Some New Classes of Consistent Risk Measures

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Risk measure vs. Insurance premium principle

- **Risk measure:** a mapping from $\mathbb{S}$ into $\mathbb{R}$
- **Insurance premium principle:** a mapping from $\mathbb{S}$ into $\mathbb{R}$ (see H. Gerber, H. Bühlmann etc.)
- **Difference?**
  IPP: “implicitly” seen to be a price, hence in monetary units
  RM: $\rho(aX) = a\rho(X)$ is also reduced to be in monetary units
  Hence: RM $\subset$ IPP

$$\rho(X) = \mathrm{E}[X] + \frac{\alpha}{u}\sigma^2(X) \qquad (\neq \text{ RM, IPP})$$

But: $Y = X + \frac{\alpha}{u}(X - \mathrm{E}[X])^2$, take $\mathrm{E}[Y]$

$$\mathrm{E}[Y] = \mathrm{E}[X] + \frac{\alpha}{u}\sigma^2(X) \qquad (= \text{ RM, IPP})$$

Set of axioms $\mathbb{S}$ characterizing (or as properties) is the topic.

Examples

- **Example 1.1** (Insurance – Reinsurance)

$$X = [X - (X-d)_+] + (X-d)_+$$

additive for comonotonic risks

$$\pi[X] = \pi[X - (X-d)_+] + \pi[(X-d)_+]$$

(characterization Yaari-Wang)

- **Example 1.2** (Premium Calculation)

$$\pi[X] \leq \pi[aX] + \pi[(1-a)X]$$

subadditive risk measures for comonotonic risks
(characterization Orlicz insurance premium)

Examples

- **Example 1.3** (Premium Calculation from Top-down) additive risk measure for independent risks

$$\pi\left[X_1^{\perp} + X_2^{\perp}\right] = \pi\left[X_1^{\perp}\right] + \pi\left[X_2^{\perp}\right]$$

(characterization by Gerber et al.)

$$\rho(X) = \frac{1}{R}\log \mathrm{E}\left[e^{RS}\right]$$

$$R = \frac{1}{u}|\log \varepsilon|$$

Examples

- **Example 1.4** (Capital Allocation)

$$\pi\left[X_1+X_2\right] \geq \pi\left[X_1\right]+\pi\left[X_2\right]$$

$$\left(X_1+X_2-\pi\left[X_1+X_2\right]\right)_+ \leq_1 \left(X_1-\pi\left[X_1\right]\right)_+ + \left(X_2-\pi\left[X_2\right]\right)_+$$

(effect of diversification)

In case of additivity

$$(i-r)\pi(X_1+X_2)=(i-r)\pi(X_1)+(i-r)\pi(X_2)$$

Subadditivity does not describe diversification.

Preservation of stochastic dominance is relevant!

Examples

- **Example 1.5** (Solvency Margin)
  - Bernoulli(q) risk B_q, $q \in [0,1]$
  - $\pi\left[aB_0\right] = \pi\left[aB_1\right] = 0$
  - $\pi\left[aB_q\right] = \pi\left[aB_{1-q}\right]$

$$\mathrm{E}\left[B_q\right](1+\lambda) = q\left(1+\alpha(1-q)\right) = q + \alpha q(1-q)$$

  - $\pi\left[B_q\right] = \pi\left[B_{1-q}\right] = \alpha q(1-q)$

Examples

- **Example 1.6** (Change of monetary unit)

$$\begin{aligned} \rho(X) &= \mathrm{E}[X] + \alpha\, \mathrm{Var}[X] \\ \rho(X) &= \mathrm{E}[X] + \frac{|\ln \varepsilon|}{u}\, \mathrm{Var}[X] \end{aligned}$$

- **Example 1.7**

$$X \leq_1 Y \Rightarrow \sout{\sigma^2(X)}$$

$$\sigma(X+Y) \leq \sigma(X) + \sigma(Y):$$ subadditivity

- **Example 1.8** (Capital allocation)

$$\rho(X_1 - u_1) + \rho(X_2 - u_2) = \rho(X_1) + \rho(X_2) - u_1 - u_2$$

Capital allocation must be based on an incoherent risk measure

Examples

- **Example 1.9** (Cost as risk measure)

$$\min_u (i-r)u + \mathrm{E}[(X-u)_+] = (i-r)F_X^{-1}(1-(i-r)) + \mathrm{E}[X - F_X^{-1}(1-(i-r))]$$

$$u = F_X^{-1}(1-(i-r)) \qquad \text{VaR!!} \quad \text{(pitfall)}$$

(dependence structure)

- **Example 1.10**

$$\rho(X^c + Y^c) < \rho(X^c) + \rho(Y^c)!!$$

- **Example 1.11**

Less capital for a group without firewalls: wrong!

Examples

- **Example 1.11** Residual or remaining risk

$$u: \text{ economic capital}$$
$$\rho((X-u)_+): \text{ remaining risk}$$

Optimizing the remaining risk is the economic criterion for determining the economic capital.
This is completely neglected in Artzner (1999, NAAJ)
Compare Physics, minimizing the energy

- **Example 1.12**

$$\rho(X+Y) \leq \rho(X) + \rho(Y) \Rightarrow \rho(X+Y-\rho(Y)) \leq \rho(X)$$

Let X, Y be B_q and assume n comonotonic B_q risks.
Surplus S: $\quad \Pr[S = u + n\rho(X)] = 1 - q$
$\Pr[S = u + n(\rho(X) - 1)] = q \qquad (\rho(X) < 1)$

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$(\mathbb{S}, \alpha)$–consistent risk measures

Let $\mathbb{S}$ be a set of axioms for risk measures, and α, $0 < \alpha < 1$, be a level. A risk measure $\pi[\cdot] = \pi_{(\mathbb{S},\alpha)}[\cdot] = \pi_\alpha[\cdot]$ is called $(\mathbb{S}, \alpha)$–consistent if $\pi[\cdot]$ is a rule that assigns a value to each risk X satisfying the axioms $\mathbb{S}$ and such that $\pi[X] \geq F_X^{-1}(\alpha)$, where $F_X^{-1}(\alpha)$ is the αth quantile of the risk X, and is defined, as usual, by $F_X^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\}$.

Sets of axioms

Set $\mathbb{S}_1$ of axioms

A1. $X \leq_1 Y \Longrightarrow \pi[X] \leq \pi[Y]$, where $\leq_1$ denotes “stochastically not greater than”;

A2. $\pi[X^c + Y^c] = \pi[X^{*c} + Y^c]$ provided that $\pi[X] = \pi[X^*]$;

A3. $\pi[X_n]$ converges to $\pi[X]$ if F_{X_n} is non-decreasing and converges weakly to F_X.

Set $\mathbb{S}_2$ of axioms

B1. $X \leq_1 Y \Longrightarrow \pi[X] \leq \pi[Y]$;

B2. $\pi[aX] = a\pi[X]$ for $a > 0$;

B3. $\pi[X_1 + X_2] \leq \pi[X_1] + \pi[X_2]$.

Orlicz risk measure

Let X be a risk variable, and let $\phi(\cdot)$ be a non-negative, strictly increasing, and continuous function on $[0,+\infty)$ with $\phi(0)=0$, $\phi(1)=1$, and $\phi(+\infty)=+\infty$. Then for any $-\infty < x < \max[X]$ and $0<\alpha<1$, the equation

$$\mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right] = 1-\alpha$$

has a unique solution $\pi_\alpha\left[X,x\right]$ satisfying the inequalities

$$\pi_\alpha\left[X,x\right] \geq F_X^{-1}(\alpha) \qquad \text{and} \qquad \pi_\alpha\left[X,x\right] > x.$$

Note that

$$\Pr\left[X > \pi\right] = \Pr\left[X - x > \pi - x\right] \leq \mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right].$$

Haezendonck risk measure

Let X be a risk variable, let $\phi(\cdot)$ be a non-negative, strictly increasing, and continuous function with $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(+\infty) = +\infty$, and let $0 < \alpha < 1$ be fixed. We consider

$$\pi_\alpha [X] = \inf_{-\infty < x < \max[X]} \pi_\alpha [X, x]$$

as the risk measure for the risk variable X, where $\pi_\alpha [X, x]$ is the unique solution of the equation

$$\mathrm{E}\left[\phi\left(\frac{(X-x)_+}{\pi - x}\right)\right] = 1 - \alpha.$$

In honor of the late J. Haezendonck we call it the *Haezendonck risk measure*, which is a minimal Orlicz risk measure.

Haezendonck risk measure: properties

The Haezendonck risk measure $\pi_\alpha[X]$ satisfies

$$F_X^{-1}(\alpha) \leq \pi_\alpha[X] \leq \max[X]$$

and

B1. Monotonicity: If $X \leq_1 Y$ then $\pi_\alpha[X] \leq \pi_\alpha[Y]$;

B2. Positive Homogeneity: $\pi_\alpha[cX] = c\pi_\alpha[X]$ for any $c > 0$;

B3. Subadditivity: If $\phi(\cdot)$ is convex, then
$\pi_\alpha[X+Y] \leq \pi_\alpha[X] + \pi_\alpha[Y]$ holds for any (X,Y) such that

$$\max[X+Y] = \max[X] + \max[Y];$$

B4. Translation Invariance: $\pi_\alpha[X+a] = \pi_\alpha[X] + a$ for any a;

B5. Preservation of convex ordering: If $\phi(\cdot)$ is convex, then
$X \leq_{cx} Y \Longrightarrow \pi_\alpha(X) \leq \pi_\alpha(Y)$.

Haezendonck risk measure: more properties

Definition: Let $\phi_1(\cdot)$ and $\phi_2(\cdot)$ be two real functions on $(0, +\infty)$. We say $\phi_2(\cdot)$ is convex (concave) in $\phi_1(\cdot)$ if and only if $\phi_2\phi_1^{-1}(\cdot)$ is convex (concave).

Haezendonck risk measure: more properties

Let $\phi_i(\cdot)$, $i = 1, 2$, be two continuous and strictly increasing functions with $\phi_i(x) = x$ for $x \in [0, 1]$ and $\phi_i(+\infty) = +\infty$, let $\pi_\alpha^{(i)}[X, x]$, $i = 1, 2$, be the solutions of $\mathrm{E}\left[\phi_i\left(\frac{(X-x)_+}{\pi - x}\right)\right] = 1 - \alpha$, and let the corresponding Haezendonck risk measures be

$$\pi_\alpha^{(i)}[X] = \inf_{-\infty < x < \max[X]} \pi_\alpha^{(i)}[X, x], \qquad i = 1, 2.$$

1. If $\phi_2(\cdot)$ is convex in $\phi_1(\cdot)$ then $\pi_\alpha^{(1)}[X, x] \leq \pi_\alpha^{(2)}[X, x]$, hence $\pi_\alpha^{(1)}[X] \leq \pi_\alpha^{(2)}[X]$;
2. If $\phi_2(\cdot)$ is concave in $\phi_1(\cdot)$ then $\pi_\alpha^{(1)}[X, x] \geq \pi_\alpha^{(2)}[X, x]$, hence $\pi_\alpha^{(1)}[X] \geq \pi_\alpha^{(2)}[X]$.

Haezendonck risk measure: more properties

The Tail-VaR, which is defined by

$$\text{TVaR}_{\alpha}\left[X\right] = F_X^{-1}(\alpha) + \frac{1}{1-\alpha}\,\mathrm{E}\left[\left(X - F_X^{-1}(\alpha)\right)_+\right], \qquad \alpha \in (0,1)\,,$$

is the smallest one among those Haezendonck risk measures $\pi_{\alpha}\left[X\right]$ that correspond to strictly increasing and convex functions $\phi(\cdot)$ satisfying $\phi(x) = x$ for $0 < x < 1$, and is the largest one among those Haezendonck risk measures $\pi_{\alpha}[X]$ that correspond to strictly increasing and concave functions $\phi(\cdot)$ satisfying $\phi(x) = x$ for $0 < x < 1$ and $\phi(+\infty) = +\infty$.