints :

(1.1) $$m < n, \quad 1 \le p \le n, \quad 1 \le q \le m.$$

This choice is reasonable (but not compelling) for the following reasons. If $p > n$ and only the short puts are exercised, then the goal of n shares will be surpassed with a possible maximum loss of amount $(p-n)\cdot(d_i - SP(i))$, which is an unnecessary waste of capital, hence a speculative investment strategy. Choosing $q \le m$ allows one to deposit all of the available shares as mortgage without going into a theoretically unlimited loss (share prices may increase arbitrarily) of amount $(q-m)\cdot\left(\max(S_t-d_j,0)-SC(j)\right)$ in case $q>m$ and the call options are exercised at time t. The fact that one buys exactly n long calls is evident. If the m short calls are exercised, all n long calls must be exercised to get at least n shares.

Neglecting interest rate effects, due to the fact that these option positions may be settled at different times, the *investment value* (say net of transaction costs) of this option strategy equals :

(1.2) $$W(i,j,k,r)=p\cdot SP(i)+q\cdot SC(j)-n\cdot LC(k)-(m-q)\cdot LP(r).$$

At maturity date T, the options have been either exercised or not. Assuming for simplicity (this is not the case in practice) that for all short positions of a given type, exactly one of these two possibilities occurs, there are four main disjoint probabilistic events of importance (up to further sub-events depending on whether or not the investor exercises some or all long option positions). Neglecting dividend effects, it is natural to make the *assumption* that an American short call on a non-dividend paying stock is never exercised early (Restriction 5 in Jarrow and Rudd(1983), p.63). Then the relevant four *option events* are described as follows :

Event E(1) : short puts exercised, short calls not exercised

$$E(1)=\{\,S_t<d_i-SP(i)\ \textit{for some}\ t,\quad S_t<d_j+SC(j)\ \textit{for all}\ t\,\}$$

Event E(2) : short puts not exercised, short calls not exercised

$$E(2)=\{\,d_i-SP(i)\le S_t<d_j+SC(j)\ \textit{for all}\ t\,\}$$

Event E(3) : short puts not exercised, short calls exercised

$$E(3)=\{\,d_i-SP(i)\le S_t\ \textit{for all}\ t,\quad S_T\ge d_j+SC(j)\,\}$$

Event E(4) : short puts exercised, short calls exercised

$$E(4)=\{\,S_t<d_i-SP(i)\ \textit{for some}\ t,\quad S_T\ge d_j+SC(j)\,\}$$

Observe that the occurrence of the events $E(\omega),\ \omega\in\{1,\dots,4\}$ are known with certainty (that is probability one) only at time T. Therefore, early exercise of the long positions is usually not possible. If $E(4)$ occurs, earlier exercise may be possible as soon as it is known that the short positions have been exercised. According to which event actually occurs, it is now necessary to prescribe how many long positions will be exercised. Such a prescription determines the *investment costs* $K(\omega)$ and the *investment value* $V(\omega,T)$ *at maturity* date T of the optioned stock (or market index).

Event E(1)

Since the short puts have been exercised and $d_i\le d_r$, it is rational to exercise the $m-q$ long puts (because this action reduces the investment costs of the option strategy). The optioned stock consists of $m+p-(m-q)=p+q$ shares at maturity date T. Two sub-cases are possible.

Sub-case (11) : $p+q \le n$

To have exactly n shares at date T, one must exercise $n-(p+q)$ long calls, and there remains $p+q$ long call positions. One has

$$K(1) = p \cdot d_i - (m-q) \cdot d_r + [n-(p+q)] \cdot d_k - W(i,j,k,r)$$
$$V(1,T) = n \cdot S_T + (p+q) \cdot (S_T - d_k)_+$$

Sub-case (12) : $p+q > n$

There are more than the n required shares at maturity date T, and there remains n long call positions. One obtains

$$K(1) = p \cdot d_i - (m-q) \cdot d_r - W(i,j,k,r)$$
$$V(1,T) = (p+q) \cdot S_T + n \cdot (S_T - d_k)_+$$

Taken together, the formulas can be summarized as follows :

(1.3) $$K(1) = p \cdot d_i - (m-q) \cdot d_r + \max(n-(p+q),0) \cdot d_k - W(i,j,k,r)$$

(1.4) $$V(1,T) = \max(p+q,n) \cdot S_T + \min(p+q,n) \cdot (S_T - d_k)_+$$

Event E(2)

To have exactly n shares at maturity date T, one must exercise $n-m$ long calls, and there remains m long call and $m-q$ long put positions at maturity date T. One has

(1.5) $$K(2) = (n-m) \cdot d_k - W(i,j,k,r)$$

(1.6) $$V(2,T) = n \cdot S_T + m \cdot (S_T - d_k)_+ + (m-q) \cdot (d_r - S_T)_+$$

Event E(3)

To have exactly n shares at date T, one must exercise $n-(m-q)$ long calls, and there remains $m-q$ long call and $m-q$ long put positions. Therefore, one obtains

(1.7) $$K(3) = (n-m+q) \cdot d_k - q \cdot d_j - W(i,j,k,r)$$

(1.8) $$V(3,T) = n \cdot S_T + (m-q) \cdot [(S_T - d_k)_+ + (d_r - S_T)_+]$$

Event E(4)

Two sub-cases must be distinguished.

Sub-case (41) : $n-p \ge m-q \ge 0$

To have exactly n shares at date T, one must exercise $n-p-(m-q) \ge 0$ long calls, and there remains $m-q+p$ long call and $m-q$ long put positions. One has

(1.9) $$K(4) = p \cdot d_i - q \cdot d_j + [(n-p)-(m-q)] \cdot d_k - W(i,j,k,r)$$

(1.10) $$V(4,T) = n \cdot S_T + p \cdot (S_T - d_k)_+ + (m-q) \cdot [(S_T - d_k)_+ + (d_r - S_T)_+]$$

Sub-case (42) : $0 \le n-p < m-q$

To get n shares at date T, exercise $(m-q)-(n-p) > 0$ long puts. There remains n long call and $n-p$ long put positions. One has

(1.11) $$K(4) = p \cdot d_i - q \cdot d_j - [(m-q)-(n-p)] \cdot d_r - W(i,j,k,r)$$

(1.12) $$V(4,T) = n \cdot S_T + n \cdot (S_T - d_k)_+ + (n-p) \cdot (d_r - S_T)_+$$

Taken together, the formulas are summarized as follows :

(1.13) $$\begin{aligned} K(4) &= p \cdot d_i - q \cdot d_j + \max\{(n-p)-(m-q),0\} \cdot d_k \\ &- \max\{(m-q)-(n-p),0\} \cdot d_r - W(i,j,k,r) \end{aligned}$$

(1.14) $$\begin{aligned} V(4,T) &= n \cdot S_T + [n - \max\{(n-p)-(m-q),0\}] \cdot (S_T - d_k)_+ \\ &+ [(m-q) - \max\{(m-q)-(n-p),0\}] \cdot (d_r - S_T)_+ \end{aligned}$$

Remark 1.1.

If one allows $p > n$ (for the sole goal of a speculative investment strategy), there is by occurrence of event $E(4)$ a further sub-case :

Sub-case (43) : $n - p < 0 \le m - q$

Exercise the $m-q$ long puts to have at date T a number of $m+p-q-(m-q) = p > n$ shares. There remains n long call and $n-p$ long put positions. One has

(1.15) $$K(4) = p \cdot d_i - q \cdot d_j - (m-q) \cdot d_r - W(i,j,k,r)$$

(1.16) $$V(4,T) = p \cdot S_T + n \cdot (S_T - d_k)_+$$

The three sub-cases of event $E(4)$ can be further summarized to the formulas :

(1.17) $$\begin{aligned} K(4) &= p \cdot d_i - q \cdot d_j + \max\{(n-p)-(m-q),0\} \cdot d_k \\ &- \max\{(m-q) - \max(n-p,0),0\} \cdot d_r - W(i,j,k,r) \end{aligned}$$

(1.18) $$\begin{aligned} V(4,T) &= \max(n,p) \cdot S_T + [n - \max\{(n-p)-(m-q),0\}] \cdot (S_T - d_k)_+ \\ &+ [(m-q) - \max\{(m-q) - \max(n-p,0),0\}] \cdot (d_r - S_T)_+ \end{aligned}$$

It is interesting to ask whether there exists a subclass of the defined class of option strategies for which the investment costs $K(\omega)$ of the optioned stock are *constant* in each state of the world $\omega \in \{1,2,3,4\}$. These so-called *fixed costs option strategies* are characterized by the following result.

Theorem 1.1. (*Fixed costs option strategies*) Given is the class of option strategies satisfying the constraints (1.1), whose investment value is given by (1.2). Then one has $K(\omega)=const.$ for all $\omega\in\{1,2,3,4\}$ if, and only if, one of the following two cases is fulfilled :

Case 1 : $p+q\le n$

The exercise prices of all options are equal, that $d_r=d_k=d_j=d_i$, and for each feasible choice (m,n,p,q) there exist s different option strategies with fixed costs

$$K(\omega)=(n-m)\cdot d_i-W(i,i,i,i),\quad i\in\{1,...,s\},\ \text{for all}\ \ \omega\in\{1,2,3,4\} \tag{1.19}$$

Case 2 : $p+q>n$

Sub-case (21) : $n-p\ge m-q\ge 0$

The exercise prices $d_k=d_j=d_i$ are equal, and necessarily

$$d_r=d_i+\left(\frac{p+q-n}{m-q}\right)\cdot d_i,\ \text{ with }\ m>q\,. \tag{1.20}$$

For each feasible choice (m,n,p,q) there exist s different option strategies with fixed costs

$$K(\omega)=(n-m)\cdot d_i-W(i,i,i,r),\quad i\in\{1,...,s\},\ \text{for all}\ \ \omega\in\{1,2,3,4\} \tag{1.21}$$

Sub-case (22) : $0<n-p<m-q$

The exercise prices $d_k=d_j$ are equal, and necessarily

$$(n-p)\cdot d_r=q\cdot d_j\,,\quad \left[n-m+\frac{q\cdot(m-q)}{n-p}\right]\cdot d_j=p\cdot d_i\,. \tag{1.22}$$

For each feasible choice (m,n,p,q) there exist s different option strategies with fixed costs

$$K(\omega)=(n-m)\cdot d_j-W(i,j,j,r),\quad i\in\{1,...,s\},\ \text{for all}\ \ \omega\in\{1,2,3,4\} \tag{1.23}$$

Proof. In order that $K(\omega)=const.$ in Case 1, the following quantities must be equal :

$$
\begin{aligned}
\{\omega=1\}:&\quad p\cdot d_i-(m-q)\cdot d_r+[n-(p+q)]\cdot d_k\\
\{\omega=2\}:&\quad (n-m)\cdot d_k\\
\{\omega=3\}:&\quad (n-m)\cdot d_k+q\cdot(d_k-d_j)\\
\{\omega=4\}:&\quad (n-m)\cdot d_k+q\cdot(d_k-d_j)+p\cdot(d_i-d_k)\text{, for sub-case (41),}\\
&\quad (n-m)\cdot d_r+q\cdot(d_r-d_j)+p\cdot(d_i-d_r)\text{, for sub-case (42).}
\end{aligned}
$$

In sub-case (41) compare $\{\omega=2\}$ with $\{\omega=3\}$ and $\{\omega=4\}$ to get immediately $d_k=d_j=d_i$. Comparing now $\{\omega=2\}$ with $\{\omega=1\}$ one must have $(m-q)\cdot(d_r-d_i)=0$. If $m=q$ there are no long put positions, and r can be chosen arbitrarily, say $r=i$. If $m>q$ then necessarily $d_r=d_i$. This settles sub-case (41). In sub-case (42) compare $\{\omega=2\}$ with $\{\omega=3\}$ to get $d_k=d_j$. Comparing now $\{\omega=1\}$ with $\{\omega=4\}$ one must have $(n-p)\cdot(d_r-d_j)=0$. Since $p+q\le n$ and $q\ge 1$, one has necessarily $p<n$, hence $d_r=d_j$. Comparison of $\{\omega=2\}$ and $\{\omega=1\}$ implies $d_j=d_i$. This settles sub-case (42), and Case 1 is shown.
In Case 2 the above quantity for $\{\omega=1\}$ must be replaced by

$$\{\omega=1\}\ :\qquad p\cdot d_i-(m-q)\cdot d_r$$

In sub-case (41), proceeding as above, one must have $d_k=d_j=d_i$. Comparing now $\{\omega=2\}$ with $\{\omega=1\}$ yields the condition $(m-q)\cdot d_r=(m-q)\cdot d_i+[p+q-n]\cdot d_i$. Since $p+q>n$ one must have $m>q$ and thus (1.20) holds. In sub-case (42) compare $\{\omega=2\}$ with $\{\omega=3\}$ to get $d_k=d_j$. Comparing $\{\omega=1\}$ with $\{\omega=4\}$ one gets $(n-p)\cdot d_r=q\cdot d_j$, that is the first part of (1.22), and comparing $\{\omega=2\}$ with $\{\omega=1\}$ yields $(n-m)\cdot d_j=p\cdot d_i-(m-q)\cdot d_r$. Inserting the first relation into the second one yields the second part of (1.22). ◊

3. <u>Probabilities of the option events</u>.

To calculate the expected return and related risk measures of an optioned stock or market index investment under the class of option strategies considered in Section 1, which will be done in Section 3, it is first necessary to evaluate the probabilities of the events $E(\omega)$, $\omega\in\{1,2,3,4\}$.

Consider a short put on the stock with exercise price a and option price $SP(a)$, and a short call with exercise price $b\ge a$ and option price $SC(b)$. As shown below in Lemma 2.1, it suffices to consider the following three probabilities :

(2.1) $P_1(a)=\Pr\{\,S_t\ge a-SP(a)\ \ \textit{for all}\ t\in[0,T]\}$, the probability that a short put with exercise price a is not exercised in the time interval $[0,T]$

(2.2) $P_2(b)=\Pr\{\,S_T>b+SC(b)\,\}$, the probability that a short call with exercise price b is exercised (under the made assumption that American short calls are never exercised before maturity)

(2.3) $P_3(a,b)=\Pr\{\,S_t\ge a-SP(a)\ \ \textit{for all}\ t\in[0,T],\,S_T>b+SC(b)\,\}$, the probability that simultaneously a short call with exercise price b is exercised but a short put with exercise price $a\le b$ is not exercised

For arbitrary stock price processes, the evaluation of (2.1) and (2.3) belongs to the so-called topic of "crossing probabilities". In the ubiquitous special case that S_t follows a geometric

Brownian motion (e.g. Black-Scholes model), these probabilities follow from classical results on absorption problems. In the following, assume that S_t has a log-normal distribution such that the accumulated logarithmic return at time t equals :

(2.4) $$R_t = \ln\left(\frac{S_t}{S_0}\right) = \mu t + \sigma W_t,$$ with W_t the standard Wiener process.

In practical calculations, it is useful to set $\mu = \ln(r) - \frac{1}{2}\sigma^2$. The relation $E[S_T] = S_0 \cdot r^T$ identifies then the parameter r as the accumulated return per unit of time. For convenience, set in the following

(2.5) $$a' = \ln\left(\frac{S_0}{a - SP(a)}\right), \quad b' = \ln\left(\frac{S_0}{b + SC(b)}\right).$$

It is immediate that

(2.6) $$P_2(b) = 1 - \Phi\left(\frac{-b' - \mu T}{\sigma\sqrt{T}}\right),$$

where $\Phi(z)$ is the standard normal distribution. To calculate (2.1) and (2.3), consider the stopping time

(2.7) $$\tau_a = \inf\{t \mid R_t < -a'\}.$$

It is not difficult to see that $P_1(a) = \Pr(\tau_a > T)$ and $P_3(a,b) = \Pr(\tau_a > T, R_T \geq -b')$. Under the assumption (2.4), it is well-known that

(2.8) $$P_1(a) = \Phi\left(\frac{a' + \mu T}{\sigma\sqrt{T}}\right) - \exp\left\{\frac{-2\mu a'}{\sigma^2}\right\} \cdot \Phi\left(\frac{-a' + \mu T}{\sigma\sqrt{T}}\right),$$

(2.9) $$P_3(a,b) = \Phi\left(\frac{b' + \mu T}{\sigma\sqrt{T}}\right) - \exp\left\{\frac{-2\mu a'}{\sigma^2}\right\} \cdot \Phi\left(\frac{b' - 2a' + \mu T}{\sigma\sqrt{T}}\right).$$

In general, the probabilities of the four relevant option events are simply determined from (2.1)-(2.3) as follows.

Lemma 2.1. Given is a short put with exercise price d_i and option price $SP(d_i)$ and a short call with exercise price d_j and option price $SC(d_j)$ as in Section 1. Then the probabilities of the events $E(\omega)$, $\omega \in \{1,...,4\}$, satisfy the following relations :

(2.10) $$\Pr(E(1)) = 1 - P_2(d_j) - P_1(d_i) + P_3(d_i, d_j)$$

(2.11) $$\Pr(E(2)) = P_1(d_i) - P_3(d_i, d_j)$$

(2.12) $$\Pr(E(3)) = P_3(d_i, d_j)$$

(2.13) $$\Pr(E(4)) = P_2(d_j) - P_3(d_i, d_j)$$

Proof. By definition of the events $E(\omega)$ in Section 1, and elementary probability, one has immediately

$$\begin{aligned}
&\Pr(E(2) \cup E(3)) = P_1(d_i), \\
&\Pr(E(1) \cup E(2)) = 1 - P_2(d_j), \\
&\Pr(E(3) \cup E(4)) = P_2(d_j).
\end{aligned}$$

Since $\Pr(E(3)) = P_3(d_i, d_j)$ and the events $E(\omega)$ are disjoint, the relations (2.10)-(2.13) follow without difficulty. ◊

4. The expected return of an optioned stock investment.

Under the assumption (2.4) of a geometric Brownian price process, the expected return of our structured optioned stock investment can be obtained as follows. First, one calculates the *conditional expected return* at maturity date T given an option event, which is denoted and defined by

(3.1) $$ER(\omega, T) = \frac{E[V(\omega,T) | E(\omega)]}{m \cdot S_0 + K(\omega)} - 1, \quad \omega \in \{1, ..., 4\},$$

where $E(\omega), V(\omega,T)$ and $K(\omega)$ have been defined in Section 1. The *expected return* at maturity date T of an optioned stock investment is then given by

(3.2) $$ER(T) = \sum_{\omega=1}^{4} ER(\omega, T) \cdot \Pr(E(\omega)),$$

where the probabilities of the option events are evaluated as in Section 2. Of related interest for decision purposes are the *mean absolute deviation of the conditional expected returns* from the expected return, defined by

(3.3) $$MADR(T) = \sum_{\omega=1}^{4} |ER(\omega, T) - ER(T)| \cdot \Pr(E(\omega)),$$

or similarly the *variance of the conditional expected returns*, defined by

(3.4) $$VarR(T) = \sum_{\omega=1}^{4} (ER(\omega, T) - ER(T))^2 \cdot \Pr(E(\omega)).$$

Example 3.1 : the uniform exercise price option strategies

Explicit expressions for the conditional expected returns (3.1) are in general rather complicated. In the important special case $d_r = d_k = d_j = d_i = d$, relatively simple formulas can be given. In the notations of Section 2, let $P_2(d) = \Pr(S_T \geq d + SC(d))$ the probability that the short calls are exercised, and consider the univariate functions $\pi(x), \bar{\pi}(x)$, called stop-loss and conjugate stop-loss transforms of S_T, which are given by

(3.5)
$$\pi(x) = E[(S_T - x)_+] = \pi(0)\cdot\Phi(u(x)+\sigma\sqrt{T}) - x\cdot\Phi(u(x)-\sigma\sqrt{T}),$$
$$\bar{\pi}(x) = E[(x - S_T)_+] = x - \pi(0) + \pi(x),$$

(3.6)
$$u(x) = \frac{\ln(S_0) + \mu T - \ln(x)}{\sigma\sqrt{T}},$$
$$\pi(0) = E[S_T] = S_0 \cdot \exp(\mu T + \tfrac{1}{2}\sigma^2 T) = S_0 \cdot r^T.$$

Through straightforward calculations, one obtains for the numerators in (3.1) the formulas

(3.7)
$$E[V(1,T)|E(1)] = E[\max(p+q,n)\cdot S_T + \min(p+q,n)\cdot(S_T - d)_+ | S_T < d + SC(d)]$$
$$= \frac{1}{[1-P_2(d)]}\cdot\left\{\begin{array}{l}\max(p+q,n)\cdot([d+SC(d)]\cdot[1-P_2(d)] - \bar{\pi}(d+SC(d))) \\ +\min(p+q,n)\cdot(\pi(d) - \pi(d+SC(d) - SC(d)\cdot P_2(d))\end{array}\right\}$$

(3.8)
$$E[V(2,T)|E(2)] = E[n\cdot S_T + m\cdot(S_T - d)_+ + (m-q)\cdot(d - S_T)_+ | S_T < d + SC(d)]$$
$$= \frac{1}{[1-P_2(d)]}\cdot\left\{\begin{array}{l}n\cdot([d+SC(d)]\cdot[1-P_2(d)] - \bar{\pi}(d+SC(d)) \\ +m\cdot(\pi(d) - \pi(d+SC(d) - SC(d)\cdot P_2(d)) + (m-q)\cdot\bar{\pi}(d)\end{array}\right\}$$

(3.9)
$$E[V(3,T)|E(3)] = E[n\cdot S_T + (m-q)\cdot[(S_T - d)_+ + (d - S_T)_+]| S_T > d + SC(d)]$$
$$= \frac{1}{P_2(d)}\cdot\{n\cdot([d+SC(d)]\cdot P_2(d) + \pi(d+SC(d)) + (m-q)\cdot\pi(d)\}$$

(3.10)
$$E[V(4,T)|E(4)] = E[n\cdot S_T + (m+p-q)\cdot(S_T - d)_+ + (m-q)\cdot(d - S_T)_+ | S_T > d + SC(d)]$$
$$= \frac{1}{P_2(d)}\cdot\{n\cdot([d+SC(d)]\cdot P_2(d) + \pi(d+SC(d)) + (m+p-q)\cdot\pi(d)\}$$

To get the expected return of an uniform exercise price optioned stock investment, insert (3.7)-(3.10) in (3.2) and use the expressions for the probabilities of the option events given in Lemma 2.1. In our special case, these probabilities are given by

(3.11)
$$\Pr(E(1)) = 1 - P_2(d) - P_1(d) + P_3(d,d),$$
$$\Pr(E(2)) = P_1(d) - P_3(d,d),\quad \Pr(E(3)) = P_3(d,d),$$
$$\Pr(E(4)) = P_2(d) - P_3(d,d),$$

where according to (2.5), (2.6), (2.8) and (2.9) one has

(3.12)
$$P_2(d) = 1 - \Phi\left(\frac{-d_2 - \mu T}{\sigma\sqrt{T}}\right)$$

(3.13)
$$P_1(d) = \Phi\left(\frac{d_1 + \mu T}{\sigma\sqrt{T}}\right) - \exp\left\{\frac{-2\mu d_1}{\sigma^2}\right\}\cdot\Phi\left(\frac{-d_1 + \mu T}{\sigma\sqrt{T}}\right)$$

(3.14)
$$P_3(d,d) = \Phi\left(\frac{d_2 + \mu T}{\sigma\sqrt{T}}\right) - \exp\left\{\frac{-2\mu d_1}{\sigma^2}\right\}\cdot\Phi\left(\frac{d_2 - 2d_1 + \mu T}{\sigma\sqrt{T}}\right)$$

(3.15)
$$d_1 = \ln\left(\frac{S_0}{d - SP(d)}\right),\quad d_2 = \ln\left(\frac{S_0}{d + SC(d)}\right).$$

4. Optioned portfolio selection with a shortfall risk criterion.

For a financial practitioner, the main interest will not be the expected return of a single optioned stock investment but that of a whole optioned portfolio investment. To simplify the analysis, we restrict ourselves to the special fixed costs options strategies obtained setting $p+q=n$ in Theorem 1.1. A similar but more complex analysis could also be done for our general structured option strategies.

Given is a portfolio manager, which invests in r different stocks and options on these stocks, all with the same maturity date T. The basic data of our optioned portfolio consists of the following items :

$S_t(i)$	: price at time t of the i-th stock, $i=1,...,r$
$d(i,j)$	: j-th exercise price of an option on the i-th stock, $j=1,...,s$
$SP(i,j)$	: price of a short put on the i-th stock with exercise price $d(i,j)$
$SC(i,j)$	: price of a short call on the i-th stock with exercise price $d(i,j)$
$LC(i,j)$	: price of a long call on the i-th stock with exercise price $d(i,j)$
$LP(i,j)$	: price of a long put on the i-th stock with exercise price $d(i,j)$
m_i	: number of shares at time $t=0$ of the i-th stock
$n_i > m_i$	: number of desired shares at time $t=T$ of the i-th stock
$m_i \ge q_i \ge \frac{1}{2} m_i$	: number of short call positions on the i-th stock
$p_i = n_i - q_i$	: number of short put positions on the i-th stock

For the i-th stock the portfolio manager chooses a fixed cost option strategy with uniform exercise price $d(i,j_i),\ j_i \in \{1,...,s\}$, whose investment value is by (1.2) equal to

$$W(i,j_i) = (n_i - q_i)\cdot SP(i,j_i) + q_i \cdot SC(i,j_i) - n_i \cdot LC(i,j_i) - (m_i - q_i)\cdot LP(i,j_i). \tag{4.1}$$

Including the price of the m shares, the fixed cost of the i-th optioned stock is thus

$$K(i,j_i) = m_i \cdot S_0(i) + (n_i - m_i)\cdot d(i,j_i) - W(i,j_i). \tag{4.2}$$

For each pair (i,j) there are four relevant option events defined as follows (see Section 1) :

$$\begin{aligned} E_1(i,j) &= \{\, S_t(i) < d(i,j) - SP(i,j) \text{ for some } t,\quad S_t(i) < d(i,j) + SC(j) \text{ for all } t \,\} \\ E_2(i,j) &= \{\, d(i,j) - SP(i,j) \le S_t(i) < d(i,j) + SC(i,j) \text{ for all } t \,\} \\ E_3(i,j) &= \{\, d(i,j) - SP(i,j) \le S_t(i) \text{ for all } t,\quad S_T(i) \ge d(i,j) + SC(i,j) \,\} \\ E_4(i,j) &= \{\, S_t(i) < d(i,j) - SP(i,j) \text{ for some } t,\quad S_T(i) \ge d(i,j) + SC(i,j) \,\} \end{aligned} \tag{4.3}$$

The whole *optioned portfolio* is characterized by the chosen vector $\boldsymbol{j} = (j_1,...,j_r)$, for which there are s^r possible choices. For each optioned portfolio $\boldsymbol{j}$ exactly 4^r different *optioned vector events* may occur, each characterized by a vector $(\omega, j) = (\omega_1,...,\omega_r, j_1,...,j_r)$, which are denoted and defined by

$$E(\omega, j) = \left(E_{\omega_1}(1,j_1), E_{\omega_2}(2,j_2),..., E_{\omega_r}(r,j_r)\right). \tag{4.4}$$

Denoting by $\Pr_{\omega}(i,j)$ the probability of the event $E_{\omega}(i,j),\ \omega\in\{1,...,4\}$, which can be calculated applying the method of Section 2 (as will be done later in Section 5), the probability $\Pr(\omega,j)$ of a vector event (4.4) will depend in some multivariate model on the probabilities $\Pr_{\omega_i}(i,j_i)$ of the single events. For example, if the events are independent, an assumption which must be tested, the probabilities are given by

(4.5) $$\Pr(\omega,j)=\prod_{i=1}^{r}\Pr_{\omega_i}(i,j_i).$$

Note that under some more general assumption of "positive dependence" between the events, the left-hand side will always be at least equal to the right-hand side. Moreover, under reasonable assumptions, there exist quite simple upper bounds (e.g. Xie and Lai(1998), see also Chow and Liu(1968)). As a direct consequence, the probability $\Pr(\omega,j)$ can always be two-sided bounded in ways, which presumably will suffice for most practical situations.

To get the *investment value* $V(\omega,j,T)$ of the optioned portfolio at time T given that $E(\omega,j)$ occurs, it is necessary to display first the investment value $V_{\omega}(i,j,T)$ of an i-th optioned stock given that $E_{\omega}(i,j)$ occurs. From the analysis made in Section 1, one has

(4.6) $$\begin{aligned}
&V_1(i,j,T)=n_i\cdot\left[S_T(i)+\left(S_T(i)-d(i,j)\right)_+\right]\\
&V_2(i,j,T)=n_i\cdot S_T(i)+m_i\cdot\left(S_T(i)-d(i,j)\right)_+ +(m_i-q_i)\cdot\left(d(i,j)-S_T(i)\right)_+\\
&V_3(i,j,T)=n_i\cdot S_T(i)+(m_i-q_i)\cdot\left[\left(S_T(i)-d(i,j)\right)_+ +\left(d(i,j)-S_T(i)\right)_+\right]\\
&V_4(i,j,T)=n_i\cdot S_T(i)+(n_i-q_i)\cdot\left(S_T(i)-d(i,j)\right)_+\\
&\quad+(m_i-q_i)\cdot\left[\left(S_T(i)-d(i,j)\right)_+ +\left(d(i,j)-S_T(i)\right)_+\right]
\end{aligned}$$

Through addition, the investment value of the optioned portfolio, given that $E(\omega,j)$ occurs, is equal to

(4.7) $$V(\omega,j,T)=\sum\nolimits_{i=1}^{r}V_{\omega_i}(i,j_i,T).$$

Using (4.2) the fixed cost of the optioned portfolio $\boldsymbol{j}$ equals

(4.8) $$K(j)=\sum\nolimits_{i=1}^{r}K(i,j_i).$$

Similarly to Section 3, consider now the *conditional expected return* given a *vector event* (ω,j) of the optioned portfolio j at date *T*, denoted and defined by

(4.9) $$ER(\omega,j,T)=\frac{E\left[V(\omega,j,T)\middle|E(\omega,j)\right]}{K(j)}-1.$$

Then the *expected return* of the optioned portfolio $\boldsymbol{j}$ at date T equals

(4.10) $$ER(j,T)=\sum_{\omega}ER(\omega,j,T)\cdot\Pr(\omega,j).$$

A related mean absolute deviation and variance of the conditional expected returns similar to (3.3) and (3.4) can also be defined. For practical calculations of these quantities, there remains the question of the evaluation of the probabilities $\Pr(\omega, j)$ (see Section 5).

Of main importance in portfolio selection is the trade-off between expected return and risk. Since the variance of the return should presumably not be an appropriate risk measure (due to the asymmetric optioned portfolio return distribution), an alternative risk measure of the optioned portfolio must be used. Following the shortfall method, which is a slight extension of the well-known value-at-risk method, we propose to use a special *shortfall risk measure* on the investment value of the optioned portfolio known as *CVaR measure* (conditional value-at-risk measure) (e.g. Hürlimann(2001a/b,2003), Rockafellar and Uryasev(2001)).

The CVaR measure may be defined as follows. Let X be the random variable of the *negative return* of the optioned portfolio $\boldsymbol{j}$, which depends on the events $E(\omega, j)$, given by

(4.11) $$X = 1 - \frac{V(\omega, j, T)}{K(j)} \quad \text{if } E(\omega, j) \text{ occurs.}$$

Given a small loss tolerance level ε (usually $\varepsilon = 0.05$ or 0.01), the CVaR measure of X to the level ε is defined by the quantity

(4.12) $$CVaR_\varepsilon[X] = Q_X(1-\varepsilon) + \frac{1}{\varepsilon} \cdot \pi_X[Q_X(1-\varepsilon)],$$

where $Q_X(u)$ is the quantile function of X, and $\pi_X(x) = E[(X-x)_+]$ is the stop-loss transform of X. The CVaR measure of risk satisfies numerous desirable properties. For example, it is a coherent risk measure in the sense of Arztner et al.(1997/99). It generates an ordering of risk, called *CVaR order*, which is equivalent to the stop-loss order or equivalently the increasing convex order (Hürlimann(2001a), Theorem 1.1, Hürlimann(2003), Proposition 2.2). Therefore, the CVaR order is compatible with the common preferences of risk averse decision makers, which use concave non-decreasing utility functions. Similarly to the classical mean-variance portfolio selection, one can consider a mean-CVaR portfolio selection, in particular a mean-CVaR optioned portfolio selection. For practical calculations of the CVaR measure (4.12), the probabilities $\Pr(\omega, j)$ remain to be evaluated. Indeed, the distribution function $F_X(x)$, and the stop-loss transform $\pi_X(x)$ of X, which must be known for the evaluation of (4.12), are obtained from the conditional distributions of X given $E(\omega, j)$ using the theorem on total probability :

(4.13) $$F_X(x) = \sum_\omega F_X(x | E(\omega, j)) \cdot \Pr(\omega, j),$$

(4.14) $$\begin{aligned} \pi_X(x) &= \int_x^\infty \overline{F}_X(t)dt = \sum_{\omega=1}^4 \left[\int_x^\infty \overline{F}_X(t | E(\omega, j))dt \right] \cdot \Pr(\omega, j) \\ &= \sum_{\omega=1}^4 E[(X-x)_+ | E(\omega, j)] \cdot \Pr(\omega, j) = \sum_{\omega=1}^4 \pi_X(x | E(\omega, j)) \cdot \Pr(\omega, j). \end{aligned}$$

The required conditional distributions and stop-loss transforms can be obtained from (4.7), (4.8) using (4.2), (4.6), and the specification of a joint distribution of $(S_T(1), \ldots, S_T(r))$ using the stock price dynamic described in Section 5. Unless $r = 1$ (e.g. the single uniform exercise price optioned stock investment of Example 3.1) the analytical evaluation of the required conditional quantities in (4.13) and (4.14) might be rather cumbersome and numerical methods should be considered.

5. Probabilities of the optioned vector events.

The expected return (4.10) and the CVaR-measure (4.12) required for a mean-CVaR optioned portfolio selection depend on the probabilities (4.5). Using the results of Section 2, a geometric Brownian motion assumption for the behaviour of stock prices, and the continuous Capital Asset Pricing Model by Merton(1972), we show how the probabilities $\Pr_{\omega}(i,j)$ of the elementary option events $E_{\omega}(i,j)$ in (4.3) can be evaluated.

The individual stock prices are linked to the general trend of the financial market, which is described by a stock market index $S_t(M)$, whose dynamic is supposed to follow an Ito-process

(5.1) $$dS(M) = r_M S(M)dt + \sigma_M S(M)dW(M),$$

where r_M is the instantaneous rate of return of the market index, σ_M^2 is the instantaneous variance rate, and $W_t(M)$ is a standard Wiener process. The index linked continuous price processes $S_t(i)$ of the individual stocks are described by

(5.2) $$dS(i) = r_i S(i)dt + \eta_{iM} S(i)dW(M) + \eta_{ii} S(i)dW(i),$$

where r_i is the instantaneous rate of return of the i-th stock, $\eta_{iM} = \frac{c_{iM}}{\sigma_M}$ is the relationship between the instantaneous covariance-rate c_{iM} of $S_t(i)$ and $S_t(M)$ and the volatility σ_M of the market index, and $W_t(i)$ is a standard Wiener process. One assumes that the parameters $r_M, \sigma_M, r_i, \eta_{ii}$ and η_{iM} are constant over the time horizon T, and that the processes $W_t(M)$ and $W_t(i)$ are independent for all $i = 1,...,r$. It follows that

(5.3) $$dS(i) = r_i S(i)dt + \sigma_i S(i)dW'(i),$$

where $\sigma_i^2 = \eta_{iM}^2 + \eta_{ii}^2$ is the instantaneous variance rate of the i-th stock, and $W_t'(i)$ is another standard Wiener process. Furthermore, one has $c_{iM} = \rho_{iM}\sigma_i\sigma_M$, where ρ_{iM} is the correlation coefficient between $S_t(i)$ and $S_t(M)$. The rate of return of the i-th stock is linked to the market rate of return following the continuous CAPM relationship :

(5.4) $$r_i - r_f = \beta_i \cdot (r_M - r_f),$$

where r_f is the instantaneous risk-free rate of return, and

(5.5) $$\beta_i = \rho_{iM} \cdot \left(\frac{\sigma_i}{\sigma_M}\right)$$

is the beta coefficient of the i-th stock. The knowledge of $\sigma_M, \sigma_i, \rho_{iM}, r_f$ and r_M, suffices to determine r_i using (5.4) and (5.5), and thus the stock price process (5.3). In particular, the price process $S_t(i)$ follows a lognormal distribution such that the accumulated logarithmic return at time t is given by

(5.6) $$R_t(i) = \ln\left\{\frac{S_t(i)}{S_0(i)}\right\} = \mu_i t + \sigma_i W_t^{'}(i), \quad \mu_i = r_i - \tfrac{1}{2}\sigma_i^2 .$$

The above analysis has reduced the portfolio situation to the standard situation (2.4) of a single stock, which allows one to calculate the probabilities $\Pr_\omega(i,j)$ of the events $E_\omega(i,j)$.

For each $i = 1,...,r,\ j = 1,...,s$ set similarly to (2.5)

(5.7) $$d_1(i,j) = \ln\left\{\frac{S_0(i)}{d(i,j) - SP(i,j)}\right\}, \quad d_2(i,j) = \ln\left\{\frac{S_0(i)}{d(i,j) + SC(i,j)}\right\}.$$

In analogy to (2.6), (2.8) and (2.9) define

(5.8) $$P_2(i,j) = 1 - \Phi\left(\frac{-d_2(i,j) - \mu_i T}{\sigma_i\sqrt{T}}\right),$$

(5.9) $$P_1(i,j) = \Phi\left(\frac{d_1(i,j) + \mu_i T}{\sigma_i\sqrt{T}}\right) - \exp\left\{\frac{-2\mu_i d_1(i,j)}{\sigma_i^2}\right\}\Phi\left(\frac{-d_1(i,j) + \mu_i T}{\sigma_i\sqrt{T}}\right),$$

(5.10) $$P_3(i,j) = \Phi\left(\frac{d_2(i,j) + \mu_i T}{\sigma_i\sqrt{T}}\right) - \exp\left\{\frac{-2\mu_i d_1(i,j)}{\sigma_i^2}\right\}\Phi\left(\frac{d_2(i,i) - 2d_1(i,j) + \mu_i T}{\sigma_i\sqrt{T}}\right).$$

Applying Lemma 2.1, one obtains finally the desired probabilities :

(5.11) $$\Pr_1(i,j) = 1 - P_2(i,j) - P_1(i,j) + P_3(i,j),$$

(5.12) $$\Pr_2(i,j) = P_1(i,j) - P_3(i,j),$$

(5.13) $$\Pr_3(i,j) = P_3(i,j),$$

(5.14) $$\Pr_4(i,j) = P_2(i,j) - P_3(i,j).$$

6. CVaR for an optioned stock investment with uniform exercise prices.

It is useful to illustrate our theoretical analysis for the special case of an optioned stock investment with uniform exercise prices, which includes in particular fixed cost option strategies (Case 1 of Theorem 1.1).

The expected return of such an investment can be calculated with the formulas (3.1), (3.2) using (3.7)-(3.11). To compute CVaR according to formula (4.12), it is necessary to obtain first the distribution of the negative return

(6.1) $$F_X(x) = \sum_{\omega=1}^{4} F_X(x \mid E(\omega)) \cdot \Pr(E(\omega)),$$

and its stop-loss transform

(6.2) $$\pi_X(x) = \sum_{\omega=1}^{4} \pi_X(x|E(\omega)) \cdot \Pr(E(\omega)),$$

where the negative return is described by

(6.3) $$X = 1 - \frac{V(\omega,T)}{V(\omega)} \quad \text{if } E(\omega) \text{ occurs, with}$$
$$V(\omega) = m \cdot S_0 + K(\omega) \quad \text{the initial cost of the optioned stock investment.}$$

We distinguish between two cases. The distribution of S_T is denoted by $F_T(x) = \Pr(S_T \le x)$, and $\overline{F}_T(x) = 1 - F_T(x)$ denotes the survival function.

<u>Case 1</u> : events $E(\omega), \omega = 1,2$

Decomposing the event $\{S_T < d + SC(d)\}$ into the two disjoint events $\{0 \le S_T < d\}$ and $\{d \le S_T < d + SC(d)\}$ and applying the rules for conditional probabilities, one obtains using the formulas of Section 1 after some calculations

(6.4) $$\begin{aligned} F_X(x|E(\omega)) &= \Pr\left(1 - \frac{V(\omega,T)}{V(\omega)} \le x \middle| S_T < d + SC(d)\right) \\ &= \frac{\Pr(\{S_T \ge \beta_{\omega 1}(x)\} \cap \{0 \le S_T \le d\}) + \Pr(\{S_T \ge \beta_{\omega 2}(x)\} \cap \{d \le S_T < d + SC(d)\})}{\Pr(S_T < d + SC(d))} \\ &= \frac{\Pr_{\omega 1}(x) + \Pr_{\omega 2}(x)}{F_T(d + SC(d))} \end{aligned}$$

(6.5) $$\begin{aligned} \beta_{11}(x) &= \frac{1}{\max(p+q,n)} \cdot (1-x)V(1) \\ \beta_{12}(x) &= \frac{1}{p+q+n} \cdot [(1-x)V(1) + \min(p+q,n) \cdot d] \\ \beta_{21}(x) &= \frac{1}{n-m+q} \cdot [(1-x)V(2) - (m-q) \cdot d] \\ \beta_{22}(x) &= \frac{1}{n+m} \cdot [(1-x)V(2) + m \cdot d] \end{aligned}$$

(6.6) $$\begin{aligned} \Pr_{\omega 1}(x) &= \begin{cases} 0, & \beta_{\omega 1}(x) \ge d \\ F_T(d) - F_T(\max\{\beta_{\omega 1}(x), 0\}, & \beta_{\omega 1}(x) < d \end{cases} \\ \Pr_{\omega 2}(x) &= \begin{cases} 0, & \beta_{\omega 2}(x) \ge d + SC(d) \\ F_T(d + SC(d)) - F_T(\max\{\beta_{\omega 2}(x), d\}, & \beta_{\omega 2}(x) < d + SC(d) \end{cases} \end{aligned}$$

Similarly, one obtains the decomposition

$$
\begin{aligned}
\pi_X(x|E(\omega)) &= E\left((1-\frac{V(\omega,T)}{V(\omega)}-x)_+ \middle| S_T < d + SC(d) \right) \\
&= \alpha_{\omega 1}\cdot E[(\beta_{\omega 1}(x)-S_T)_+ \cdot I\{0 \le S_T < d\}] \\
&\quad + \alpha_{\omega 2}\cdot E[(\beta_{\omega 2}(x)-S_T)_+ \cdot I\{d \le S_T < d+SC(d)\}] \\
&= \alpha_{\omega 1}\cdot \pi_{\omega 1}(x) + \alpha_{\omega 2}\cdot \pi_{\omega 2}(x)
\end{aligned} \tag{6.7}
$$

with

$$
\begin{aligned}
&\alpha_{11} = \frac{\max(p+q,n)}{V(1)}, \quad \alpha_{12} = \frac{p+q+n}{V(1)}, \\
&\alpha_{21} = \frac{n-m+q}{V(2)}, \quad \alpha_{22} = \frac{n+m}{V(2)},
\end{aligned} \tag{6.8}
$$

the $\beta_{\omega j}(x),\ j=1,2,$ as above, and $I\{\}$ the indicator function of the event $\{\}$. After some straightforward but tedious calculations, one gets the expressions

$$
\pi_{\omega 1}(x) = \begin{cases} E[(\beta_{\omega 1}(x)-S_T)_+], & \beta_{\omega 1}(x) < d \\ (\beta_{\omega 1}(x)-d)\cdot F_T(d) + E[(d-S_T)_+], & \beta_{\omega 1}(x) \ge d) \end{cases} \tag{6.9}
$$

$$
\pi_{\omega 2}(x) = \begin{cases} 0, \quad \beta_{\omega 2}(x) < d \\ E[(\beta_{\omega 2}(x)-S_T)_+] - E[(d-S_T)_+] - (\beta_{\omega 2}(x)-d)\cdot F_T(d), \\ d \le \beta_{\omega 2}(x) < d+SC(d) \\ (\beta_{\omega 2}(x)-d-SC(d))\cdot (F_T(d+SC(d)) - F_T(d)) \\ + E[(d+SC(d)-S_T)_+] - E[(d-S_T)_+] - SC(d)\cdot F_T(d), \\ \beta_{\omega 2}(x) \ge d+SC(d) \end{cases} \tag{6.10}
$$

<u>Case 2</u> : events $E(\omega), \omega = 2,3$

It is not difficult to see that

$$
\begin{aligned}
F_X(x|E(\omega)) &= \Pr\left(1-\frac{V(\omega,T)}{V(\omega)} \le x \middle| S_T \ge d + SC(d) \right) \\
&= \frac{\Pr(\{S_T \ge \beta_\omega(x)\} \cap \{S_T \ge d+SC(d)\})}{\Pr(S_T \ge d+SC(d))} = \frac{\Pr_\omega(x)}{\overline{F}_T(d+SC(d))}
\end{aligned} \tag{6.11}
$$

with

$$
\begin{aligned}
\beta_3(x) &= \frac{1}{n+m-q}\cdot [(1-x)V(3) + (m-q)\cdot d] \\
\beta_4(x) &= \frac{1}{n+m+p-q}\cdot [(1-x)V(4) + (m+p-q)\cdot d]
\end{aligned} \tag{6.12}
$$

$$
\Pr_\omega(x) = \overline{F}_T(\max\{\beta_\omega(x), d+SC(d)\} \tag{6.13}
$$

Similarly, one gets

$$\pi_X(x|E(\omega)) = E\left((1-\frac{V(\omega,T)}{V(\omega)} - x)_+ \middle| S_T \geq d + SC(d) \right) \tag{6.14}$$
$$= \alpha_\omega \cdot E\left[(\beta_\omega(x) - S_T)_+ \cdot I\{S_T \geq d + SC(d)\} \right] = \alpha_\omega \cdot \pi_\omega(x),$$

with

$$\alpha_3 = \frac{n+m-q}{V(3)}, \quad \alpha_4 = \frac{n+m+p-q}{V(4)}, \tag{6.15}$$

and

$$\pi_\omega(x) = E\left[(\beta_\omega(x) - S_T)_+ \right] - E\left[(d + SC(d) - S_T)_+ \right] \tag{6.16}$$
$$- \left(\beta_\omega(x) - d - SC(d) \right) \cdot F_T(d + SC(d)).$$

Now, to get practical values of $CVaR_\varepsilon[X]$ for given ε, say $\varepsilon = 0.05$ and/or 0.01, first find an accurate approximation x_ε to $Q_X(1-\varepsilon) = \inf\{x : F_X(x) \geq 1-\varepsilon\}$ through numerical computation of (6.1) using (6.4) and (6.11). Then, based on (6.7) and (6.14), calculate $\pi_X(x_\varepsilon)$ according to (6.2). The obtained value $x_\varepsilon + \frac{1}{\varepsilon} \cdot \pi_X(x_\varepsilon)$ will be an accurate approximation to $CVaR_\varepsilon[X] = Q_X(1-\varepsilon) + \frac{1}{\varepsilon} \cdot \pi_X[Q_X(1-\varepsilon)]$.

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